Journal of the Society of Korea Industrial and Systems Engineering
Vol. 35, No. 1, pp.169－180, March 2012.

복수제품의 순환생산을 위한 생산배송계획 수립연구

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Production and Shipment Planning for a Multi-Product Inventory Model with Cyclic Production Incorporated

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본 논문은 (1 : 1 : N) 재고모형에 대한 복수제품의 순환생산 및 배송 일정계획을 수립하는 연구를 수행하였다. 세부적으로 공급자가 원자재를 생산자에게 배송하면, 생산자는 순환생산방식을 활용하여 복수의 제품을 생산하여 N 구매자에게 배송하는 상황을 고려하고 있다. 본 연구의 목적은 공급자, 생산자, 구매자를 포함하는 시스템 전체의 비용을 최소화하는 계획을 수립하는 것이다. 최적해가 가지는 몇 가지 특성들을 분석하고, 이를 통해서 단계적인 휴리스틱 절차를 제시할 것이다. 테스트 실험을 통해서 제시된 휴리스틱 절차가 매우 효과적이면서도 효율적이라는 점을 입증하게 될 것이다.

Keywords : Inventory, Multi-Product, Cyclic Production Policy, Integrated Vendor-Buyer Model, Heuristic

1. Introduction

With growing focus on supply chain management, firms realize that inventories across any associated entire supply chain can be more efficiently managed through greater cooperation and better coordination among the associated decision activities. Recalling inventory theory, it is well known that the economic order quantity (EOQ) formula can give an optimal solution for each of the associated problems if any vendor and buyer inventory problems are treated in separation. However, it is often the case that the optimal EOQ solution for the separate buyer may be unacceptable to the vendor. Likewise, the optimal EOQ solution for the separate vendor may be unacceptable to the buyer [17].

In a typical purchasing situation where the vendor negotiates with the buyer about both price and lot size, the outcome often depends on balance of market power between them. Sometimes, collaborative arrangement between them is enforced in some contractual agreement. For example, from the vendor's point of view, Monahan [20] has developed a model offering quantity

논문접수일 : 2012년 02월 03일　　게재확정일 : 2012년 03월 12일
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※ This study was supported by research funds from Dong-A University.

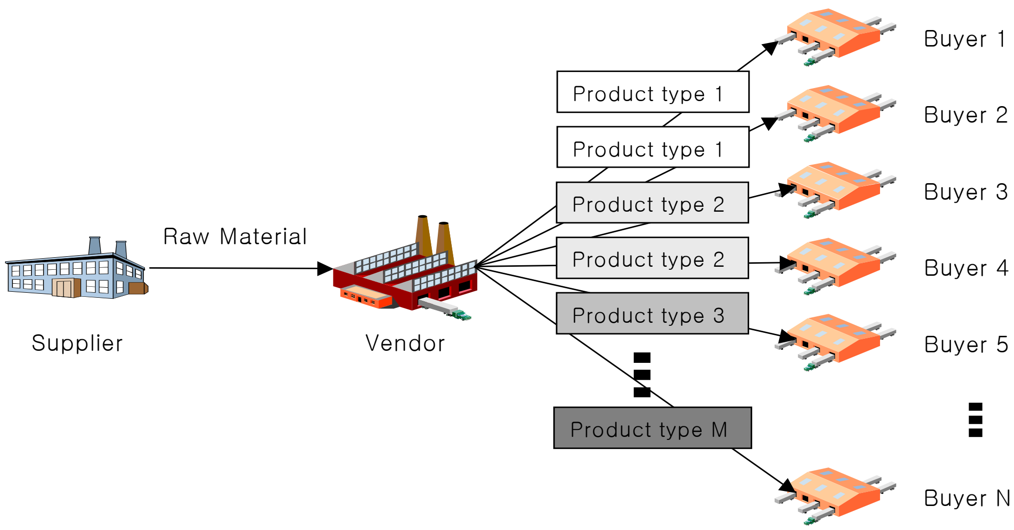
discount so as to induce the buyer to order in a quantity that maximizes the vendor's profit. Banerjee [2] has generalized Monahan's model by incorporating a vendor's inventory carrying cost. Rosenblatt and Lee [22] have also considered the model, by removing the assumption of Monahan [20] and Banerjee [2] that the vendor performs his part of negotiation based on a lot-for-lot policy.

Goyal [11] has considered a joint economic lot size model where the objective is to minimize total relevant cost for a one-vendor one-buyer system. In the lot size model, the lot-for-lot policy is incorporated, and everybody can contribute himself to the whole system rather than his own benefit. Banerjee [2] has generalized Goyal's model by incorporating a finite production rate for the vendor. Goyal [9] has also generalized Banerjee's model by removing the lot-for-lot policy for the vendor, and showed that his model provides a lower (or equal) joint total relevant cost.

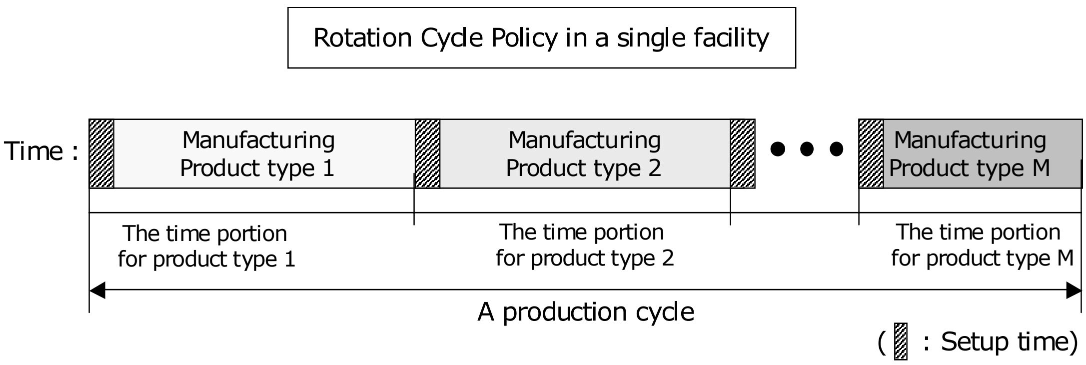
Goyal [10, 12] and Hill [13, 14] have studied very interesting problems associated with production and shipment policy for a one-vendor one-buyer system. Sharafali [24] has presented some stochastic models of cooperating one vendor and one buyer. In other words, they have considered only the single-product (1 : 1) supply chain network.

Lu [17] has considered heuristics for a single-vendor multi-buyer multi-product problem with infinite production rate allowed under the assumption that one type of product should be delivered to only one buyer and each buyer can purchase only one type of product. In the problem, neither lot-for-lot policy nor inventory cost at the buyer's side is incorporated, but only the vendor can have market power (negotiation power), and the objective of minimizing the total cost of the vendor is considered.

This paper considers an extension of the single-vendor single-buyer (1 : 1) supply chain network to a multi-product (1 : 1 : N) supply chain network with a cyclic production policy incorporated, under the assumption that the time range on prod-



<Figure 1> An Example of the (1 : 1 : N) Supply Chain Network



<Figure 2> An Example of the Multi-Product Production System with the Cyclic Production Policy Incorporated

uct shipment interval for each buyer is provided through negotiation between the vendor and each buyer. The cyclic production policy is a manufacturing policy concerned with producing multiple types of products sequentially in a fixed manufacturing order so as to produce multiple products type by type. Given that only one facility is available for the vendor's manufacturing system, multiple types of products will be produced via the cyclic production policy on a single facility, where their manufacturing sequence is fixed in every production cycle. <Figure 1> depicts a simple example of the associated ($1 : 1 : N$) supply chain network and <Figure 2> depicts an example of the multi-product production system with the cyclic production policy incorporated.

The cyclic production policy has been practically considered in flexible manufacturing situations where multiple products need to be produced to meet with a variety of different small-sized demands frequently occurring at a single facility. The policy has gained increasing acceptance in practice due to its scheduling simplicity, ease of control, and increased system visibility, along with the prevalence of the just-in-time (JIT) management philosophy in repetitive manufacturing, which has been discussed in Bahl and Ritzman [1], Campbell and Mabert [5], McClain and Trigeiro [19], Kim and Mabert [15]. The policy has been studied for various lot-sizing problems such as dynamic lot sizing problem, discrete lot sizing and sequencing problem and economic lot scheduling problem, referring to Bahl and Ritzman [1], Campbell and Mabert [5], Kim and Mabert [15], Elmaghraby [7], and Kim, Mabert and Pinto [16].

As seen in the literature, any integrated model of the vendor-and-buyer coordination and the cyclic production policy has not been considered yet. Moreover, the integrated model is known as complex to solve in any existing approach. This provides the motivation for us to consider the addressed problem.

The addressed problem considers a kind of more realistic extension of Lu's model, by considering the assumption of a finite production rate for the multi-product production system with cyclic production policy incorporated and also the assumption that each type of products can be delivered to any number of buyers but each buyer can purchase only one product type. The objective is to minimize total cost per unit time of the whole system (supplier, vendor and multiple buyers involved). In other words, this paper considers an integrated multi-product ($1 : 1 : N$) inventory model in which a supplier supplies raw material to a vendor who produces multiple products via a cyclic production policy (Chan and Song [6]) and supplies them to N buyers.

Coordinating the production planning (associated with the vendor) and the shipment planning (associated with the buyer) within a single model has been a promising research trend in recent years. It has been discussed that any solution model considering buyer and vendor inventory problems in separate (e.g. individual EOQ models) cannot provide a good solution for the entire system, referring to Goyal [9, 10, 12], Hill [13, 14], and Lu [17]. Therefore, this paper considers an integrated problem of establishing both production planning and shipment planning simultaneously in a multi-product ($1 : 1 : N$) inventory model by incorporating both vendor-and-buyer coordination and cyclic production policy. As seen in the literature, any research associated with the cyclic production policy has not considered yet the vendor-and-buyer coordination issue but only one-vendor situation (in ($1 : 1$) network) in separate. The addressed integrated model is an extension of the vendor-and-buyer coordination problem with the cyclic production policy additionally incorporated, which can also be viewed as an extension of the cyclic production policy consideration into a ($1 : 1 : N$) supply chain network with the vendor-and-buyer coordination additionally incorporated.

The proposed model is applicable to some manufacturing processes with injection molding process involved, such as paper product manufacturing processes (Bankston and Harnett [4]), glass-containers (bottle and jars) manufacturing processes (Fransoo et al. [8]), and most automobile-part (nuts, bolts, washers, springs etc.) manufacturing processes (Nori and Sarker [21]). As an example, a manufacturing process having its individual sub-processes of heating, forming, and cooling of multi-products can be considered, where all the associated products are produced via cast in a mold at high heat and pressure. They are all made of the same material, but may differ in size, shape, function, or combinations thereof. Once any desired mold is put into its manufacturing equipment's slot, it is heated up to its required production temperature. Such heating time for the equipped mold is regarded as the production setup time for each product type manufacturing.

The proposed model of this paper is also applicable to automobile manufacturing processes. For example, as stated in Mabert and Pinto [18], Kim, Mabert and Pinto [16], "*Ford Motor Company's Plant assembles eight tail lamp models using two flexible assembly systems (FASs), one dedicated to the right tail lamps and the other to the left. The critical issue in this repetitive manufacturing environment is how to reduce inventory levels and meet shipping schedules.*" By treating each automobile with tail lamps not installed yet as a type of raw material, the proposed model can be applied to this Ford Motor Company's plant.

Moreover, this paper considers a circumstance under which the time range on product shipment interval is provided through negotiation between the vendor and each buyer. Thereby, the time range on product shipment interval is given as the following inequality;

$$\left(\begin{array}{c}\text{buyer } i\text{'s}\\ \text{lowest bound}\end{array} \leq \begin{array}{c}\text{buyer } i\text{'s}\\ \text{shipment interval}\end{array} \leq \begin{array}{c}\text{buyer } i\text{'s}\\ \text{highest bound}\end{array}\right) \quad (1)$$

While the inventory level associated with a buyer can be reduced by shortening the shipment interval, the associated delivery cost of the vendor may increase, since the shortened shipment interval may incur to increase the frequency of shipments. Thus, the buyer will want short shipment interval, but the vendor will want long shipment interval. Therefore, the associated shipment interval decision is important, for which negotiation may be needed between the vendor and the buyer to resolve such conflict of interest. Accordingly, the objective of the proposed integrated multi-product (1 : 1 : N) inventory model is now restated in more compact terms as to minimize the total cost per unit time of the whole system (supplier, vendor and multiple buyers involved) subject to some time range constraints on product shipment interval for each buyer. In order to have a better cooperation with the buyer, the vendor may need to pass part of his saving to the buyer. Imposing any time range constraints for each buyer is a way for the vendor to pass part of his savings to the buyer, since any portion of vendor's savings is determined based on negotiation between the vendor and each buyer for the time range on product shipment interval. That is, the wider the time range is, the more part of saving the vendor will pass.

The remainder of this paper is organized as follows. Section 2 describes the detail assumptions and notation, and presents the model formulation for the proposed integrated multi-product (1 : 1 : N) inventory problem. Section 3 characterizes some solution properties, with which a solution algorithm is derived. Section 4 gives a numerical example and the validation of the proposed algorithm, and Section 5 states conclusions.

2. Assumptions and Model Formulation

The proposed integrated multi-product (1 : 1 : N) inventory problem considers a single raw material supplier, a single vendor (facility), N buyers, a single type of raw material, and M different types of products ($M \leq N$). It is assumed that demand rates, production rates and all the associated procurement and inventory costs are known and constant, and also that no shortages are allowed for raw material and finished products. Moreover, replenishment of raw material and finished products are all made instantaneously so as to have no lead time. It is further assumed that during every production cycle, one setup associated with each product type is required with negligible setup time, and the product-manufacturing sequence is fixed. Moreover, if each product demand is fixed, then the product will have the same amount of production in each consecutive cycle, once the amount is determined based on the associated demand. The assumption "single facility" in this paper can be represented by a single machine, a critical or bottleneck work center, or an entire assembly line with a set of machines, referring to Kim and Mabert [15]. The assumption "constant demand" has often been considered in many researches. For example, McClain and Trigeiro [19] have commented as "There are many situations where constant demand is not a bad assumption. For example, the assembly schedule for automobile production is likely to be constant over long periods." There are many other studies which have considered the constant demand assumption; referring to Lu [17], Goyal [9, 10, 12], and Hill [13, 14].

Let index i denote the i-th buyer, index j denote the j-th product type, and m_j denote the set of buyers purchasing product type j, so that $\sum_{j=1}^{M} |m_j| = N$, where $|\cdot|$ denotes the number of elements included in a set of buyers. Buyer i purchases only one product type from the vendor at annual demand D_i, ordering cost A_i and annual inventory holding cost Hb_i. Let T be the vendor's time interval between two consecutive setups of one product type, which is the time length of one production cycle, and let the assigned time proportion for manufacturing product type j be r_j and the vendor's annual production rate of product type j be P_j such that $P_j r_j \geq \sum_{i \in m_j} D_i$ for $\forall j$ and $\sum_{j=1}^{M} r_j = 1$. Then, each associated feasible r_j value can be determined as follows;

$$r_j = \sum_{i \in m_j} D_i / P_j + \left[1 - \sum_{j=1}^{M} \left(\sum_{i \in m_j} D_i / P_j \right) \right] \Big/ M, \text{ for } j = 1, 2, \cdots, M. \quad (2)$$

Note that if the term $T \sum_{i \in m_j} D_i / P_j$ is interpreted as the actual manufacturing time of product type j during the given cycle, then the remaining portion $T\left[1-\sum_{j=1}^{M}\left(\sum_{i\in m_j} D_i/P_j\right)\right]\Big/M$ can be interpreted as the facility idle time, given the assigned manufacturing time r_jT.

This paper considers an integer-ratio delivery policy where each type of finished goods are delivered to the buyer in an integer number (or reciprocal of an integer) of delivery runs during the vendor's production cycle time T (Lu [17]). For example, k delivery runs indicate that the specific goods are delivered k times during a given cycle T, and $1/k$ delivery runs indicate that the goods are delivered once during k cycles. Then, T/k_i is defined as the time interval between two consecutive purchases of buyer i (that is, per repetitive purchase interval), where $k_i \in \{1, 2, 3, \cdots\} \cup \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$. Buyer i purchases the product at the amount D_iT/k_i each time. Therewith, the inequality (1) can be expressed as in the following constraint;

$$l_i \le T/k_i \le u_i, \text{ for } l_i,\ u_i \in R^+ \text{ and } i = 1, 2, \cdots, N \quad (3)$$

where l_i and u_i denote buyer i's lowest and highest bounds of the shipment interval range, respectively, which is provided through negotiation between the vendor and each buyer.

The objective of the proposed integrated multi-product (1 : 1 : N) inventory problem is to minimize total cost per unit time of the integrated model subject to the constraints (3), where the total cost per unit time is represented by the sum of the supplier's total cost, the vendor's total cost and the buyers' total cost per unit time. The total cost per unit time of each buyer i is derived as

$$TC_{bi} = \frac{k_iA_i}{T} + \frac{1}{2}Hb_i \cdot T \cdot D_i/k_i \quad (4)$$

The first term on the right side in Eq. (4) represents the ordering cost per unit time and the second term represents the holding cost per unit time of the product. Then, the total cost per unit time of N buyers is

$$\sum_{i=1}^{N} TC_{b_i} = \sum_{t=1}^{N}\left(\frac{k_iA_i}{T} + \frac{1}{2}Hb_i \cdot T \cdot D_i/k_i\right)$$
$$= \sum_{j=1}^{M}\sum_{i\in m_j}\left(\frac{k_iA_i}{T} + \frac{1}{2}Hb_i \cdot T \cdot D_i/k_i\right) \quad (5)$$

Similarly, the total cost per unit time of the raw material supplier can be derived under the assumption that one shipment of the raw material is made from the supplier to the vendor at the beginning of each production cycle, which is commonly adapted in the most references in the literature. The annual inventory holding cost of the raw material cost shipped from the suppler is Hs, and the required raw material amount for the vendor to manufacture one unit of product type j is f_j, so that the supplier makes shipping of the amount $T \cdot \sum_{j=1}^{M}\sum_{i\in m_j}(D_i \cdot f_j)$ each time to the vendor. Let g be the sum of the fixed transportation cost of the raw material and the supplier's production setup cost. Then, the supplier's total cost per unit time is derived as

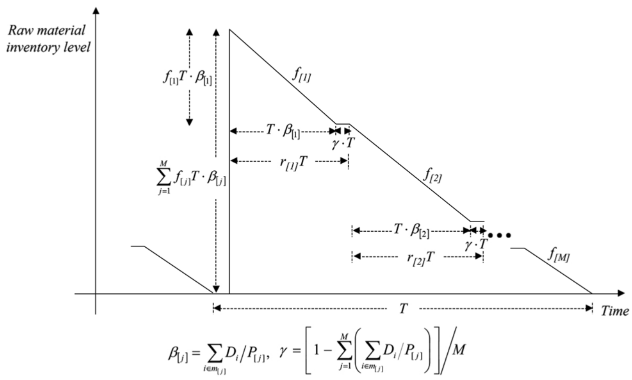
$$TC_{supplier} = \frac{g}{T} + \frac{1}{2} \cdot Hs \cdot T \cdot \sum_{j=1}^{M}\sum_{i\in m_j}(D_i \cdot f_j) \quad (6)$$

The first term on the right side in Eq. (6) represents the sum of the fixed transportation cost and the setup cost per unit time, and the second term represents the holding cost per unit time of raw material. Finally, the total cost per unit time of the vendor shall be derived. Note that the total cost of the vendor is represented by the sum of the raw material ordering cost, the finished product transportation cost, the production setup cost, the holding cost of raw material and finished products. The vendor's ordering cost for raw material is B, the fixed transportation cost for buyer i is e_i, and the setup cost for manufacturing product type j is s_j. Therefore, the sum of the vendor's raw material ordering cost, the transportation cost and the production setup cost per unit time is derived as

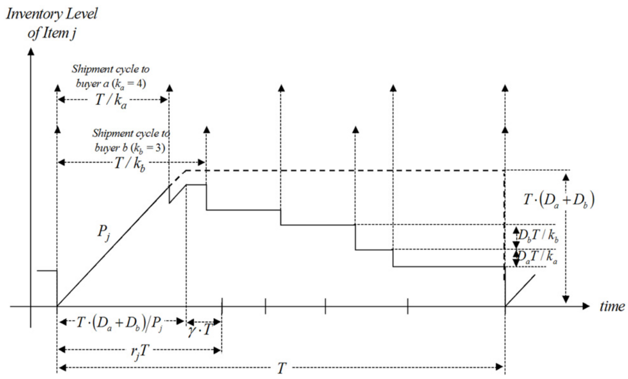
$$\frac{B}{T} + \frac{1}{T}\sum_{j=1}^{M}\sum_{i\in m_j} e_i \cdot k_i + \frac{1}{T}\sum_{j=1}^{M} s_j \quad (7)$$

In order to calculate the vendor's raw material holding cost per unit time, the manufacturing sequence for M products should be decided, since the associated raw material inventory amount depends on the manufacturing sequence.

In this paper, the manufacturing sequence is represented by the decreasing order of the required raw material amount, f_j, of each product. It can be easily seen that raw material inventory cost can be minimized by considering the associated manufacturing sequence in decreasing order of the required raw material amount of each product. Thus, letting $f_{[j]}$ be the rearrangement of f_j in the decreasing order, the manufacturing sequence is decided as in the order of $f_{[j]}$, where $f_{[1]} \ge f_{[2]} \ge \cdots \ge f_{[M]}$, as depicted in <Figure 3>.



<Figure 3> Raw Material Inventory Levels for the Vendor



<Figure 4> Inventory Levels of Item j for Two Buyers and 5 Items with k_a = 4 and k_b = 3.

This paper determines, for convenience, the manufacturing sequence in decreasing order of the required raw material amount of each product. However, the proposed model and solution algorithm procedure in this paper can be used without any modification, but only with modifying the associated input sequence of the required raw material amount f_j, in any situation where some more important factors (e.g., customer service) are additionally considered so as to require any original manufacturing sequence modification.

Let the annual raw material inventory holding cost for the vendor be Hr. Then, the vendor's raw material holding cost per unit time is

$$\alpha \cdot T \cdot H_r,$$

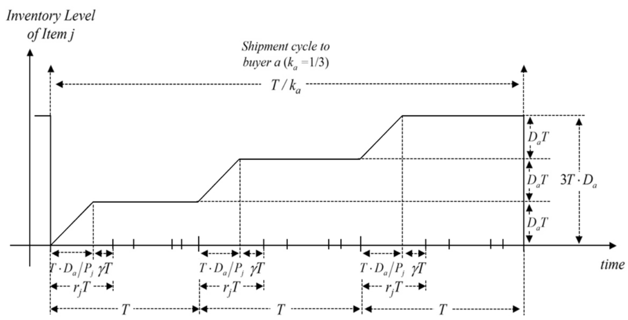
$$\text{where }\ \alpha=\sum_{k=1}^{M-1}\left[\frac{1}{2}\beta_{[k]}\cdot\left(\sum_{j=k}^{M} f_{[j]}\cdot\beta_{[j]}+\sum_{j=k+1}^{M} f_{[j]}\cdot\beta_{[j]}\right)+r\cdot\sum_{j=k+1}^{M} f_{[j]}\cdot\beta_{[j]}\right]+\frac{1}{2}f_{[M]}(\beta_{[M]})^2 \tag{8}$$

Now, the vendor's holding cost per unit time of each product type j shall be derived under the assumption that production of product type j starts at the beginning of the assigned manufacturing time for product type j, which is commonly adapted in the most references in the literature. This paper considers two cases of $k_i > 1$ (partial delivery allowed before a lot production completion) and $k_i \le 1$ (no partial delivery allowed) together in the total cost minimization. For the case where $k_i > 1$, an illustration of inventory level is depicted as in <Figure 4>.

The vendor's annual inventory holding cost for product type j is Hv_j, so that the vendor's holding cost per unit time for the case of $k_i > 1$ is derived as

$$\sum_{j=1}^{M} Hv_j\cdot\left[\sum_{i\in m_j}\frac{D_iT(k_i+1)}{2k_i}-\frac{T}{2P_j}\left(\sum_{i\in m_j}D_i\right)^2\right] \tag{9}$$

For the case of $k_i \le 1$, it is assumed that the setup associated with each product type occurs once in each production cycle and the production amount of each product is constant in every production cycle, which is commonly adapted in the most references in the literature, as depicted in <Figure 5>.



<Figure 5> Inventory Level of Item j for One Buyer and 5 Items with ka = 1/3

The vendor's holding cost per unit time for the case of $k_i \le 1$ is derived as

$$\sum_{j=1}^{M} Hv_j\cdot\left[\sum_{i\in m_j}\frac{D_iT(k_i+1)}{2k_i}-\frac{T}{2P_j}\left(\sum_{i\in m_j}D_i\right)^2\right] \tag{10}$$

Combining expressions (9) and (10) together, the vendor's holding cost per unit time is expressed as

$$\sum_{j=1}^{M} Hv_j\cdot\left[\sum_{i\in m_j}\frac{D_iT(k_i+1)}{2k_i}-\frac{T}{2P_j}\left(\sum_{i\in m_j}D_i\right)^2\right] \tag{11}$$

Then, the vendor's total cost per unit time is derived as

$$TC_{vendor}=\frac{B}{T}+\frac{1}{T}\sum_{j=1}^{M}\sum_{i\in m_j}e_i\cdot k_i+\frac{1}{T}\sum_{j=1}^{m}s_j+\alpha\cdot Hr\cdot T$$
$$+\sum_{j=1}^{M} Hv_j\cdot\left[\sum_{i\in m_j}\frac{D_iT(k_i+1)}{2k_i}-\frac{T}{2P_j}\left(\sum_{i\in m_j}D_i\right)^2\right] \tag{12}$$

Then, the total cost per unit time of the integrated (1 : 1 : N) model is derived as

Total Cost (TC) =

$$\sum_{j=1}^{M}\sum_{i\in m_j} TC_{b_i} + TC_{supplier} + TC_{vendor} \tag{13}$$

This total cost expression is formulated as the following mixed integer programming problem (SVB) which can be used to determine T and k_i's.

(SVB)

Min TC(T, k_1, k_2, $\cdots$, k_N) =

$$\delta \cdot T + \frac{n}{T} + \sum_{j=1}^{M}\sum_{i\in m_j}\left[\frac{1}{2}(Hb_i + Hv_j)\cdot D_i \cdot \frac{T}{k_i} + \frac{k_i}{T}(A_i + e_i)\right]$$

s.t. $\quad T \ge 0$

$$l_i \le T/k_i \le u_i, \text{ for } l_i,\ u_i \in R^+ \text{ and } i = 1, 2, \cdots, N \tag{14}$$

$$k_i \in \{1, 2, 3, \cdots\} \cup \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}, \text{ for } i = 1, 2, \cdots, N \tag{15}$$

where $n = g + B + \sum_{j=1}^{M} s_j$

$$\delta = \alpha \cdot Hr + \sum_{j=1}^{m}\left[\frac{1}{2}Hs\sum_{i\in m_j}(D_i \cdot f_j) - \frac{Hv_j}{2P_j}\left(\sum_{i\in m_j} D_i\right)^2 + \frac{Hv_j \cdot \sum_{i\in m_j} D_i}{2}\right]$$

3. Solution Procedure for the Integrated Multi-Product(1 : 1 : *N*) Inventory Model

It appears very difficult to find the optimal solution of the (1 : 1 : N) problem (SVB), since the mathematical formulation is a nonlinear programming of decision variables. Thereupon, in this section, an iterative heuristicsolution procedure will be proposed based on Theorem 1.

Theorem 1 : *Given any particular value of T > 0, the objective function TC(T, k_1, k_2, $\cdots$, k_N) of the problem (SVB) is convex with respect to each k_i. On the other hand, given any particular values of k_1, k_2, $\cdots$, and k_N, the function TC(T, k_1, k_2, $\cdots$, k_N) is convex with respect to T.*

Proof : It is straightforward.

According to Theorem 1, with any given particular value of T > 0, the optimal values of k_i's can be determined easily, and so can the optimal value of T when ki's values are all known. In other words, k_i's cannot be determined easily without knowing T, and T cannot be determined easily without knowing k_i's either. Therefore, the iterative heuristic procedure for the problem (SVB) will be derived.

Some feasibility conditions for the problem (SVB) are specified in Theorem 2 and Theorem 3. Note that given a value of T > 0, from the constraints (14), all k_i's should satisfy the relation

$$l_i \le T/k_i \le u_i,\ i = 1, 2, \cdots, N. \tag{16}$$

Theorem 2 : *Under the condition that $u_i \ge 2 \cdot l_i$ and with any given value of T > 0, there is at least one integer or the reciprocal of integer k_i satisfying the relation* (16).

Proof : Consider two cases where $T/u_i > 1$ and $T/l_i < 1$. First, consider the case where $T/u_i > 1$. It holds that $\frac{T}{l_i} - \frac{T}{u_i} = \frac{(u_i - l_i)T}{u_i \cdot l_i} \ge \frac{l_i \cdot T}{u_i \cdot l_i} = \frac{T}{u_i} > 1$, due to the condition $u_i \ge 2 \cdot l_i$. This implies that there is at least one integer k_i satisfying the relation (16). Now, consider the case where $T/l_i < 1$. It holds that $\frac{u_i}{T} - \frac{l_i}{T} = \frac{(u_i - l_i)}{T} \ge \frac{l_i}{T} > 1$, due to the condition $u_i \ge 2 \cdot l_i$.

This also implies that there is at least one reciprocal of integer k_i satisfying the relation (16).

This completes the proof.

Theorem 3 : *Let T^*/k_i^* be the optimal repetitive shipment interval for the proposed integrated multi product inventory problem. In the situation where $EOQ_{v_i} \ge 2EOQ_{b_i}$, it holds that $T^*/k_i^* \in [EOQ_{b_i}, EOQ_{v_i}]$. Otherwise, it holds that $T^*/k_i^* \in [\frac{EOQ_{v_i}}{2}, EOQ_{v_i}]$, or $T^*/k_i^* \in [EOQ_{b_i}, 2 \cdot EOQ_{b_i}]$, where EOQ_{b_i} and EOQ_{v_i} denote the optimal shipment interval* (*EOQ solution*) *for buyer i and for the vendor associated with buyer i, respectively, each of which is found from the associated vendor-and-buyer* (1 : 1) *inventory problem by treating the associated buyer and vendor sub-problems in separation.*

Proof : Denoting by $\widetilde{T}_i$ the shipment interval for buyer i, the total ordering and inventory cost per unit time of buyer i is derived as $\overline{TC}_{bi} = \frac{A_i}{\widetilde{T}_i} + \frac{1}{2}Hb_i \cdot \widetilde{T}_i \cdot D_i$, which is an

EOQ formula for buyer i. The total cost function is characterized as a convex function of $\widehat{T}_i$. Also, denoting by $\widehat{T}_i$ the shipment interval for the vendor associated with buyer i, the total setup and inventory cost per unit time of the vendor is derived as $\overline{TC}_{vi} = \frac{(e_i+s_j)}{\widehat{T}_i} + \frac{1}{2}\frac{Hv_j \cdot \widehat{T}_i \cdot D_i}{r_j p_j}$, which is an EOQ formula for the vendor associated with buyer i, and also characterized as a convex function of $\widehat{T}_i$. Let EOQ_{b_i} and EOQ_{v_i} be the optimal shipment interval (EOQ solution) for buyer i, and the optimal shipment interval (EOQ solution) for the vendor, respectively, which are the results from separate solution search ($EOQ_{b_i} < EOQ_{v_i}$). Then, the optimal shipment interval, T^*/k_i^*, for the proposed integrated multi-product vendor- and-buyer (1 : 1) inventory problem can be seen as belonging to the range $[EOQ_{b_i}, EOQ_{v_i}]$, since $\overline{TC}_{bi}$ and $\overline{TC}_{vi}$ are the convex function of $\widetilde{T}_i$ and $\widetilde{T}_i$, respectively, and $EOQ_{b_i} < EOQ_{v_i}$.

It is seen from the results of Theorem 2 that when $EOQ_{v_i} \geq 2EOQ_{v_i}$, there is at least one integer or the reciprocal of integer k_i^*, for any $T^* > 0$, satisfying the relation (16) with $u_i = EOQ_{v_i}$ and $l_i = EOQ_{b_i}$. This leads to $T^*/k_i^* \in [EOQ_{b_i}, EOQ_{v_i}]$.

Similarly, it is seen that when $EOQ_{v_i} < 2EOQ_{b_i}$, there is at least one integer or the reciprocal of integer k_i^*, for any $T^* > 0$, satisfying the relation (16) with $u_i = EOQ_{v_i}$ and $l_i = \frac{EOQ_{v_i}}{2}$, or with $u_i = 2 \cdot EOQ_{b_i}$ and $l_i = EOQ_{b_i}$, which leads to $T^*/k_i^* \in \left[\frac{EOQ_{v_i}}{2}, EOQ_{v_i}\right]$, or $T^*/k_i^* \in [EOQ_{b_i}, 2 \cdot EOQ_{b_i}]$. This completes the proof.

The results of Theorem 3 can be used to provide a guideline for negotiation between the vendor and buyers. Moreover, in the situation where there is no negotiation between the vendor and buyers, the results of Theorem 3 can be used to provide the time range on product shipment interval.

Now, consider part of the third term in the objective function of the problem (SVB), which is denoted by the following function $f_i(k_i)$;

$$f_i(k_i) = \frac{1}{2}(Hb_i + Hv_j) \cdot D_i \cdot \frac{T}{k_i} + \frac{k_i}{T}(A_i + e_i),$$
$$T/u_i \leq k_i \leq T/l_i. \tag{17}$$

Note that $f_i(k_i)$ is a convex function of k_i. Then, consider three associated cases, including $T/l_i \leq 1$, $T/u_i \geq 1$, and $T/u_i < 1 < T/l_i$.

Case 1 : $T/l_i \leq 1$: k_i can only be the reciprocal of an integer. Using Schwarz's result [23], the function $f_i(k_i)$ achieves the minimum at $\hat{k}_i(T)$, where $\frac{1}{\hat{k}_i(T)}$ is an integer satisfying the relation

$$\frac{1}{\hat{k}_i(T)}\left(\frac{1}{\hat{k}_i(T)} - 1\right) < \frac{2(A_i+e_i)}{(Hb_i+Hv_j) \cdot D_i \cdot T^2} \leq \frac{1}{\hat{k}_i(T)}\left(\frac{1}{\hat{k}_i(T)} + 1\right), \text{ where } T/u_i \leq \hat{k}_i(T) \leq T/l_i.$$

Thus, the optimal solution of $f_i(k_i)$ is found at

$$k_i^*(T) = \max\left\{1\Big/\left\lfloor \frac{u_i}{T} \right\rfloor, \min\left\{1\Big/\left\lceil \frac{l_i}{T} \right\rceil, \hat{k}_i(T)\right\}\right.$$

Case 2 : $T/u_i \geq 1$: k_i can only be an integer. Then, the function $f_i(k_i)$ achieves the minimum at $\bar{k}_i(T)$, where $\bar{k}_i(T)$ is an integer satisfying the relation

$$\bar{k}_i(T)(\bar{k}_i(T)-1) < \frac{(Hb_i+Hv_j) \cdot D_i \cdot T^2}{2(A_i+e_i)} \leq \bar{k}_i(T)(\bar{k}_i(T)+1), \text{ where } T/u_i \leq \bar{k}_i(T) \leq T/l_i.$$

Thus, the optimal solution of $f_i(k_i)$ is found at

$$k_i^*(T) = \max\{\lceil T/u_i \rceil, \min\{\lfloor T/l_i \rfloor, \bar{k}_i(T)\}\}.$$

Case 3 : $T/u_i < 1 < T/l_i$: k_i can be an integer and also the reciprocal of an integer. Therefore, applying the results of Cases 1 and 2, the function $f_i(k_i)$ achieves the minimum at $k_i^*(T)$, which is found as

$$k_i^*(T) = \arg\min\{f_i(y) | y \in G_i\},$$

where

$$G_i = \left\{\max\left\{1\Big/\left\lfloor \frac{u_i}{T} \right\rfloor, \hat{k}_i(T)\right\}, \min\left\{\hat{k}_i(T), \lfloor T/l_i \rfloor\right\}\right\}.$$

The results of the three cases provide the basis for finding the optimal $k_i(T)$ for the associated part of the third term in the objective function of the problem (SVB), as specified in Theorem 4.

Theorem 4 : *The optimal $k_i(T)$ values associated with any given value of $T > 0$ and Eq. (17) can be found by use of the following search procedure which is derived from the above three cases including $T/l_i \le 1$, $T/u_i \le 1$, and $T/u_i < 1 < T/l_i$.*

Search Procedure : Finding the optimal $k_i(T)$ with a given particular value of T.

Step 0 : If $T/l_i \le 1$, go to Step 1. If $T/u_i \ge 1$, go to Step 2. Otherwise, go to Step 3.

Step 1 : Calculate the integer $\frac{1}{\hat{k}_i(T)}$ satisfying the relation $\frac{1}{\hat{k}_i(T)}\left(\frac{1}{\hat{k}_i(T)}-1\right) < \frac{2(A_i+e_i)}{(Hb_i+Hv_j)\cdot D_i \cdot T^2} \le \frac{1}{\hat{k}_i(T)}\left(\frac{1}{\hat{k}_i(T)}+1\right)$ and then compute the associated optimal
$k_i^*(T) = \max\left\{1/\left\lfloor \frac{u_i}{T}\right\rfloor, \min\left\{1/\left\lceil\frac{l_i}{T}\right\rceil, \hat{k}_i(T)\right\}\right\}$.Stop.

Step 2 : Calculate the integer $\bar{k}_i(T)$ satisfying the relation $\bar{k}_i(T)(\bar{k}_i(T)-1) < \frac{(Hb_i+Hv_j)\cdot D_i \cdot T^2}{2(A_i+e_i)}$, $\le \bar{k}_i(T)(\bar{k}_i(T)+1)$, and then compute the associated optimal $k_i^*(T) = \max\{\lceil T/u_i\rceil, \min\{\lfloor T/l_i \rfloor, \bar{k}_i(T)\}\}$. Stop.

Step 3 : Calculate the integer $\frac{1}{\hat{k}_i(T)}$ satisfying the relation $\frac{1}{\hat{k}_i(T)}\left(\frac{1}{\hat{k}_i(T)}-1\right) < \frac{2(A_i+e_i)}{(Hb_i+Hv_j)\cdot D_i \cdot T^2} \le \frac{1}{\hat{k}_i(T)}\left(\frac{1}{\hat{k}_i(T)}+1\right)$ and calculate the integer $\bar{k}_i(T)$ satisfying the relation $\bar{k}_i(T)(\bar{k}_i(T)-1) < \frac{(Hb_i+Hv_j)\cdot D_i \cdot T^2}{2(A_i+e_i)}$, $\le \bar{k}_i(T)(\bar{k}_i(T)+1)$ and then compute the associated optimal $k_i^*(T) = \arg\min\{f_i(y) \mid y \in G_i\}$, where $G_i = \left\{\max\left\{1/\left\lfloor\frac{u_i}{T}\right\rfloor, \hat{k}_i(T)\right\}, \min\left\{\bar{k}_i(T), \lfloor T/l_i \rfloor\right\}\right\}$. Stop.

Once all k_i's values $K = (k_1, k_2, \cdots, k_N)$ are determined, the range of T can then be determined from the constraint (16) as in

$$l_i \cdot k_i \le T \le u_i \cdot k_i,\ i = 1, 2, \cdots, N.$$

Therewith, the necessary and sufficient conditions for a feasible value $T(K) > 0$ are derived as in Theorem 5.

Theorem 5 : Given any $K = (k_1, k_2, \cdots, k_N)$, the following feasible value T(K) > 0 can be found as $T(K) = \max\{\alpha(K), \min\{\hat{\alpha}(K), \overline{T}(K)\}\}$ if and only if $\alpha(K) \le \hat{\alpha}(K)$, where

$$a(K) = \max_{1\le i\le N}\{l_i \cdot k_i\},\quad \hat{a}(K) = \min_{1\le i\le N}\{u_i \cdot k_i\}, \tag{18}$$

and
$$\overline{T}(K) = \sqrt{\frac{\left[n + \sum_{j=1}^{M}\sum_{i\in m_j} k_i(A_i+e_i)\right]}{\left[\delta + \frac{1}{2}\sum_{j=1}^{m}\sum_{i\in m_j}\frac{(Hb_i+Hv_j)D_i}{k_i}\right]}} \tag{19}$$

Proof : The results can be easily derived via derivative approach in association with K. This completes the proof.

The results of Theorem 5 can immediately be applied to the initial stage of the following heuristic procedure HSVB which starts with the initial setting $k_i = 1$, $\forall i$.

Now, these all procedures for finding T and k_i's are put together to derive a heuristic procedure, called HSVB, for the proposed integrated multi product (1 : 1 : N) inventory problem.

Heuristic Procedure HSVB

Step 0 : Set $k_i = 1$, $i = 1, 2, \cdots, N$.
Find $a(K)$ and $\hat{a}(K)$ by Eq. (18).
If $a(K) \le \hat{a}(K)$, find $\overline{T}(K)$ by Eq. (19), and compute
$T(K) = \max\{a(K), \min\{\hat{a}(K), \overline{T}(K)\}\}$.
Otherwise, let $T(K) = \frac{1}{2}(a(K) + \hat{a}(K))$.

Step 1 : Find k_i's by using the Search Procedure.
If all k_i's values are the same as those obtained in the preceding iteration (indicating a local solution finding), stop. Otherwise, go to Step 2.

Step 2 : Find $a(K)$ and $\hat{a}(K)$ by Eq. (18), and find $\overline{T}(K)$ by Eq. (19), and compute $\overline{T}(K) = \max\{a(K), \min\{\hat{a}(K), \overline{T}(K)\}\}$. Go to Step 1.

4. Computational Results

Although the range parameters u_i and l_i can be obtained through negotiation between the vendor and each buyer, this paper simply arranges the parameters, for this experiment, based on the results of Theorem 3 such as having $l_i = EOQ_{b_i}$ and $u_i = EOQ_{v_i}$ in the situation where $EOQ_{v_i} \geq 2EOQ_{b_i}$; otherwise, having $l_i = \frac{EOQ_{v_i}}{2}$ and $u_i = 2 \cdot EOQ_{b_i}$, which, in fact, satisfies the feasibility conditions.

A Numerical Example

An experiment was made with a numerical example of 5 buyers and 3 product types to illustrate the heuristic procedure HSVB which was implemented in the Computer Language C++ on IBM PC (Pentium IV Processor/1.4GHz, 512MB memory). <Table 1> gives the relevant data and buyer's optimal EOQ solutions.

<Table 1> Data for a Numerical Example with M = 3 and N = 5.

Buyer (N)	Product type Purchase	A_i (\$)	D_i	Hb_i (\$)	e_i (\$)	EOQ_{bi}	EOQ_{vi}	l_i	u_i
1	1	20	100	20	15	0.14	6.21	0.14	6.21
2	1	40	50	10	15	0.4	8.79	0.4	8.79
3	2	25	120	15	15	0.17	6.29	0.17	6.29
4	2	20	80	25	15	0.14	7.71	0.14	7.71
5	3	15	30	20	50	0.22	5.33	0.22	5.33

Product type (M)	P_j	f_j	s_j	Hv_j	r_j
1	450	4	80	8	0.36
2	400	3	120	12	0.53
3	360	3	110	15	0.11

H_r = 6
H_s = 5
g = 150
B = 30

<Table 2> Summary of the Results of the Procedure HSVB

Buyer(N)	Buyer's order interval, $k_i^* T^*$
1	0.1667
2	0.6666
3	0.1667
4	0.1667
5	0.3333
Vendor's production cycle length, T^* = 0.3333 Total cost of the integrated model per unit time = \$ 5112.32	

The numerical example shows that the heuristic procedure HSVB converges after the third iteration, giving the solution summarized as in <Table 2>.

For comparison, this paper solves three integrated single-product (1 : 1 : N) inventory models individually for each product type, and presents their solutions in <Table 3>. It can be seen from the solutions that by coordinating replenishments of all the different product types, the proposed integrated multiproduct (1:1:N) inventory model can reduce the total cost per unit time by about 16.3%. Thus, coordinating the replenishments of all the different product types is more effective than determining individual replenishments separately.

Validation of the heuristic procedure HSVB

The proposed heuristic procedure HSVB was also tested for its effectiveness and efficiency with more 25 numerical problems, including 15 three-product (1 : 1 : 5) numerical problems (models) and 10 four-product (1 : 1 : 7) numerical problems (models). In order to evaluate the performance of the procedure HSVB, the obtained heuristic solutions are compared with the associated optimal solutions which are found through full enumeration searches for all the potential k_i's values. <Table 4> presents the experimental results for all the 25 numerical problems. The column "cost reduction (%)" represents the percentage of the cost reduction of HSVB solution compared with the cost sum of the single-product (1 : 1 : N) inventory models for each product type.

<Table 3> Solutions of Three Integrated Single-Product (1 : 1 : N) Inventory Models for Each Product Type

	Product type	Buyer	Buyer's order interval, $k_i^* T^*$	Vendor's production cycle length, T^*	Total cost unit time (\$)
Inventory Model 1	1	1	0.1598	0.3195	2237.61
		2	0.639		
Inventory Model 2	2	3	0.2685	0.2685	2793.45
		4	0.2685		
Inventory Model 3	3	5	0.389	0.778	1079.63
Total cost per unit time(\$) = \$ 6110.7					

<Table 4> The Experimental Results for 25 Numerical Problems

	#of buyer	#of item type	Procedure HSVB		Obtained Optimal Solution			Single product(1 : 1 : N)	
			#of iterations	Cost	Time (sec.)	Optimal cost	Gap (%)	Cost sum of inventory models	Cost reduction (%)
1	5	3	5	7096	4.17	6938	2.2	8496	16.4
2	5	3	2	6365	4.18	6365	0.0	7688	17.2
3	5	3	1	7615	4.17	7594	0.3	9116	16.5
4	5	3	2	5990	4.17	5970	0.3	7411	19.2
5	5	3	2	6860	4.17	6773	1.3	8039	14.7
6	5	3	6	8305	4.15	7980	4.1	9729	14.6
7	5	3	2	8587	4.14	8587	0.0	10474	18.0
8	5	3	3	7669	4.15	7641	0.3	9404	18.4
9	5	3	3	7276	4.15	7184	1.3	8744	16.8
10	5	3	4	8202	4.14	8056	1.8	9957	17.6
11	5	3	2	7622	4.17	7352	3.6	9112	16.3
12	5	3	2	6626	4.15	6577	0.7	7875	15.9
13	5	3	3	5929	4.15	5929	0.0	7260	18.3
14	5	3	2	7721	4.18	7684	0.4	9378	17.7
15	5	3	3	7570	4.17	7401	2.2	8753	13.5
16	7	4	7	9597	290.2	9405	2.0	12041	20.2
17	7	4	2	9870	204.5	9870	0.0	11882	16.9
18	7	4	2	9403	253.9	9297	1.1	11734	19.8
19	7	4	2	11227	312.3	9859	13.8	12600	10.8
20	7	4	5	10993	259.5	10895	0.8	13888	20.8
21	7	4	2	10977	352.5	10801	1.6	13450	18.3
22	7	4	3	10425	395.5	10317	1.0	13060	20.1
23	7	4	2	10003	299.5	10003	0.0	12796	21.8
24	7	4	2	11739	375.5	10485	11.9	13269	11.5
25	7	4	2	9053	354.5	9049	0.04	11108	18.5

H_r~U(10, 15), H_s~U(5, 10), g~U(150, 200), B~U(30, 60), A_i~U(10, 50), D_i~U(30, 150), Hb_i~U(15, 30), e_i~U(15, 30), P_j~U(600, 800), f_j~U(3, 8), s_j~U(80, 150), Hv_j~U(8, 12).

As observed from <Table 4>, the numbers of iterating the procedure HSVB to find all the required feasible solutions are small, and the solution gaps between the heuristic solutions and the associated optimal solutions are also small. So, it may be claimed that the proposed procedure HSVB is efficient and effective. Also, the average percentage of the cost reduction of HSVB compared with the sum of the cost of the single product (1 : 1 : N) inventory models is about 17.2%. Therefore, it can be claimed that coordinating the replenishments of all the different product types is more effective than determining individual replenishments separately.

Now, a comment is made on the robustness of the proposed solution algorithm in its practical application. In the real field, a variety of different unanticipated production bottlenecks can destroy their associated production schedules, and any repeating production schedule by use of such cyclic production policy is not exceptional. For example, some unanticipated production bottlenecks include breakdowns of machines, failure of a vendor to deliver on schedule or in sufficient quantity, low production yield, or absence of a key worker. In such bottleneck situations, any associated analytical methods for re-establishing any repeating production schedules may be very hard to derive and vary with bottleneck situations. Therefore, it could be better to invest in prevention methods, rather than adapting any (complex) methods of recovering from any unanticipated production bottlenecks. In this connection, safety stock and safety lead time have been considered as two common preventive measures that can be used to protect such production schedules (McClain and Trigeiro [19]). Of course, such safety stock and safety lead time measures can immediately be adapted in the model of this paper. In this way, the proposed algorithm can take care of those production bottlenecks.

5. Conclusions

This paper considers an integrated multi-product (1 : 1 : N) inventory model where a supplier supplies raw material to a vendor who produces multiple products via a cyclic production policy and supplies them to N buyers. The objective of the model is to minimize total annual cost per unit time of the whole system (supplier, vendor and multiple buyers involved) subject to some constraints of time range on product shipment interval for each buyer, under the assumption that the time range is provided through negotiation between the vendor and each buyer. Some solution properties are characterized, with which an iterative heuristic solution procedure, called HSVB, is derived, and tested with 25 numerical problems to show that the total cost per unit time can be reduced by coordinating replenishments of all the different product types by 17.2% on the average. Based on the computational experiments, it is concluded that the derived procedure HSVB is efficient and effective.

The derived heuristic procedure may be immediately applied to manufacturing processes with injection molding processes involved, such as paper product manufacturing processes, glass-containers manufacturing processes (bottle and jars), and most automobile-part (nuts, bolts, washers, springs etc.) manufacturing processes. Its application for automobile assembly processes is also possible.

Further research may focus on development of any quantity discount pricing policy to maximize total profit of the integrated multi-product (1 : 1 : N) inventory system.

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