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The Nonexistence of Conformal Deformations on Riemannian Warped Product Manifolds

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Abstract

In this paper, when N is a compact Riemannian manifold, we discuss the nonexistence of conformal deformations on Riemannian warped product manifold $M = (a, \infty)\times_f N$ with prescribed scalar curvature functions.

Key words : Warped Product, Scalar Curvature, Partial Differential Equation

1. Introduction

In a recent study[7-9], M.C. Leung has studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of Riemannian warped metric with some prescribed scalar curvature function. He has studied the uniqueness of positive solution to equation

$$\Delta_{g_0} u(x) + d_n u(x) = d_n u(x)^{\frac{n+2}{n-2}} \qquad (1.1)$$

where Δ_{g_0} is the Laplacian operator for an n-dimensional Riemannian manifold (N, g_0) and $d_n = n-2/4(n-1)$. Equation (1.1) is derived from the conformal deformation of Riemannian metric[1,4-6,8,9].

Similarly, let (N, g_0) be a compact Riemannian dimensional manifold. We consider the $(n+1)$–dimensional Riemannian warped product manifold $M = (a, \infty)\times_f N$ with the metric $g = dt^2 + f(t)^2 g_0$, where f is a positive function on (a, ∞). Let $u(t, x)$ be a positive smooth function on M and let g have a scalar curvature equal to $r(t, x)$. If the conformal metric $g_c = u(t, x)^{4/n-1}$ has a prescribed function $R(t, x)$ as a scalar curvature, then it is well known that $u(t, x)$ satisfies equation

$$\frac{4n}{n-1}\Box_g u(t,x) - r(t,x)u(t,x) + R(t,x)u(t,x)^{\frac{n+3}{n-1}} = 0 \qquad (1.2)$$

where $\Box_g$ is the d'Alembertian for a Riemannian warped manifold $M = (a, \infty)\times_f N$.

In this paper, we study the nonexistence of a positive solution to equation (1.2). This paper contains the results of Riemannian version of [3].

2. Main Results

First of all, in order to prove the nonexistence of solutions of some partial differential equations, we need brief results about Young's inequality. The following proposition is well known(cf. Theorem 1 in [10, p.48]).

Proposition 1. Let f be a real-valued, continuous and strictly increasing function on $[0, c]$ with $c > 0$. If $f(0) = 0$, $a \in [0, c]$ and $b \in [0, f(c)]$, then

$$\int_0^a f(t)dt + \int_0^b f^{-1}(t)dt \geq ab$$

where f^{-1} is the inverse function of f. Equality holds if and only if $b = f(a)$.

Corollary 2. Let $a, b \geq 0$ and $p > 1$ such that $1/p + 1/q = 1$. For any $\varepsilon > 0$,

$$\varepsilon\frac{1}{p}a^p + \frac{1}{\varepsilon^{\frac{1}{p-1}}}\frac{1}{q}b^q \geq ab$$

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Proof. In Proposition 1, we choose $f(t) = \varepsilon\, t^{p-1}$ for $p > 1$.

From now on, we let (N, g_0) be a compact Riemannian n–dimensional manifold with $n \geq 3$ and without boundary. The following proposition is also well known(cf. Theorem 5.4 in [2]).

Proposition 3. Let $M = (a, \infty)\times_f N$ have a Riemannian warped product metric $g = dt^2 + f(t)^2 g_0$. Then the Laplacian $\Box_g$ is given by

$$\Box_g = \frac{\partial^2}{\partial t^2} + \frac{nf'(t)}{f(t)}\frac{\partial}{\partial t} + \frac{1}{f(t)^2}\Delta_x$$

where Δ_x is the Laplacian on fiber manifold N.

By Proposition 3, equation (1.2) is changed into the following equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} + \frac{nf'(t)}{f(t)}\frac{\partial u(t,x)}{\partial t} + \frac{1}{f(t)^2} + \frac{1}{f(t)^2}\Delta_x u(t,x) - \frac{n-1}{4n} r(t,x)u(t,x) + \frac{n-1}{4n}R(t,x)u(t,x)^{\frac{n+3}{n-1}} = 0 \tag{2.1}$$

If $u(t, x) = u(t)$ is a positive function with only variable t(we call it a time-scale conformal deformation) and if $R(t, x) = R(t)$ and $r(t)$ are also functions of only variable t, then equation (2.1) becomes

$$u''(t) + \frac{nf'(n)}{f(n)}u'(t) = h(t)u(t) - H(t)u(t)^{\frac{n+3}{n+1}} \tag{2.2}$$

where $h(t) = \frac{n-1}{4n}r(t)$ and $H(t) = \frac{n-1}{4n}R(t)$.

The proof of the following theorem is similar to that of Theorem 2 in [4].

Theorem 4. Let $u(t)$ be a positive solution of equation (2.2). And let $h(t)$, $H(t)$ satisfy the following condition : $h(t) \geq 0$ and $H(t) \leq -c_1$, where c_1 is positive constant. Assume that there exist positive constants t_0 and c_0 such that $|f'(t)/f(t)| \leq C_0$ for all $t > t_0$. Then $u(t)$ is bounded from above.

Proof. From equation (2.2) we have

$$\frac{(f^n u')'}{f^n} = h(t)u - H(t)u^{\frac{n+3}{n-1}} \tag{2.3}$$

Let $\chi \in C_0^\infty((a,\infty))$ be a cut-off function. Multiplying both sides of equation (2.3) by $\chi^{2(n+1)/n-1}u$ and then using integration by parts we obtain

$$-\int_a^\infty (f^n u')\left\{\frac{\chi^{\frac{2(n+1)}{n-1}}u}{f^n}\right\}dt = h(t)\int_a^\infty \chi^{\frac{2(n+1)}{n-1}}u^2 dt - \int_a^\infty H(t)\chi^{\frac{2(n+1)}{n-1}}u^{\frac{2n+2}{n-1}}dt \tag{2.4}$$

Since $h(t) \geq 0$ and $H(t) \leq -c_1$ where c_1 is a positive constant, equation (2.4), implies

$$-\int_a^\infty (f^n u')\left\{\frac{\chi^{\frac{2(n+1)}{n-1}}u}{f^n}\right\}dt \geq c_1\int_a^\infty \chi^{\frac{2(n+1)}{n-1}}u^{\frac{2n+2}{n-1}}dt$$

From the left side of the above equation, we have

$$-(f^n u')\left\{\frac{\chi^{\frac{2(n+1)}{n-1}}u}{f^n}\right\} = -\frac{2(n+1)}{n-1}\chi^{\frac{n+3}{n-1}}u\chi'u' - \chi^{\frac{2(n+1)}{n-1}}|u'|^2 + n\chi^{\frac{2(n+1)}{n-1}}uu'\frac{f'}{f}$$

Applying the Cauchy inequality, we get

$$-\frac{2(n+1)}{n-1}\chi^{\frac{n+3}{n-1}}u\chi'u' = -2\left\{\frac{2(n+1)}{n-1}\chi^{\frac{2}{n-1}}u\chi'\right\}\left\{\frac{1}{2}\chi^{\frac{n+1}{n-1}}u'\right\} \leq \frac{4(n+1)^2}{(n-1)^2}\chi^{\frac{4}{n-1}}u^2|\chi'|^2 + \frac{1}{4}\chi^{\frac{2(n+1)}{n-1}}|u'|^2$$

and

$$n\chi^{\frac{2(n+1)}{n-1}}uu'\frac{f'}{f} = 2\left(n\chi^{\frac{n+1}{n-1}}u\frac{f'}{f}\right)\left(\frac{1}{2}\chi^{\frac{n+1}{n-1}}u'\right) \leq n^2\chi^{\frac{2(n+1)}{n-1}}\left(\frac{f'}{f}\right)^2u^2 + \frac{1}{4}\chi^{\frac{2n+1}{n-1}}|u'|^2$$

Together with the above equations, we obtain

$$n^2\int_a^\infty\left(\frac{f'}{f}\right)^2\chi^{\frac{2(n+1)}{n-1}}u^2dt + \frac{4(n+1)^2}{(n-1)^2}\int_a^\infty \chi^{\frac{4}{n-1}}u^2|\chi'|^2dt \geq c_1\int_a^\infty \chi^{\frac{2(n+1)}{n-1}}u^{\frac{2(n+1)}{n-1}}dt + \frac{1}{2}\int_a^\infty \chi^{\frac{2(n+1)}{n-1}}|u'|^2dt$$

Applying Corollary 2 and using the bound $\left|\frac{f'}{f}\right| \le C_0$, we have

$$\int_a^\infty \chi^{\frac{2(n+1)}{n-1}} u^2 dt = \int_a^\infty \chi^{\frac{4}{n-1}} \chi^2 u^2 dt \le \varepsilon \frac{2}{n+1}$$
$$\int_a^\infty \chi^{\frac{2(n+1)}{n-1}} dt + \frac{1}{\varepsilon^{\frac{2}{n-1}}} \frac{n-1}{n+1} \int_a^\infty \chi^{\frac{2(n+1)}{n-1}} u^{\frac{2(n+1)}{n-1}} dt$$

and

$$\int_a^\infty \chi^{\frac{4}{n-1}} u^2 |\chi'|^2 dt = \int_a^\infty |\chi'|^2 \chi^{\frac{-2n+6}{n-1}} \chi^2 u^2 dt \le \varepsilon \frac{2}{n+1}$$
$$\int_a^\infty \chi^{\frac{-(n+1)(n-3)}{n-1}} |\chi'|^{n+1} dt + \frac{1}{\varepsilon^{\frac{2}{n-1}}} \frac{n-1}{n+1} \int_a^\infty \chi^{\frac{2(n+1)}{n-1}} u^{\frac{2(n+1)}{n-1}} dt$$

For large $\varepsilon > 0$ we obtain

$$C' \int_a^\infty \chi^{\frac{2(n+1)}{n-1}} u^{\frac{2(n+1)}{n-1}} dt + \frac{1}{2} \int_a^\infty \chi^{\frac{2(n+1)}{n-1}} |u'|^2 dt \le C'' \int_a^\infty \left(\chi^{\frac{2(n+1)}{n-1}} + \chi^{\frac{-(n+1)(n-3)}{n-1}} |\chi'|^{n+1} \right) dt \tag{2.5}$$

where C′, C″ are positive constants. Let $\chi \equiv 0$ on $(a, r) \cup [r+3, \infty]$ with $r > t_0$, and $\chi \equiv 1$ on $[r+1, r+2]$, $\chi \ge 0$ on $[a, \infty]$ and $|\chi'| \le 1/2$. From equation (2.5) we have

$$C' \int_{r+1}^{r+2} u^{\frac{2(n+1)}{n-1}} dt + \frac{1}{2} \int_{r+1}^{r+2} |u'|^2 dt \le C'''$$

for all $r > t_0$, where C‴ is a constant independent on r.

Therefore u is bounded from above.

Theorem 5. Let (M, g) be a Riemannian manifold with scalar curvature equal $h(t)$. Assume that there exist positive constants t_0 and C_0 such that $|f'(t)/f(t)| \le C_0$ for all $t > t_0$. For a smooth function $H(t)$, let $h(t)$, $H(t)$ satisfy the following condition : $h(t) \ge 0$, $H(t) \le -c_1$, where c_1 is positive constant. Then equation (2.2) has no positive solution.

Proof. If $u = u(t)$ is a positive solution of equation (2.2), then by Theorem 4 $u(t)$ is bounded from above on (a, ∞) Then, by Omori-Yau maximum principle(c.f. [11]), there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} u(t_k) = \sup_{t\in(a,\infty)} u(t)$, $|u'(t_k)| \le 1/k$ and $u''(t_k) \le 1/k$ Since $\sup_{t\in(a,\infty)} u(t) = c_2 > 0$, there exist a number $\varepsilon > 0$ and K such that

$$h(t_k)u(t_k) - H(t_k)u(t_k)^{\frac{n+3}{n-1}} > \varepsilon$$

for all $k > K$, which is a contradiction to the fact that

$$u''(t_k) + \frac{nf'(t_k)}{f(t_k)} u'(t_k) \le \frac{1 + nC_0}{k}$$

for all $k > K$. Therefore equation (2.2) has no positive solution.

The following corollary is derived easily from the previous theorems.

Corollary 6. Let $(M, g) = ((a, \infty) \times_f N, g)$ be a Riemannian manifold with scalar curvature equal to $r(t)$. Assume that there exist positive constants t_0 and C_0 such that $|f'(t)/f(t)| \le C_0$ for all $t > t_0$. For a smooth function let $R(t)$, let $r(t)$, $R(t)$ satisfy the following condition : $r(t) \ge 0$, $R(t) \le -c_0$ where c_1 is positive constant. Then there does not exist a time-scale conformal deformation on M with scalar curvature $R(t)$.

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