On 2-Absorbing and Weakly 2-Absorbing Ideals of Commutative Semirings

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Abstract. Let R be a commutative semiring. We define a proper ideal I of R to be 2-absorbing (resp., weakly 2-absorbing) if $abc \in I$ (resp., $0 \neq abc \in I$) implies $ab \in I$ or $ac \in I$ or $bc \in I$. We show that a weakly 2-absorbing ideal I with $I^3 \neq 0$ is 2-absorbing. We give a number of results concerning 2-absorbing and weakly 2-absorbing ideals and examples of weakly 2-absorbing ideals. Finally we define the concept of $\mathfrak{o} - (\mathfrak{1}-, \mathfrak{2}-, \mathfrak{3}-)2$-absorbing ideals of R and study the relationship among these classes of ideals of R.

1. Introduction

The semiring is an important structure that has gained importance in recent decades as its usefulness to many disciplines has been discovered and exploited. The concept of semirings was introduced by H. S. Vandiver in 1935. A nonempty set R together with two associative binary operations called addition and multiplication (denoted by + and ., respectively) will be called a semiring provided that:

(1) $(R, +)$ is a commutative monoid with identity element 0.

(2) $(R, .)$ is a monoid with identity element $1 \neq 0$.

(3) The multiplication is distributive with respect to the addition both from the left and from the right.

(4) $a.0 = 0.a = 0$ for all $a \in R$.

A semiring R is commutative if $(R, .)$ is a commutative semigroup. In this paper all semirings are considered to be commutative.

A nonempty subset I of a semiring R will be called an ideal if $a, b \in I$ and $r \in R$ imply that $a + b \in I$ and $ra \in I$. Ideal theory in semirings has been studied extensively by many authors (e.g. [2, 4, 7, 8, 9, 10, 11, 12, 15]). A k-ideal (also called a subtractive ideal) I is an ideal of R such that if $a, a + b \in I$ then $b \in I$. Let A be a subset of R. The semiring ideal generated by A, denoted by A_d, is $\{r_1a_1 + r_2a_2 + ...r_sa_s | r_i \in R, a_i \in A, s \in \mathbb{N}_0\}$. The k-ideal generated by A, denoted by A_{sub}, is $\bigcap\{I | A \subseteq I$ and I is a subtractive ideal of $R\}$. It is proved in

[17, Theorem 1.2] that $A_{sub} = \{x \in R | \exists a \in A_d$ such that $a + x \in A_d\}$. A prime ideal of R is a proper ideal P of R for which whenever $a, b \in R$ with $ab \in P$ then either $a \in P$ or $b \in P$. So P is a prime ideal of R if and only if for ideals A, B of R with $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$, where $AB = \{ab | a \in A$ and $b \in B\}$ (see [4, Theorem 5]). R is called a prime semiring if 0 is a prime ideal of R.

The concept of weakly prime ideals in a semiring R (not necessarily commutative) was introduced in [15], where a proper ideal P of R is called weakly prime if whenever $a, b \in R$ with $0 \neq aRb \subseteq P$, then $a \in P$ or $b \in P$.

Badawi in [5] generalized the concept of prime ideals in a different way. He defined a nonzero proper ideal I of a commutative ring R to be a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. This definition can obviously be extended to include the zero ideal of R. This concept has a generalization, called weakly 2-absorbing ideals, which has been studied in [6]. A proper ideal I of R to be a weakly 2-absorbing ideal of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. This paper will be focused on 2-absorbing and weakly 2-absorbing ideals of commutative semirings and some basic properties of these concepts.

2. 2-absorbing and weakly 2-absorbing ideals

In this section we give two definitions which are generalizations of prime ideals and weakly prime ideals in semirings.

Definition 2.1. Let R be a commutative semiring and I a proper ideal of R.
(1) I is said to be a 2-absorbing ideal of R if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.
(2) I is said to be a weakly 2-absorbing ideal of R provided that $0 \neq abc \in I$ with $a, b, c \in R$ implies that $ab \in I$ or $ac \in I$ or $bc \in I$.

Lemma 2.2. *Let R be a commutative semiring.*
(1) *Every prime ideal of R is 2-absorbing.*
(2) *Every weakly prime ideal of R is weakly 2-absorbing.*
(3) *Every 2-absorbing ideal of R is weakly 2-absorbing.*

Theorem 2.3. *Let R be a commutative semiring. Then the intersection of each pair of distinct prime ideals of R is 2-absorbing.*

Proof. Assume that I_1 and I_2 are two distinct prime ideals of R. Suppose that $a, b, c \in R$ with $abc \in I_1 \cap I_2$ but $ab \notin I_1 \cap I_2$ and $ac \notin I_1 \cap I_2$. Assume that $ab \notin I_1$ and $ac \notin I_1$. Then from $abc \in I_1$ and I_1 prime we get $c \in I_1$. In this case $ac \in I_1$ which is a contradiction. A similar argument for the case where $ab \notin I_2$ and $ac \notin I_2$ leads us to a contradiction. So either $ab \notin I_1$ and $ac \notin I_2$, or $ab \notin I_2$ and $ac \notin I_1$. Assume that the former case holds. Then from $abc \in I_1$ and I_1 prime we have $c \in I_1$; so $bc \in I_1$. Also from $abc \in I_2$ we get $b \in I_2$; so $bc \in I_2$. Therefore $bc \in I_1 \cap I_2$. Likewise, if the latter case holds then $bc \in I_1 \cap I_2$. Consequently $I_1 \cap I_2$ is a 2-absorbing ideal of R. □

Example 2.4. By Lemma 2.2, every prime ideal of a commutative semiring is 2-absorbing, but the reverse implication is not necessarily true. Let $\mathbb{N} = \{0, 1, 2, ...\}$ be the natural numbers with the usual addition and multiplication. Then $\mathbb{N}$ is a local semiring with maximal ideal $M = N \setminus \{1\} = (2, 3)$. The prime ideals of $\mathbb{N}$ are $0, p\mathbb{N}$ (p is prime), and M. Every ideal of $\mathbb{N}$ has the form dA where A is an ideal of $\mathbb{N}$ containing all natural numbers larger than some natural number (see [3] for details). Now, for every pair of distinct prime numbers p and q, $pq\mathbb{N} = p\mathbb{N} \cap q\mathbb{N}$ is a 2-absorbing ideal of $\mathbb{N}$ by Theorem 2.3, while it is not a prime ideal.

An ideal I of a semiring R is called a Q-ideal (also called a partitioning ideal) if there exists a subset Q of R such that
(1) $R = \cup\{q + I : q \in Q\}$, and
(2) If $q_1, q_2 \in Q$, then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$.
Let I be a Q-ideal of a semiring R and let $R/I = \{q + I : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows:

$$(q_1 + I) \oplus (q_2 + I) = q_3 + I$$

where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$.

$$(q_1 + I) \odot (q_2 + I) = q_4 + I$$

where $q_4 \in Q$ is the unique element such that $q_1 q_2 + I \subseteq q_4 + I$. This semiring R/I is called the quotient semiring of R by I. By definition of a Q-ideal, there exists a unique $q' \in Q$ such that $0 + I \subseteq q' + I$. Then $q' + I$ is a zero element of R/I. Clearly, if R is commutative, then so is R/I. Let R be a semiring, I a Q-ideal of R and A a k-ideal of R with $I \subseteq A$. Then $A/I = \{q + I : q \in A \cap Q\}$ is a k-ideal of R/I. Conversely, if I a Q-ideal of R and L a k-ideal of R/I, then $L = J/I$, where $J = \{r \in R : q_1 + I \in L\}$ (note that if $r \in R$, then there exists the unique element $q_1 \in Q$ such that $r \in q_1 + I$) is a k-ideal of R (see [8, 13, 14]). In the next result we give the relations between the 2-absorbing (resp., weakly 2-absorbing) ideals of R and 2-absorbing (resp., weakly 2-absorbing) ideals of R/I.

Theorem 2.5. *Let R be a commutative semiring, I a Q-ideal of R and A a k-ideal of R with $I \subseteq A$. Then*
(1) A is a 2-absorbing ideal of R if and only if A/I is a 2-absorbing ideal of R/I.
(2) If A is a weakly 2-absorbing ideal of R, then A/I is a weakly 2-absorbing ideal of R/I.
(3) If I and A/I are both weakly 2-absorbing ideals, then A is weakly 2-absorbing.

Proof. (1) Let A be a 2-absorbing ideal of R. Suppose that $q_1+I, q_2+I, q_3+I \in R/I$ are such that $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in A/I$ where $q_1, q_2, q_3 \in Q$. Then there is a unique element $q_4 \in Q \cap A$ such that $q_1 q_2 q_3 + I \subseteq q_4 + I \in A/I$, so $q_1 q_2 q_3 \in A$. Then A 2-absorbing gives either $q_1 q_2 \in A$ or $q_1 q_3 \in A$ or $q_2 q_3 \in A$. Assume that $q_1 q_2 \in A$. If $(q_1 + I) \odot (q_2 + I) = q_5 + I$, then $q_5 \in Q$ is the unique element with

$q_1q_2 + I \subseteq q_5 + I$. In this case $q_5 + f = q_1q_2 + e \in A$ for some $e, f \in I$. Since A is a k-ideal, we have $q_5 \in A \cap Q$. Consequently, $(q_1 + I) \odot (q_2 + I) \in A/I$. In a similar way we can show that if $q_1q_3 \in A$ or $q_2q_3 \in A$, then $(q_1 + I) \odot (q_3 + I) \in A/I$ or $(q_2 + I) \odot (q_3 + I) \in A/I$. It follows that A/I is 2-absorbing in R/I.

Conversely, suppose that A/I is 2-absorbing. Let $a, b, c \in R$ be such that $abc \in A$. Then there are elements $q_1, q_2, q_3 \in Q$ such that $a \in q_1 + I$, $b \in q_2 + I$ and $c \in q_3 + I$, so $a = q_1 + c$, $b = q_2 + d$ and $c = q_3 + e$ for some $c, d, e \in I$. Since $abc = q_1q_2q_3 + q_1q_2e + q_1q_3d + q_1de + cq_2q_3 + ceq_2 + cdq_3 + cde \in A$, and A is a k-ideal of R, we must have $q_1q_2q_3 \in A$. Let q be the unique element in Q such that $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q + I$ where $q_1q_2q_3 + I \subseteq q + I$, so $q + f = q_1q_2q_3 + g$ for some $f, g \in I$. Since A is a k-ideal of R, we get $q \in Q \cap A$ and $q + I \in A/I$; hence A/I 2-absorbing gives either $(q_1 + I) \odot (q_2 + I) \in A/I$ or $(q_1 + I) \odot (q_3 + I) \in A/I$ or $(q_2 + I) \odot (q_3 + I) \in A/I$. It follows that either $q_1q_2 \in A$ (so $ab \in A$) or $q_1q_3 \in A$ (so $ac \in A$) or $q_2q_3 \in A$ (so $bc \in A$). Thus A is 2-absorbing.

(2) Note that if $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \neq 0$ in R/I, then $q_1q_2q_3 \neq 0$ in R. Now the proof is completely similar to that of part (1).

(3) Suppose that I and A/I are weakly 2-absorbing ideals. Let $0 \neq abc \in A$ for some $a, b, c \in R$. If $abc \in I$, then $ab \in I \subseteq A$ or $ac \in I \subseteq A$ or $bc \in I \subseteq A$ since I is assumed to be weakly 2-absorbing. So assume that $abc \notin I$. There are elements $q_1, q_2, q_3 \in Q$ such that $a \in q_1 + I$, $b \in q_2 + I$ and $c \in q_3 + I$, so $a = q_1 + c$, $b = q_2 + d$ and $c = q_3 + e$ for some $c, d, e \in I$. Since $abc = q_1q_2q_3 + q_1q_2e + q_1q_3d + q_1de + cq_2q_3 + ceq_2 + cdq_3 + cde \in A$, and A is a k-ideal of R, we must have $q_1q_2q_3 \in A$. Let q be the unique element in Q such that $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q + I$ where $q_1q_2q_3 + I \subseteq q + I$, so $q + f = q_1q_2q_3 + g$ for some $f, g \in I$. Since A is a k-ideal of R, we get $q \in Q \cap A$ and $q + I \in A/I$. Let $q' \in Q$ be the unique element such that $q' + I$ is the zero element in R/I. If $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = 0_{R/I} = q' + I$, then $q_1q_2q_3 + s = q' + t \in I$ for some $s, t \in I$. Since I is a Q-ideal of R it is a k-ideal of R by [14, Lemma 2]. Therefore $q_1q_2q_3 \in I$ and hence $abc \in I$ which is a contradiction. Hence $0 \neq (q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in A/I$ and A/I 2-absorbing imply that $(q_1 + I) \odot (q_2 + I) \in A/I$ or $(q_1 + I) \odot (q_3 + I) \in A/I$ or $(q_2 + I) \odot (q_3 + I) \in A/I$. It follows that either $q_1q_2 \in A$ (so $ab \in A$) or $q_1q_3 \in A$ (so $ac \in A$) or $q_2q_3 \in A$ (so $bc \in A$). Thus A is weakly 2-absorbing.   □

Let R be a commutative semiring. It is mentioned in Lemma 2.2 that every 2-absorbing ideal of R is weakly 2-absorbing, but a weakly 2-absorbing ideal of R need not be 2-absorbing. For example let $R = \mathbb{Z}_6$. Then 0 is a weakly 2-absorbing ideal of $R_{n \times n}$, the semiring of all $n \times n$ matrices with arrays in R, while it is not 2-absorbing. The following theorem provides a condition under which the concepts of 2-absorbing and weakly 2-absorbing ideals of R are identical.

Theorem 2.6. *Let R be a commutative semiring. If I is a weakly 2-absorbing k-ideal of R, then either $I^3 = 0$ or I is 2-absorbing.*

Proof. Suppose that $I^3 \neq 0$. We show that I is 2-absorbing. Let $a, b, c \in R$ be such that $abc \in I$. If $abc \neq 0$, then I weakly 2-absorbing gives $ab \in I$ or $ac \in I$ or $bc \in I$.

So assume that $abc = 0$. Assume first that $abI \neq 0$, say $abr_0 \neq 0$ for some $r_0 \in I$. Then $0 \neq abr_0 = ab(c+r_0) \in I$. Since I is weakly 2-absorbing, $ab \in I$ or $a(c+r_0) \in I$ or $b(c + r_0) \in I$. Hence $ab \in I$ or $ac \in I$ or $bc \in I$. So we can assume that $abI = 0$. Likewise we can assume that $acI = 0$ and $bcI = 0$. Since $I^3 \neq 0$, there exist $a_0, b_0, c_0 \in I$ with $a_0 b_0 c_0 \neq 0$. If $ab_0 c_0 \neq 0$, then $0 \neq ab_0 c_0 = a(b + b_0)(c + c_0) \in I$ implies that $a(b + b_0) \in I$ or $a(c + c_0) \in I$ or $(b + b_0)(c + c_0) \in I$. Hence $ab \in I$ or $ac \in I$ or $bc \in I$. So we can assume that $ab_0 c_0 = 0$. Likewise we can assume that $a_0 b_0 c = 0$ and $a_0 b c_0 = 0$. Then from $0 \neq a_0 b_0 c_0 = (a + a_0)(b + b_0)(c + c_0) \in I$ we get $(a + a_0)(b + b_0) \in I$ or $(a + a_0)(c + c_0) \in I$ or $(b + b_0)(c + c_0) \in I$. Therefore $ab \in I$ or $ac \in I$ or $bc \in I$, and so I is 2-absorbing. □

Let I be an ideal of a commutative semiring R. Then the radical of I, denoted by $rad(I)$ or $\sqrt{I}$, is the set of all $x \in R$ for which $x^n \in I$ for some positive integer n. This is an ideal of R containing I, and it is the intersection of all prime ideals of R that contain I (See [1]). If $I = 0$, then the radical of I is called the nilradical of R.

Corollary 2.7. *Let I be a weakly 2-absorbing k-ideal of R. If I is not 2-absorbing, then $I \subseteq \sqrt{0}$ and $\sqrt{I} = \sqrt{0}$.*

Let R ba a commutative ring with non-zero identity. A non-zero element $a \in R$ is said to be semi-unit in R if there exist $r, s \in R$ such that $1 + ra = sa$. R is said to be a local semiring if and only if R has a unique maximal k-ideal, say M. In this case we say that (R, M) is a local semiring. It has been shown in [10] that the semiring R is a local semiring if and only if the set of non-semi-unit elements of R forms a k-ideal. It has been also proved that if R is a local semiring then the unique maximal k-ideal of R is precisely the set of non-semi-units of R.

Theorem 2.8. *Let (R, M) be a local semiring with $M^3 = 0$. Then every proper k-ideal of R is weakly 2-absorbing.*

Proof. Assume that $M^3 = 0$, and let I be a proper k-ideal of R. Suppose that $0 \neq abc \in I$. Then we have $a \in M$ or $b \in M$ or $\in M$. But a, b and c can not lie in M at the same time for otherwise $abc \in M^3 = 0$ a contradiction. So a or b or c must be semi-unit. Assume that a is a semi-unit. Then there exist $r, s \in R$ such that $1 + ra = sa$. So $bc + rabc = (1 + ra)bc = sabc \in I$ and $rabc \in I$ imply that $bc \in I$. Likewise one can shows that if b or c is semi-unit, then $ac \in I$ or $ab \in I$, respectively. Thus I is weakly 2-absorbing. □

Let R be a semiring. The concepts of $o - (1-, 2-)$ prime ideals in R were introduced and studied in [18], where an ideal P of R is called $o-$ prime (resp., $2-$ prime) if whenever A, B are ideals (resp., k-ideals) of R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. Also, an ideal P of R is said to be $1-$ prime if A is a k-ideal of R and B is an ideal of R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. In the remainder of this paper, we give a similar definition for 2-absorbing ideals of R and provide some basic results.
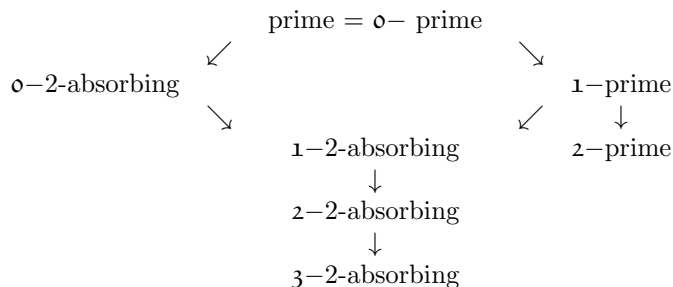
Definition 2.9. Let R be a commutative semiring and let A be an ideal of R.

(i) A is called a $0-2$-absorbing (resp., $3-2$-absorbing) ideal of R if whenever I, J and K are ideals (resp., k-ideals) of R such that $IJK \subseteq A$, then $IJ \subseteq A$ or $IK \subseteq A$ or $JK \subseteq A$.

(ii) A is called a $1-2$-absorbing ideal of R if whenever I, J and K are ideals of R such that at least one of them is a k-ideal with $IJK \subseteq A$, then $IJ \subseteq A$ or $IK \subseteq A$ or $JK \subseteq A$.

(iii) A is called a $2-2$-absorbing ideal of R if whenever I, J and K are ideals of R such that at least two of them are k-ideals with $IJK \subseteq A$, then $IJ \subseteq A$ or $IK \subseteq A$ or $JK \subseteq A$.

The following diagram shows all implications among these classes of ideals of a commutative semiring (including those relations by transitivity):

$$\text{prime} = 0- \text{ prime}$$

$$0-2\text{-absorbing} \qquad \qquad 1-\text{prime}$$

$$1-2\text{-absorbing} \qquad 2-\text{prime}$$

$$2-2\text{-absorbing}$$

$$3-2\text{-absorbing}$$

Theorem 2.10. *Let A be a k-ideal of a commutative semiring R. Then the following statements are equivalent:*

(1) A is a 2-absorbing ideal of R.

(2) A is a $0-2$-absorbing ideal of R.

(3) A is a $3-2$-absorbing ideal of R.

Proof. The proof of $(1) \Leftrightarrow (2)$ is completely to that of [5, Theorem 2.13]. If A is $0-2$-absorbing, then clearly it is $3-2$-absorbing. So it remains only to prove $(3) \Rightarrow (2)$. Assume that A is $3-2$-absorbing. Suppose that $IJK \subseteq A$ for some ideals I, J and K of R. In this case $I \subseteq (A :_R JK)$. Since A is a k-ideal, it follows that $(A :_R JK)$ is a k-ideal of R. Therefore $I_{sub} \subseteq (A :_R JK)$. In this case $I_{sub}JK \subseteq A$. Now from $JI_{sub}K \subseteq A$ we have $J \subseteq (A :_R I_{sub}K)$. In a similar way we have $J_{sub} \subseteq (A :_R I_{sub}K)$. Therefore $I_{sub}J_{sub}K \subseteq A$. Continuing this way we have $I_{sub}J_{sub}K_{sub} \subseteq A$. But by assumption, A is a $3-2$-absorbing ideal. Consequently we have $I_{sub}J_{sub} \subseteq A$ or $I_{sub}K_{sub} \subseteq A$ or $J_{sub}K_{sub} \subseteq A$. Therefore, $IJ \subseteq A$ or $IK \subseteq A$ or $JK \subseteq A$. $\square$

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