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On Two Sequences of Polynomials Satisfying Certain Recurrence

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Abstract

Bae and Kim displayed a sequence of 4th degree self-reciprocal polynomials whose maximal zeros are related in a very nice and far from obvious way. Kim showed that the auxiliary polynomials in their results are related to Chebyshev polynomials. In this paper, we study two sequences of polynomials satisfying the recurrence of the auxiliary polynomials with generalized initial conditions. We obtain same results with the auxiliary polynomials from a sequence, and some interesting conjectural properties about resultants and discriminants from another sequence.

Key words : Polynomials, Chebyshev Polynomials, Sequences

1. Introduction

Bae and Kim[2] displayed a sequence of 4 th degree self-reciprocal polynomials whose maximal zeros are related in a very nice and far from obvious way. In fact, they proved

Theorem 1 (Bae and Kim). For each real number $n > 6$, there is a sequence $\{p_k(n,z)\}_{k=1}^{\infty}$ of fourth degree self-reciprocal polynomials such that the zeros of $p_k(n, z)$ are all simple and real, and every $p_{k+1}(n, z)$ has the largest (in modulus) zero $\alpha\beta$ where α and β are the first and the second largest (in modulus) zeros of $p_k(n, z)$, respectively. One such sequence is given by $p_k(n, z)$ so that

$$p_k(n,z) = z^4 - q_{k-1}(n)z^3 + (q_k(n)+2)z^2 - q_{k-1}(n)z + 1 ,$$

where $q_0(\mathrm{n}) = 1$ and other $q_k(n)$'s are polynomials in n defined by the severely nonlinear recurrence

$$4q_{2m-1}(n) = q_{2m-2}^2(n) - (4n+1)\prod_{j=0}^{m-2} q_{2j}^2(n)$$

$$4q_{2m}(n) = q_{2m-1}^2(n) - (n-2)(n-6)\prod_{j=0}^{m-2} q_{2j+1}^2(n)$$

for $m \geq 1$, with the usual empty product conventions, i.e. $\prod_{j=0}^{-1} b_j = 1$.

For notational convenience, write q_k instead of $q_k(n)$ in the theorem. The auxiliary polynomials q_k that parametrize their coefficients are of significant independent interest and they satisfy good recurrence relations (see Lemma 3 (i) of[2])

$$q_{k+1} = q_{k-1}^2 - 2q_k - 4, \quad k \geq 1 \tag{1}$$

Kim[3] showed the polynomials q_k are some sort of Chebyshev polynomials.

Proposition 2 (Kim). Define

$$H(n) := 2T_{2^n}\left(\frac{1}{2}\sqrt{4-x}\right),$$

where $T_n(x)$ is the nth degree Chebyshev polynomial of the first kind. Then the sequence

$$H(1)^2, 2H(2), H(2)^2, 2H(3), H(3)^2, 2H(4), \ldots \tag{2}$$

comes from the recurrence

$$Q_{k+1} = Q_{k-1}^2 - 2Q_k - 4 \tag{3}$$

with initial conditions

$$\{Q_0, Q_1, Q_2\} = \{0, -x, 2x-4\} \tag{4}$$

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where the sequence (2) begins with Q_3.

The recurrence (3) for polynomials Q_k is in the same form as (1) for polynomials q_k. Proposition 2 asserts that the polynomials q_k are some sort of Chebyshev polynomials. In this paper, we study two sequences $\{Q_k\}$ of polynomials satisfying the recurrence (3) from different generalized initial conditions

$$\{Q_0, Q_1, Q_2\} = \left\{0, -x+a-\frac{3}{2}, 2x-2a-1\right\} \quad (5)$$

and

$$\{Q_0, Q_1, Q_2\} = \{0, -x, 2x-2a-1\} \quad (6)$$

where the sequence $\{Q_k\}$ from the latter initial conditions begins with Q_1. In (5), we see that $Q_2 = Q_0^2 - 2Q_1 - 4$, but $Q_2 \neq Q_0^2 - 2Q_1 - 4$ in (6). However, in both, $Q_{k+1} = Q_{k-1}^2 - 2Q_{k-4}$ if $k \geq 3$, and for $a = 3/2$, (5) is the same as (4) and Q_0, Q_2 in (6) are same as those in (4). The sequence $\{Q_k\}$ from (5) will deduce same results with Proposition 2 by replacing $H(n)$ in Proposition 2 with $H(n) = 2T_{2^n}\left(\frac{1}{2}\sqrt{a+\frac{5}{2}-x}\right)$. The proof of this easily follows from that of Proposition 2. We will provide it in Proposition 3 in Section 2·

A resultant is a scalar function of two polynomials which is non-zero if and only if the polynomials are relatively prime. The theory of resultants is an old and much studied topic in what used to be called the theory of equations. The resultant of two polynomials is in general a rather complicated function of their coefficients. However some polynomials such as any two cyclotomic polynomials have elegant formulas[1]. The discriminant of a polynomial can be computed using resultants. More specifically,

$$Disc_x(p(x)) = \frac{(-1)^{n(n-1)/2}}{a_n} Res_x(p(x), p'(x))$$

where deg $p(x) = n$ and a_n is the leading coefficient of $p(x)$. It seems that the sequence $\{Q_k\}$ from (6) has very interesting properties about factorizations and zero distributions. We will obtain some interesting conjectural properties about resultants and discriminants in Section 2.

2. Two Sequences of Polynomials from Different Initial Conditions

We begin with the sequence $\{Q_k\}$ from (5).

Proposition 3. Define

$$H(n) := 2T_{2^n}\left(\frac{1}{2}\sqrt{a+\frac{5}{2}-x}\right),$$

where $T_n(x)$ is the nth degree Chebyshev polynomial of the first kind. Then

$$H(n+1) = H(n)^2 - 2$$

and the sequence

$$H(1)^2, 2H(2), H(2)^2, 2H(3), H(3)^2, 2H(4), \ldots \quad (7)$$

comes from the recurrence

$$Q_{k+1} = Q_{k-1}^2 - 2Q_k - 4$$

with initial conditions

$$\{Q_0, Q_1, Q_2\} = \{0, -x+a-3/2, 2x-2a-1\}$$

where the sequence (7) begins with Q_3.

Proof. Using an identity

$$T_{2n}(x) = 2T_n^2(x) - 1$$

followed from double angle formula, we may compute

$$H(n+1) = H(n)^2 - 2 \quad (8)$$

We use induction on $k \geq 3$. For $k = 3$,

$$\begin{aligned} Q_3 &= Q_1^2 - 2Q_2 - 4 \\ &= \left(-x+a-\frac{3}{2}\right)^2 - 2(2x-2a-1) - 4 \\ &= \frac{1}{4}(2x-2a-1)^2 \end{aligned}$$

and

$$\begin{aligned} H(1) &= 2T_2\left(\frac{1}{2}\sqrt{a+\frac{5}{2}-x}\right) \\ &= 2\left[2\left(\frac{1}{2}\sqrt{a+\frac{5}{2}-x}\right)^2 - 1\right] \\ &= \frac{1}{2}(-2x+2a+1) \end{aligned}$$

So $H(1)^2 = Q_3$. Assume the result holds for 3,4,...,k, i.e., $Q_j(3 \leq j \leq k)$ corresponds to

$$\begin{cases} 2H(j/2), & j\text{even}, \\ H^2((j-1)/2), & j\text{odd}. \end{cases}$$

If k is even, by (8)

$$\begin{aligned} Q_{k+1} &= Q_{k-1}^2 - 2Q_k - 4 \\ &= H^4(k/2-1) - 2 \cdot 2H(k/2) - 4 \\ &= H^4(k/2-1) - 4[H^2(k/2-1)-2] - 4 \\ &= H^2(k/2) \end{aligned}$$

and if k is odd, by (8)

$$\begin{aligned} Q_{k+1} &= Q_{k-1}^2 - 2Q_k - 4 \\ &= (2H((k+1)/2-1))^2 - 2H^2((k+1)/2-1) - 4 \\ &= 2[H^2((k+1)/2-1) - 2] \\ &= 2H((k+1)/2) \end{aligned}$$

which completes the proof. □

Another main purpose of this paper is to provide some well-organized interesting conjectures. We now consider discriminants and resultants of the polynomials Q_k in the resulting sequence

$$\{Q_3, Q_4, Q_5, Q_6, \dots\} \tag{9}$$

comes from the recurrence

$$Q_{k+1} = Q_{k-1}^2 - 2Q_k - 4$$

with initial conditions (6)

$$\{Q_0, Q_1, Q_2\} = \{0, -x, 2x-2a-1\}$$

A few polynomials Q_k are as follows.

$$\begin{aligned} Q_3 &= x^2 - 4x + 2(2a-1), \\ Q_4 &= 2x^2 - 4(2a-1)x + (2a-1)^2 \\ Q_5 &= x^4 - 8x^3 + 8(a+1)x^2 - 8(2a-1)x + 2(4a^2-4a-1), \\ Q_6 &= 2x^4 - 32(a-1)x^3 + 4(20a^2-24a+1)x^2 \\ &\quad -8(2a-1)(4a^2-4a-1)x + (4a^2-4a-1)^2 \end{aligned}$$

If the parameter a is retained as a variable, the resulting two variable polynomial has discriminants and resultants (i.e. resultants of consecutive polynomials) with respect to both a and x that have remarkable factorizations. For examples,

$$\begin{aligned} &Res_x(Q_6(x), Q_7(x)) \\ &\quad = (2a-3)^4(16a^4+32a^2-200a^2+232a-79)^2 \\ &\quad\quad (16a^4+32a^3-72a^2-24a+17)^2 \\ &\quad\quad (16a^4+32a^3-40a^2-56a+17)^2 \\ &\quad\quad (16a^4+16a^3+344a^2-280a-463), \end{aligned}$$

$$\begin{aligned} &Res_a(Q_6(a), Q_7(a)) \\ &\quad = 65536(x^4-8x^2-8x-2)^2(x^4-8x^2+8x-2)^2 \\ &\quad\quad (x^4-4x^2+2)^2 \\ &\quad\quad (x^4-8x^3+20x^2-16x+2)(x^4+8x^3+20x^2+16x+2) \end{aligned}$$

and for discriminants,

$$\begin{aligned} &Res_x(Q_7(x), Q_7'(x)) \\ &\quad = 2147483648(2a-3)^4 \\ &\quad\quad (16a^4+32a^2-200a^2+232a-79)^2 \\ &\quad\quad (16a^4+32a^3-72a^2-24a+17)^2 \\ &\quad\quad (16a^4+32a^3-40a^2-56a+17)^2, \end{aligned}$$

$$\begin{aligned} &Res_a(Q_7(a), Q_7'(a)) \\ &\quad = 268435456(x^4-8x^2-8x-2)^2(x^4-8x^2+8x-2)^2 \\ &\quad\quad (x^4-4x^2+2)^2 \end{aligned}$$

From above we see that for some constants c_1 and c_2,

$$\begin{aligned} &Res_x(Q_6(x), Q_7(x)) \\ &\quad = c_1 Disc_x(Q_7(x)) \\ &\quad\quad (16a^4+160a^3+344a^2-280a-463) \end{aligned}$$

$$\begin{aligned} &Res_a(Q_6(a), Q_7(a)) \\ &\quad = c_2 Disc_a(Q_7(a))(x^4-8x^3+20x^2-16x+2) \\ &\quad\quad (x^4+8x^3+20x^2+16x+2) \end{aligned}$$

It is surprising in that it shows resultants of consecutive polynomials Q_k and Q_{k+1} have factors the discriminants of Q_{k+1} with respect to both [illegible]and [illegible]. This is unusual and it makes further studies of $Q_{k+(x)}$ worth.

Next we consider zero distributions. For $a = 3/2$ we have the previous. For $a > 3/2$, computer calculations suggest the conjecture that the complex zeros of the polynomials in the resulting sequence (9) lie alternately (depending on the parity of k) on

(a) the part of the parabola $y^2 = (2a-3)x$, where here x and y denote real and imaginary parts respectively with $0 \le x \le 4$ and on

(b) the infinite ray $(2a-3)/4 \le x < \infty$.

When $a \le 3/2$, the parabola collapses onto the real line and for $a = 3/2$ we have the Chebyshev polynomials of the first kinds described in Proposition 2.

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