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Basic Statistics in Quantile Regression

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Abstract

In this paper we study some basic statistics in quantile regression. In particular, we investigate the residual, goodness-of-fit statistic and the effect of one or few observations on estimates of regression coefficients. In addition, we compare the proposed goodness-of-fit statistic with the statistic considered by Koenker and Machado (1999). An illustrative example based on real data sets is given to see the numerical performance of the proposed basic statistics.

Keywords: Goodness-of-fit statistic, influence measure, influential observations, residual.

1. Introduction

Classical regression methods are mainly concerned about estimating conditional mean function when covariates are given; however, quantile regression methods, first introduced by Koenker and Bassett (1978) focused on the estimation of conditional quantile functions. A special case of quantile regression is the least absolute deviation(LAD) estimator by fitting medians to covariates; in addition, it is well known that LAD is more robust to outliers than the classical least squares estimators(LSE). Since then, quantile regression has emerged as a comprehensive tool in statistical regression modelling that has been widely used in many fields. Among them, Cole and Green (1992), and Heagerty and Pepe (1999) applied to reference charts in medicine and Hendricks and Koenker (1992), and Koenker and Hallock (2001) applied to economic modelling. One novel feature of quantile regression is its applicability to the censored regression that was initiated by Powell (1984, 1986) and further developed by Portnoy (2003). Subsequent works in censored regression include Peng and Huang (2008), Yin *et al.* (2008), Shim and Hwang (2009), Wang and Wang (2009), and Huang (2010). Also, Li *et al.* (2007) studied quantile regression in the reproducing kernel Hilbert space. Good papers for review are, Yu *et al.* (2003) and Koenker (2008); in addition, an excellent book in quantile regression is Koenker (2005).

In this paper, we study some basic statistics in quantile regression. In particular, we investigate residual, the goodness-of-fit statistic, and the effect of one or few observations on estimates of

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regression coefficients. Goodness-of-fit statistic in quantile regression was first considered by Koenker and Machado (1999). They derived a version of the coefficient of determination as a goodness-of-fit statistic in quantile regression. We consider another version of the goodness-of-fit statistic, and compare with the statistic suggested by Koenker and Machado (1999) through numerical studies. In addition, we suggest a quantile version of residual in quantile regression, and argue that the proposed residual can serve as a basic building block. Further, we investigate the effect of observations on the estimate of regression coefficients, and we note that it is significantly different from the case of least squares estimator. The rest of this paper is organized as follows.

In Chapter 2, review on the quantile regression features is given. In Chapter 3, we suggest a version of residual and the goodness-of-fit statistic in quantile regression; in addition, and investigation on the perturbation of observations is given. Numerical studies assessing two goodness-of-fit statistics are given in Chapter 4.

2. Quantile Regression

Consider a multiple linear model

$$y_i = \boldsymbol{x}_i^T \boldsymbol{\beta} + \epsilon_i, \quad i = 1, 2, \ldots, n$$

where y_i is a response variable, $\boldsymbol{x}_i$ is a p-vector of covariate with 1 in the first component, $\boldsymbol{\beta}$ is a p-vector of unknown coefficients, and ϵ_i is a identically and independently distributed error with mean 0 and variance σ^2. The least squares estimation(LSE) of $\boldsymbol{\beta}$ is obtained by minimizing the quadratic loss function $r(u) = u^2/2$, *i.e.*, given $\{\boldsymbol{x}_i, y_i\}_{i=1}^n$, the LSE is obtained by minimizing

$$\sum_{i=1}^n r\left(y_i - \boldsymbol{x}_i^T \boldsymbol{\beta}\right) = \frac{1}{2} \sum_{i=1}^n \left(y_i - \boldsymbol{x}_i^T \boldsymbol{\beta}\right)^2$$

over β. Therefore, the LSE is concerned with the estimation of the conditional expectation $E[Y|X = \boldsymbol{x}]$.

However, median quantile regression estimates the conditional median of Y given $X = \boldsymbol{x}$, and the corresponding loss function is $|u|/2$. The resulting estimator is called the least absolute deviation(LAD) estimator, because it minimizes

$$\sum_{i=1}^n r\left(y_i - \boldsymbol{x}_i^T \boldsymbol{\beta}\right) = \frac{1}{2} \sum_{i=1}^n \left|y_i - \boldsymbol{x}_i^T \boldsymbol{\beta}\right|.$$

Note that

$$\begin{aligned} \rho_{0.5}(u) &= 0.5|u| \\ &= 0.5uI_{[0,\infty)}(u) - (1 - 0.5)uI_{(-\infty,0)}(u), \end{aligned}$$

where $I(\cdot)$ is an indicator function. By replacing 0.5 by τ, $100\tau\%$ quantile regression $q_\tau(\boldsymbol{x})$ at $\boldsymbol{x}$ can be defined as the value of θ that minimizes

$$E[\rho_\tau(Y - \theta)|X = \boldsymbol{x}].$$

Here,

$$\rho_\tau(u) = \tau uI_{[0,\infty)}(u) - (1 - \tau)uI_{(-\infty,0)}(u) \tag{2.1}$$

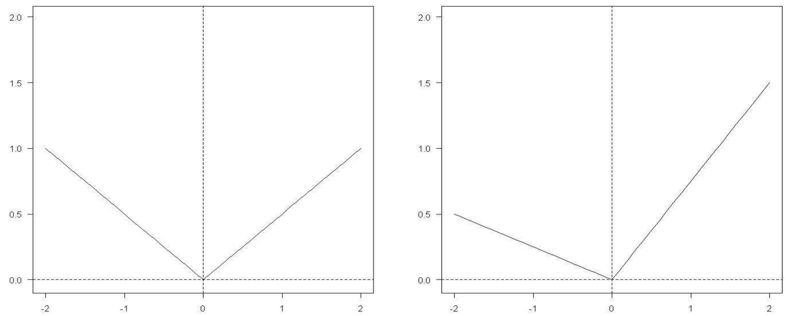

Figure 2.1. Check function($\tau = 0.5$, $\tau = 0.75$)

is called the 'check function' (see Figure 2.1), and it can also be written as

$$\rho_\tau(u) = u(\tau - I(u < 0)).$$

Let

$$\hat{\boldsymbol{\beta}}_\tau = \arg_{\boldsymbol{\beta} \in \mathbb{R}^p} \min \sum_{i=1}^{n} \rho_\tau \left(y_i - \boldsymbol{x}_i^T \boldsymbol{\beta} \right)$$

be the τ^{th} quantile estimator of $\boldsymbol{\beta}$, and let

$$\hat{y}_{\tau,i} = \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}_\tau$$

be the i^{th} fitted value. In addition, let

$$e_{\tau,i} = y_i - \hat{y}_{\tau,i} \tag{2.2}$$

be the i^{th} residual in the τ^{th} quantile regression.

3. Basic Statistics in Quantile Regression

3.1. Goddness-of-fit statistic

Note that, in the quantile regression, the residual $e_{\tau,i}$ defined in (2.2) is not appropriate since it does not reflect the aspect of the value τ. Here, we define a residual, which can reflect the characteristic of quantile regression, as

$$\begin{aligned} r_{\tau,i} &= \rho_\tau(e_{\tau,i}) \\ &= \tau e_{\tau,i} I_{[0,\infty)}(e_{\tau,i}) - (1-\tau) e_{\tau,i} I_{(-\infty,0)}(e_{\tau,i}). \end{aligned}$$

As in the classical linear model, the proposed residual $r_{\tau,i}$ in the quantile regression can be used as basic building blocks for diagnostic issues, model checking, and the goodness-of-fit statistic.

To derive a goodness-of-fit measure, we first consider decomposition of sum of squares in the LSE case, *i.e.*,

$$\begin{array}{ccccc} \sum_{i=1}^{n}(y_i - \bar{y})^2 & = & \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 & + & \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 \\ \text{SST} & = & \text{SSR} & + & \text{SSE}\,. \end{array}$$

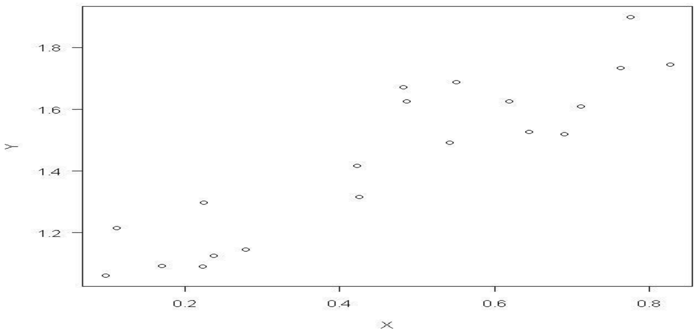

Figure 3.1. Scatter plot of 20 artificial data set

Using this, the coefficient of determination R^2 is defined as

$$R^2 = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SST}}.$$

Now, we define the goodness-of-fit statistic in the quantile regression by a similar way of decomposition. To do this, let $\bar{y}_\tau$ be the τ^{th} quantile estimate based on $Y_1, Y_2, \ldots, Y_n$, *i.e.*, the τ^{th} quantile estimate of $\boldsymbol{\beta}$ when there are no covariates. Then, we have

$$(y_i - \bar{y}_\tau) = (y_i - \hat{y}_{\tau,i}) + (\hat{y}_{\tau,i} - \bar{y}_\tau).$$

Again, to reflect the characteristic of quantile regression we apply the transformation $\rho_\tau(\cdot)$ on each term in (2.1), and call them SAT(Total Sum of Absolute deviation), SAR(Regression Sum of Absolute deviation), and SAE(Error Sum of Absolute deviation), respectively, *i.e.*,

$$\text{SAT}_\tau = \sum_{i=1}^{n} \rho_\tau(y_i - \bar{y}_\tau),$$
$$\text{SAR}_\tau = \sum_{i=1}^{n} \rho_\tau(\hat{y}_{\tau,i} - \bar{y}_\tau),$$
$$\text{SAE}_\tau = \sum_{i=1}^{n} \rho_\tau(y_i - \hat{y}_{\tau,i}) = \sum_{i=1}^{n} r_{\tau,i}.$$

Note that,

$$\text{SAT}_\tau \neq \text{SAR}_\tau + \text{SAE}_\tau.$$

Here, we suggest a goodness-of-fit statistic for the significance of the current quantile regression model as

$$R_\tau = \frac{\text{SAR}_\tau}{\text{SAT}_\tau},$$

which can be regarded as a quantile regression version of the coefficient of determination R^2. On the other hand, Koenker and Machado (1999) proposed a goodness-of-fit statistic for the significance of the current quantile regression model as

$$R_\tau^1 = 1 - \frac{\text{SAE}_\tau}{\text{SAT}_\tau}.$$

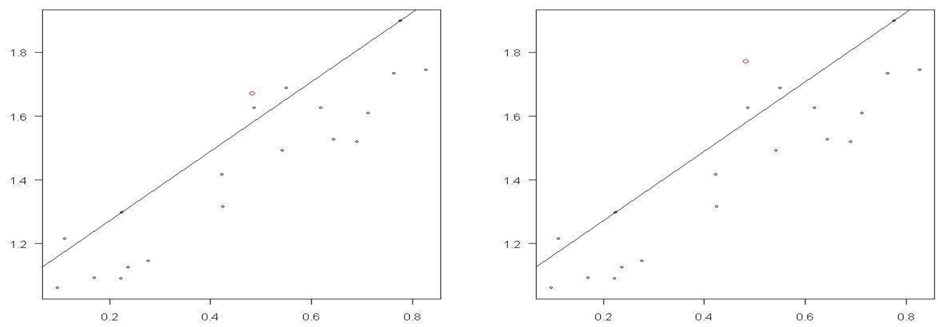

Figure 3.2. The quantile fit does not change at all even though we move an observation, which lies above the fitted line, to upwards ($\tau = 0.75$)

Note that $R_\tau + R_\tau^1 \neq 1$, and we usually have $R_\tau + R_\tau^1 > 1$. In addition, it is not possible to suggest a cutoff value for R_τ since the coefficient of determination is used only for a relative measure, not for an absolute measure.

REMARK 3.1. In computing SAT_τ and SAR_τ , there are two types; One is based on its own sign of $y_i - \bar{y}_\tau$ and $\hat{y}_{\tau,i} - \bar{y}_\tau$, respectively, and the other is based on the sign of its residual $y_i - \hat{y}_{\tau,i}$. As far as we know, there are no research results on this problem. We tried both methods in computing SAT_τ and SAR_τ in numerical studies, and we found that the method based on the sign of its residual $y_i - \hat{y}_{\tau,i}$ gave more consistent results.

3.2. Perturbation of observations

In this section we investigate several different features in quantile regression fit from those in the ordinary least squares fit using an artificial data set (see Figure 3.1). The artificial data set of size 20 were as follows; The explanatory variable X was generated from the uniform distribution $U(0, 1)$, and the error terms were generated from the standard normal distribution $N(0, 1)$. Finally, the response variable was set $Y = 1 + X + \epsilon$.

First, the quantile regression fit does not change at all even though we move an observation, which lies above the fitted line, to upwards. Similarly, the quantile regression fit does not change at all even though we move an observation, which lies below the fitted line, to downwards. This feature is illustrated in Figure 3.2. The robustness of the quantile regression is due to this phenomenon.

Second, a slight change of an observation on the fitted line does change the fitted line, however, the amount of change does not matter (see Figure 3.3).

Third, we investigate the effect of deletion of one observation on the fitted line. In the LSE case, an influential observation is usually due to an outlier with large residual and/or a high leverage point. To see whether the same phenomenon occurs in the quantile regression, we compute $\hat{\beta}_{\tau,0} - \hat{\beta}_{\tau,0(i)},\ \hat{\beta}_{\tau,1} - \hat{\beta}_{\tau,1(i)}$, residual $r_{\tau,i}$ and leverage in Table 3.1, where $\hat{\beta}_{\tau,k}$ and $\hat{\beta}_{\tau,k(i)}$ denote estimate of $\beta_{\tau,k},\ k = 0, 1$ based on n observations and $n - 1$ observations after deleting the ith observation, respectively. We note that the effect of residual and leverage in the quantile regression is quite different from the ordinary LSE case, because residual and leverage do not show any systematic effects on the influence.

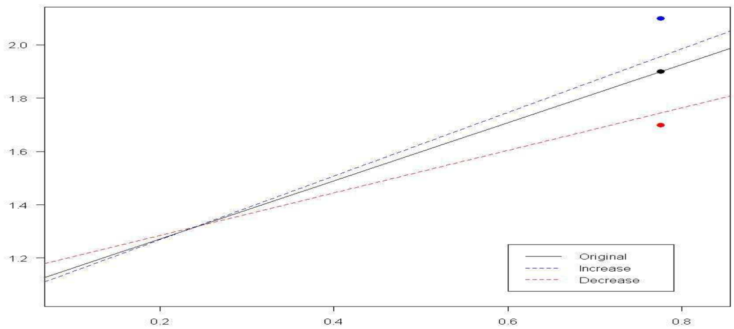

Figure 3.3. The quantile fit changes by moving an observation on the fitted line to upwards or downwards, and the amount of change does not matter ($\tau = 0.75$)

Table 3.1. Relationships between changes in estimates, residual, and leverage when one observation is deleted in the artificial data set ($\tau = 0.75$)

i	$\hat{\beta}_{\tau,0} - \hat{\beta}_{\tau,0(i)}$	$\hat{\beta}_{\tau,1} - \hat{\beta}_{\tau,1(i)}$	Residual	Leverage
1	0	0	−0.0525	0.175
2	0	0	−0.0384	0.0558
3	0.0463	−0.0597	−0.0364	0.1321
4	0.0463	−0.0597	0	0.1046
5	0.0583	−0.2604	0.0312	0.0505
6	0.0463	−0.0597	−0.0499	0.0515
7	0	0	−0.0552	0.1084
8	0	0	−0.0257	0.0729
9	−0.1115	0.1436	0.0297	0.1678
10	0.0463	−0.0597	−0.0244	0.1782
11	0	0	−0.0716	0.0987
12	0.0463	−0.0597	−0.0244	0.0516
13	0.0463	−0.0597	−0.0524	0.0830
14	0.0583	−0.2604	0.0252	0.0571
15	0.0583	−0.2604	0.0704	0.0503
16	0	0	−0.0572	0.0809
17	0	0	−0.038	0.1351
18	0.0463	−0.0597	−0.0515	0.1051
19	0.071	−0.2808	0	0.1424
20	0.0463	−0.0597	−0.0463	0.0991

3.3. Example

As an illustrative example for the goodness-of-fit measure in quantile regression, we use food expenditure data in Koenker and Bassett (1982). This data set consists of 235 observations about income and expenditures on food in Belgian working class households. The response variable is food expenditure and exploratory variable is income. In this data, variations in response for large income are larger than those for small income, and therefore, quantile regression is more appropriate than the usual least squares regression. For each τ, we compute two goodness-of-fit statistics R_τ and R_τ^1 for the linear and quadratic fit in food expenditure data. We see that both statistics suggest that the linear model is enough, *i.e.*, the extra contribution of the quadratic term is not significant. We will investigate the performance of both statistics in Section 4.

Table 3.2. Two goodness-of-fit statistics for the linear and quadratic fit in food expenditure data

τ	Linear Model		Quadratic Model	
	R_τ	R^1_τ	R_τ	R^1_τ
0.1	0.7884	0.6286	0.8026	0.6381
0.2	0.8356	0.6271	0.8489	0.6443
0.3	0.8719	0.6280	0.8920	0.6469
0.4	0.9068	0.6170	0.9247	0.6463
0.5	0.9645	0.6206	0.9545	0.6441
0.6	0.9797	0.6320	0.9796	0.6643
0.7	0.9842	0.6608	0.9558	0.6868
0.8	0.9959	0.7071	0.9676	0.7319
0.9	0.9490	0.7683	0.9551	0.7685

4. Simulation Study

To investigate numerical performance of two goodness-of-fit statistics R_τ and R^1_τ, we consider two types of models (case I and case II - see below). The artificial data set of size n were generated as follows. The explanatory variables X_1, X_2 and the error terms were generated from the standard normal distribution $N(0,1)$. Finally, the response variable was set $Y = 1 + X_1 + \epsilon$ in Case I and $Y = 1 + X_1 + X_2 + \epsilon$ in Case II. For the sample size, we consider $n = 30, 50$, and 100. Also, 1,000 replications are done.

(I) Case I

true model: X_1
postulated model: I-A (X_1), I-B (X_1, X_2)

(II) Case II

true model: X_1, X_2
postulated model: II-A (X_1), II-B (X_1, X_2)

In each case, we compute R_τ and R^1_τ for each postulated model, and results are summarized in Table 4.1–Table 4.3. Here, $R_{A,\tau}$ and $R_{B,\tau}$ denote the value of R_τ under each postulated model A and B, respectively. We see that both statistics R_τ and R^1_τ perform quite well in detecting true model; however, R_τ shows better results than R^1_τ in the sense that R_τ is quite stable for each τ while R^1_τ is not (see Figure 4.1). The variation in R^1_τ for each τ is larger than that of R_τ; however, more systematic studies using discrepancy measures such as the mean squared error should be done under various conditions.

5. Concluding Remarks

Quantile regression has increasingly become an important alternative statistical model when the classical least squares methods are not appropriate or not applicable. The Quantile regression method is quite robust to outliers compared to the least squares method. In this paper we study basic statistics in the linear quantile regression. We define a version of residual, suggest a goodness-of-fit statistic, and investigate influence of one or few observations. The residual we defined reflects the characteristics of quantile, and can be serve as a basic building block for the goodness-of-fit statistic and influence measure. The goodness-of-fit statistic we suggest is based on the amount of contribution due to regression, while the existing goodness-of-fit statistic is based on the amount

Table 4.1. Two goodness-of-fit statistics R_τ and R_τ^1 when $n = 30$

τ	Case I			Case II		
	$R_{A,\tau}$	$R_{B,\tau}$	$R_{B,\tau} - R_{A,\tau}$	$R_{A,\tau}$	$R_{B,\tau}$	$R_{B,\tau} - R_{A,\tau}$
0.1	0.6679	0.6958	0.0278	0.5492	0.7726	0.2233
0.2	0.6958	0.7158	0.0200	0.5731	0.8039	0.2309
0.3	0.7127	0.7328	0.0201	0.5941	0.8255	0.2314
0.4	0.7241	0.7472	0.0231	0.6041	0.8400	0.2359
0.5	0.7279	0.7503	0.0224	0.6073	0.8481	0.2408
0.6	0.7222	0.7417	0.0194	0.6004	0.8422	0.2418
0.7	0.7049	0.7293	0.0244	0.5822	0.8259	0.2437
0.8	0.6750	0.6983	0.0233	0.5540	0.7941	0.2401
0.9	0.6180	0.6436	0.0256	0.4979	0.7394	0.2415

τ	Case I			Case II		
	$R^1_{A,\tau}$	$R^1_{B,\tau}$	$R^1_{B,\tau} - R^1_{A,\tau}$	$R^1_{A,\tau}$	$R^1_{B,\tau}$	$R^1_{B,\tau} - R^1_{A,\tau}$
0.1	0.4178	0.4698	0.0520	0.2927	0.5805	0.2878
0.2	0.3647	0.3958	0.0311	0.2524	0.5179	0.2655
0.3	0.3354	0.3593	0.0239	0.2277	0.4831	0.2554
0.4	0.3190	0.3399	0.0209	0.2148	0.4625	0.2477
0.5	0.3114	0.3313	0.0199	0.2067	0.4536	0.2469
0.6	0.3101	0.3309	0.0208	0.2028	0.4527	0.2499
0.7	0.3132	0.3355	0.0223	0.2024	0.4596	0.2572
0.8	0.3208	0.3474	0.0266	0.2067	0.4775	0.2708
0.9	0.3306	0.3819	0.0513	0.2114	0.5207	0.3093

Table 4.2. Two goodness-of-fit statistics R_τ and R_τ^1 when $n = 50$

τ	Case I			Case II		
	$R_{A,\tau}$	$R_{B,\tau}$	$R_{B,\tau} - R_{A,\tau}$	$R_{A,\tau}$	$R_{B,\tau}$	$R_{B,\tau} - R_{A,\tau}$
0.1	0.6442	0.6620	0.0178	0.5167	0.7567	0.2400
0.2	0.6877	0.6988	0.0111	0.5583	0.7961	0.2378
0.3	0.7108	0.7210	0.0102	0.5760	0.8184	0.2424
0.4	0.7224	0.7354	0.0130	0.5928	0.8334	0.2406
0.5	0.7247	0.7384	0.0136	0.5918	0.8353	0.2434
0.6	0.7174	0.7299	0.0125	0.5820	0.8268	0.2448
0.7	0.7003	0.7110	0.0107	0.5707	0.8124	0.2417
0.8	0.6703	0.6829	0.0125	0.5462	0.7849	0.2388
0.9	0.6180	0.6337	0.0157	0.4948	0.7367	0.2420

τ	Case I			Case II		
	$R^1_{A,\tau}$	$R^1_{B,\tau}$	$R^1_{B,\tau} - R^1_{A,\tau}$	$R^1_{A,\tau}$	$R^1_{B,\tau}$	$R^1_{B,\tau} - R^1_{A,\tau}$
0.1	0.3803	0.4065	0.0262	0.2530	0.5380	0.2850
0.2	0.3433	0.3590	0.0157	0.2264	0.4902	0.2638
0.3	0.3217	0.3344	0.0127	0.2084	0.4641	0.2557
0.4	0.3109	0.3222	0.0113	0.1999	0.4492	0.2493
0.5	0.3063	0.3170	0.0107	0.1956	0.4430	0.2474
0.6	0.3064	0.3177	0.0113	0.1945	0.4444	0.2499
0.7	0.3112	0.3239	0.0127	0.1965	0.4528	0.2563
0.8	0.3178	0.3338	0.0160	0.2035	0.4710	0.2675
0.9	0.3318	0.3599	0.0281	0.2070	0.5031	0.2961

of contribution due to error. Two types of goodness-of-fit statistics reduces to the same statistics in the classical least squares estimation; however, they differ quite a lot in the quantile regression.

Table 4.3. Two goodness-of-fit statistics R_τ and R_τ^1 when $n = 100$

τ	Case I			Case II		
	$R_{A,\tau}$	$R_{B,\tau}$	$R_{B,\tau} - R_{A,\tau}$	$R_{A,\tau}$	$R_{B,\tau}$	$R_{B,\tau} - R_{A,\tau}$
0.1	0.6359	0.6431	0.0072	0.5105	0.7445	0.2340
0.2	0.6747	0.6801	0.0054	0.5489	0.7824	0.2334
0.3	0.6992	0.7045	0.0053	0.5701	0.8066	0.2365
0.4	0.7130	0.7188	0.0058	0.5825	0.8210	0.2385
0.5	0.7157	0.7214	0.0057	0.5832	0.8259	0.2427
0.6	0.7119	0.7168	0.0049	0.5819	0.8212	0.2393
0.7	0.6932	0.6991	0.0059	0.5663	0.8052	0.2389
0.8	0.6673	0.6714	0.0041	0.5436	0.7781	0.2345
0.9	0.6222	0.6283	0.0061	0.5000	0.7347	0.2347

τ	Case I			Case II		
	$R^1_{A,\tau}$	$R^1_{B,\tau}$	$R^1_{B,\tau} - R^1_{A,\tau}$	$R^1_{A,\tau}$	$R^1_{B,\tau}$	$R^1_{B,\tau} - R^1_{A,\tau}$
0.1	0.3568	0.3717	0.0149	0.2286	0.5074	0.2788
0.2	0.3301	0.3389	0.0088	0.2115	0.4724	0.2609
0.3	0.3134	0.3200	0.0066	0.1998	0.4509	0.2511
0.4	0.3036	0.3097	0.0061	0.1927	0.4385	0.2458
0.5	0.2998	0.3053	0.0055	0.1896	0.4336	0.2440
0.6	0.3007	0.3063	0.0057	0.1897	0.4357	0.2460
0.7	0.3058	0.3123	0.0065	0.1932	0.4440	0.2508
0.8	0.3158	0.3236	0.0078	0.1993	0.4590	0.2597
0.9	0.3334	0.3467	0.0133	0.2052	0.4857	0.2805

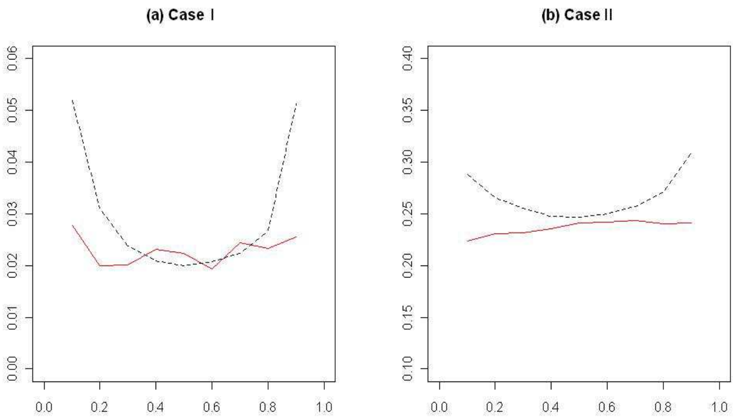

Figure 4.1. $R_{B,\tau} - R_{A,\tau}$ and $R^1_{B,\tau} - R^1_{A,\tau}$ for $\tau = 0.1(0.1)0.9$ when $n = 30$. (a) Case I, (b) Case II

Through numerical studies based on artificial data sets, we found that our statistic is more stable than the existing one as far as determining the threshold. In addition, we noticed that one or few observations can be influential in the quantile regression; however, they showed quite different aspects from the classical regression diagnostics which is mostly based on the residual and leverage.

For future research, we need a more systematic approach to analyze the difference between two types of goodness-of-fit statistics. In addition, it will be quite important to derive a version of Cook's distance in quantile regression. Further, it is worth pursuing to study basic statistics in the nonparametric quantile regression.

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