KYUNGPOOK Math. J. 52(2012), 61-79
http://dx.doi.org/10.5666/KMJ.2012.52.1.61

Soft Hemirings Related to Fuzzy Set Theory

Huajun Wu and Jianming Zhan*
Department of Mathematics, Hubei University for Nationalities, Enshi, Hubei Province, 445000*, P. R. China*
e-mail : 307478661@qq.com *and* zhanjianming@hotmail.com

Abstract. In this paper, we investigate soft hemirings by fuzzy theory. Some characterizations of hemirings are introduced by means of soft sets. In particular, the h-hemiregular hemirings and h-intra-hemiregular hemirings are also characterized.

1. Introduction

To solve complicated problems in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which have been pointed out in [21]. Maji et al. [19] and Molodtsov [21] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [21] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. At present, works on soft set theory are progressing rapidly. Maji et al. [20] discussed the application of soft set to a decision making problem. Chen et al. [5] presented a new definition of soft set parametrization reduction. Research and application about soft sets can be also found in [1, 2, 3, 6, 7, 9, 12, 14, 15, 21, 23, 24, 25, 33]. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [28, 29], there are many papers (see [4, 16]) devoted to fuzzify the classical mathematics into fuzzy mathematics.

---

* Corresponding Author.
Received December 20, 2010; accepted April 25, 2011.
2010 Mathematics Subject Classification: 16Y60, 13E05.
Key words and phrases: h-idealistic soft hemiring, h-bi-idealistic soft hemiring, h-quasi-idealistic soft hemirng, h-hemiregular-hemiring, h-intra-hemiregular hemiring.

Semirings is also a useful tool dealing with problems in different areas of applied mathematics and information sciences and have the same properties as rings except that $(S, +)$ is assumed to be a semigroup. But ideals in semirings don't coincide with the usual ring ideals. Consequently some more restricted concepts of ideals such as k-ideals [10] and h-ideals [11] have been introduced in the study of the semiring theory. Some researchers have investigated some important results of semirings, the notions of fuzzy semirings, fuzzy (prime) ideals, fuzzy h-ideals and obtained many good results (see [13,18,26,30,31,32,34]). Feng et.al [8] have started to investigate the structure of soft semirings.

Hemiring is a semiring with zero and a commutative semigroup $(S, +)$. In this paper, we will study soft hemirings by fuzzy set theory and introduce h-idealistic soft hemirings, h-bi-idealistic soft hemirings and h-quasi-idealistic soft hemirings generated by soft set theory. Finally, we will give some characterizations of h-hemiregular hemirings and h-intra-hemiregular hemirings by soft set.

2. Preliminaries

A semiring is an algebra system $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations on S called addition and multiplication such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws $a(b+c) = ab+ac$, and $(a + b)c = ac + bc$ are satisfied for all $a, b, c \in S$.

A zero element of a semiring S is an element 0 such that $0 \cdot x = x \cdot 0$, and $0 + x = x + 0$ for all $x \in S$. A hemiring is a semiring with zero and a commutative semigroup $(S, +)$. Throughtout this paper, H is a hemiring. A subhemiring of H is a subset A that is a hemiring under the addition and multiplication in H. A left (right) ideal I of H is a subhemiring such that $HI \subseteq I$ $(IH \subseteq I)$. A subhemiring I is called an ideal if it is both a left ideal and a right ideal of H.

A subset B of H is called a bi-ideal of H. If B is closed under addition and multiplication such that $BSB \subseteq B$. A subset Q of H is called a quasi-ideal of H if Q is closed under addition and $HQ \cap QH \subseteq Q$.

A left ideal (right ideal, ideal and bi-ideal) A of H is called a left h-ideal (right h-ideal, h-ideal and h-bi-ideal) of H if for any $x, y \in H$, and $a, b \in A$, $x + a + z = b + z \to x \in A$.

A quasi-ideal Q of H is called an h-quasi-ideal of H if $\overline{HQ} \cap \overline{QH} \subseteq Q$ and for any $x, z \in H$ and $a, b \in Q$ from $x + a + z = b + z$, it follows that $x \in Q$.

A fuzzy set μ of a set A is a function μ: $A \to [0, 1]$.

If λ and μ are two fuzzy subsets of A, the intrinsic product $\lambda * \mu$ is a fuzzy subset of A defined as

$$(\lambda * \mu)(x) = \sup_{x=\sum_{finite} a_i b_i} (\min\{\lambda(a_i), \mu(b_i)\}).$$

A fuzzy subset μ of a set X of the form

$$\mu(y) = \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. A fuzzy point x_t is said to belong to (resp., be quasicoincident with) a fuzzy set μ, written as $x_t \in \mu$ (resp., $x_t q \mu$) if $\mu(x) \geq t$ (resp., $\mu(x) + t > 1$). If $x_t \in \mu$ or $x_t q \mu$, then we write $x_t \in \vee q \mu$. If $\mu(x) < t$ (resp., $\mu(x) + t \leq 1$), then we call $x_t \overline{\in} \mu$ (resp., $x_t \overline{q} \mu$). We denote that the symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold.

A fuzzy set μ of H is called a fuzzy subhemiring of H, if for any $x, y \in H$, the following requirements are met: (1) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$, and (2) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

A fuzzy set μ of H is called a fuzzy left (right) h-ideal of H if all $a, b, x, y, z \in H$, the following requirements are met: (1) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$, (3) $\mu(xy) \geq \mu(y)(\mu(xy) \geq \mu(x))$ and (4) $x + a + z = b + z \rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}$.

A fuzzy set μ is called a fuzzy h-bi-ideal of H if it satisfies (1), (4) and (5) $\mu(xyz) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y, z \in H$.

A fuzzy set μ is called a fuzzy h-quasi-ideal of H if it satisfies (1), (4) and (6) $((\mu * \chi_H) \cap (\chi_H * \mu))(x) \leq \mu(x)$ for all $x \in H$, where χ_H is the characteristic function of H.

Definition 2.1([22]). A pair (F, A) is called a soft set (over U) if and only if F is a mapping of A into the set of all subsets of the set U.

Definition 2.2([22]). Let (F, A) and (G, B) be two soft sets over U. Then (F, A) is said to be a soft subset of (G, B) if:

(1) $A \subseteq B$ and

(2) $\forall x \in A$, $F(x) \subseteq G(x)$.

This relationship is denoted by $(F, A) \widetilde{\subset} (G, B)$.

Definition 2.3([19]). The product of two soft sets (F, A) and (G, B) over U is the soft set $(H, A \times B)$, where $H(x, y) = F(x)G(y)$, for all $(x, y) \in A \times B$. This is denoted by $(F, A) * (G, B) = (H, A \times B)$.

Definition 2.4([19]). If (F, A) and (G, B) are soft sets over U, then (F, A) AND (G, B) is denoted $(F, A) \wedge (G, B)$, where $(F, A) \wedge (G, B)$ is defined as $(H, A \times B)$, where $H(x, y) = F(x) \cap G(y)$, $x, y \in A \times B$.

Lemma 2.5([32]). *A fuzzy set μ defined on U has the property ψ if and only if all non-empty level subsets $U(\mu; t)$ have the property ψ.*

Theorem 2.6([17]). *A fuzzy set μ of H is a fuzzy h-ideal (fuzzy left h-ideal, fuzzy right h-ideal, fuzzy h-bi-ideal, fuzzy h-quasi-ideal) of H if and only if each non-empty level subset $U(\mu; t)$ is an h-ideal (left h-ideal, right h-ideal, h-bi-ideal, h-quasi-ideal) of H, respectively.*

Given a fuzzy set μ of H and $A \subseteq [0,1]$, consider two set-valued functions

$$\mathscr{F} : A \to P(H), t \mapsto \{x \in H | x_t \in \mu\}$$

and

$$\mathscr{F}_q : A \to P(H), t \mapsto \{x \in H | x_t q \mu\}.$$

Then $(\mathscr{F}, A)$ and $(\mathscr{F}_q, A)$ are called an $\in$-*soft set* and *q-soft set* over H, respectively.

3. Some kinds of h-idealistic soft hemirings

This section will be divided into three subsections. In section 3.1, we study h-idealistic soft hemirings. In section 3.2, we consider h-quasi-idealistic soft hemirings. Finally, we investigate h-bi-idealistic soft hemirings in section 3.3.

3.1. h-idealistic soft hemirings

In this subsection, we define the notion of h-idealistic soft hemirings and investigate some properties of them.

Definition 3.1.1. Let (F, A) be a soft set over H. Then (F, A) is said to be a left (right) h-idealistic soft hemiring over H if and only if $F(x)$ is a left (right) h-ideal of H for all $x \in A$, (F, A) is said to be an h-idealistic soft hemiring over H if and only if (F, A) is both a right h-idealistic soft hemiring over H and a left h-idealistic soft hemiring over H.

Example 3.1.2. Let H and A be the set of all non-negative integers, H is a hemiring with respect to the usual addition and multiplication of integers. $\forall x \in A$, let $F(x) = \{y \mid y\rho x \Leftrightarrow y = 2xa,\ a \in A\}$. If $y_1,\ y_2 \in F(x)$, then there exist a_1, $a_2 \in A$ such that $y_1 = 2xa_1$, $y_2 = 2xa_2$, $y_1 + y_2 = 2xa_1 + 2xa_2 = 2x(a_1 + a_2)$, then $y_1 + y_2 \in F(x)$. Let $z_1 \in H$, $y_1 z_1 = 2xa_1 z_1 = 2x(a_1 z_1) \in F(x)$. Similarly, $z_1 y_1 \in F(x)$, so $F(x)$ is an ideal of H. It is easy to check that $x + a + z = b + z$ implies $x \in F(x)$ for any $x, z \in H$ and $a, b \in F(x)$, then $F(x)$ is an h-ideal of H and (F, A) is an h-idealistic soft hemiring.

Proposition 3.1.3. *Let μ be a fuzzy set of H and $A = [0,1]$. Then $(\mathscr{F}, A)$ is a left (right) h-idealistic soft hemiring over H if and only if μ is a fuzzy left (right) h-ideal of H.*

Proof. The proof is from Theorem 2.5. □

Proposition 3.1.4. *Let μ be a fuzzy set of H and $A = [0,1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is a left (right) h-idealistic soft hemiring over H if and only if μ is a fuzzy left (right) h-ideal of H.*

Proof. We only prove the case of left ideals, the other is similar.

Assume that $(\mathscr{F}_{\mathrm{q}}, A)$ is a left h-idealistic soft hemiring. Then $\mathscr{F}_{\mathrm{q}}(t)$ is a left h-ideal of H. If there exist $x, y \in H$ such that $\mu(x+y) < \min\{\mu(x), \mu(y)\}$, then we can select $t \in A$ such that $\mu(x+y)+t \leq 1 < \min\{\mu(x),\mu(y)\}+t$, hence $\mu(x)+t > 1$, $\mu(y)+t > 1$ and $\mu(x+y)+t \leq 1$, *i.e.*, $x, y \in \mathscr{F}_{\mathrm{q}}(t)$, $x+y \overline{\in} \mathscr{F}_{\mathrm{q}}(t)$, but $x+y \in \mathscr{F}_{\mathrm{q}}(t)$, this is a contradiction. So $\mu(x+y) \geq \min\{\mu(x),\mu(y)\}$ for all $x, y \in H$. In the same way, we can also prove that $\mu(xy) \geq \mu(y)$ for all $x, y \in H$. On the other hand, if there exist $x, z, a, b \in H$ such that $x+a+z = b+z$, we have $\mu(x) < \min\{\mu(a),\mu(b)\}$, there exists t such that $\mu(x)+t \leq 1 < \min\{\mu(a),\mu(b)\}+t$, hence $\mu(a)+t > 1$, $\mu(b)+t > 1$, $\mu(x)+t \leq 1$, i.e., $a \in \mathscr{F}_{\mathrm{q}}(t)$, $b \in \mathscr{F}_{\mathrm{q}}(t)$ and $x \overline{\in} \mathscr{F}_{\mathrm{q}}(t)$, but $x \in \mathscr{F}_{\mathrm{q}}(t)$, it is a contradiction, so $\mu(x) \geq \min\{\mu(a),\mu(b)\}$. Therefore μ is a fuzzy left h-ideal of H.

Conversely, μ is a fuzzy left h-ideal of H. Let $t \in A$ and $x, y \in \mathscr{F}_{\mathrm{q}}(t)$, then $\mu(x+y)+t \geq \min\{\mu(x),\mu(y)\}+t > 1$, i.e., $x+y \in \mathscr{F}_{\mathrm{q}}(t)$. Let $z \in H$, $x \in \mathscr{F}_{\mathrm{q}}(t)$, $\mu(zx)+t \geq \mu(x)+t > 1$, then $zx \in \mathscr{F}_{\mathrm{q}}(t)$, so $\mathscr{F}_{\mathrm{q}}(t)$ is a left ideal of H. As $x+a+z = b+z \to \mu(x) \geq \min\{\mu(a),\mu(b)\}$, it is clear that $a, b \in \mathscr{F}_{\mathrm{q}}(t)$ implies $x \in \mathscr{F}_{\mathrm{q}}(t)$, so $\mathscr{F}_{\mathrm{q}}(t)$ is a left h-ideal of H and $(\mathscr{F}_{\mathrm{q}}, A)$ is a left h-idealistic soft hemiring. □

Definition 3.1.5([17]). A fuzzy set μ of H is called an $(\in, \in \vee q)$-fuzzy left (right) h-ideal of H if for all $t, r \in (0,1]$ and $a, b, x, y, z \in H$, the following conditions are satisfied:

(A1) $x_t \in \mu$ and $y_r \in \mu$ imply $(x+y)_{\min\{t,r\}} \in \vee q\mu$;

(A2) $y_t \in \mu (x_t \in \mu)$ implies $(xy)_t \in \vee q\mu$;

(A3) $a_t \in \mu$ and $b_r \in \mu$ imply $x_{\min\{t,r\}} \in \vee q\mu$ with $x+a+z = b+z$.

Lemma 3.1.6([17]). *A fuzzy set μ of H is an $(\in, \in \vee q)$-fuzzy left (right) h-ideal of H if and only if for all $a, b, x, y, z \in H$, the following conditions are satisfied:*

(B1) $\mu(x+y) \geq \min\{\mu(x),\mu(y),0.5\}$;

(B2) $\mu(xy) \geq \min\{\mu(y),0.5\}$ $(\mu(xy) \geq \min\{\mu(x),0.5\})$;

(B3) $\mu(x) \geq \min\{\mu(a),\mu(b),0.5\}$ *with* $x+a+z = b+z$.

Theorem 3.1.7. *Let μ be a fuzzy set of H and $A = (0, 0.5]$. Then $(\mathscr{F}, A)$ is a left (right) h-idealistic soft hemiring over H if and only if μ is an $(\in, \in \vee q)$-fuzzy left (right) h-ideal of H.*

Proof. We only prove the case of left ideals.

Assume that $(\mathscr{F}, A)$ is a left h-idealistic soft hemiring over H, then $\mathscr{F}(t)$ is a left h-ideal for all $t \in A$. If there exist $x, y \in H$ such that $\mu(x+y) < \min\{\mu(x),\mu(y),0.5\}$, then we can select $t \in A$ such that $\mu(x+y) < t \leq \min\{\mu(x),\mu(y),0.5\}$. Thus $0 < t \leq 0.5$, $\mu(x) \geq t$, $\mu(y) \geq t, \mu(x+y) < t$. i.e., $x, y \in \mathscr{F}(t)$, $x+y \overline{\in} \mathscr{F}(t)$, but $x+y \in \mathscr{F}(t)$, this is a contradiction, so $\mu(x+y) \geq \min\{\mu(x),\mu(y),0.5\}$ for all $x, y \in H$. In the same way, we can prove that $\mu(xy) \geq \min\{\mu(y),0.5\}$ and $\mu(x) \geq \min\{\mu(a),\mu(b),0.5\}$ with $x+a+z = b+z$ for all $a, b, x, y, z \in H$, so μ is an $(\in, \in \vee q)$-fuzzy left h-ideal of H.

Conversely, suppose that μ is an $(\in, \in \vee q)$-fuzzy left h-ideal of H. Let $t \in A$, by Lemma 3.1.6, $\forall a, b, x, y, z \in H$, we can get $\mu(x+y) \geq \min\{\mu(x),\mu(y),0.5\}$, $\mu(xy) \geq \min\{\mu(y),0.5\}$ and $\mu(x) \geq \min\{\mu(a),\mu(b),0.5\}$ with $x+a+z = b+z$. If $x, y \in$

$\mathscr{F}(t)$, then $\mu(x) \geq t$, $\mu(y) \geq t$. It follows that $\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\} \geq \min\{t, 0.5\} = t$, and so $x + y \in \mathscr{F}(t)$. Let $z \in H$, $x \in \mathscr{F}(t)$, then $\mu(zx) \geq \min\{\mu(x), 0.5\} \geq \min\{t, 0.5\} = t$, so $zx \in \mathscr{F}(t)$ and $\mathscr{F}(t)$ is a left ideal of H. It is easy to check that $\mathscr{F}(t)$ is a left h-ideal of H, so $(\mathscr{F}, A)$ is a left h-idealistic soft hemiring over H. □

Definition 3.1.8([17]). A fuzzy set μ of H is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy left (right) h-ideal* of H if for all $t, r \in (0, 1]$ and $a, b, x, y, z \in H$, the following conditions are satisfied:

(C1) $(x+y)_{\min\{t,r\}} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q} \mu$ or $y_r \overline{\in} \vee \overline{q} \mu$;

(C2) $(xy)_t \overline{\in} \mu$ implies $y_t \overline{\in} \vee \overline{q} \mu$ $(x_t \overline{\in} \vee \overline{q} \mu)$;

(C3) $x_{\min\{t,r\}} \overline{\in} \mu$ implies $a_t \overline{\in} \vee \overline{q} \mu$ or $b_r \overline{\in} \vee \overline{q} \mu$ with $x + a + z = b + z$.

Lemma 3.1.9([17]). *A fuzzy set μ of H is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) h-ideal of H if and only if the following conditions hold for all $a, b, x, y, z \in H$:*

(D1) $\max\{\mu(x+y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$;

(D2) $\max\{\mu(xy), 0.5\} \geq \mu(y)$ $(\max\{\mu(xy), 0.5\} \geq \mu(x))$;

(D3) $x + a + z = b + z$ *implies* $\max\{\mu(x), 0.5\} \geq \min\{\mu(a), \mu(b)\}$.

Theorem 3.1.10. *Let μ be a fuzzy set of H and $A = (0.5, 1]$. Then $(\mathscr{F}, A)$ is a left (right) h-idealistic soft hemiring over H if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) h-ideal of H.*

Proof. We only prove the case of left ideals.

Assume that $(\mathscr{F}, A)$ is a left h-idealistic soft hemiring over H, then $\mathscr{F}(t)$ is a left h-ideal for all $t \in A$. If there exist $x, y \in H$ such that $\max\{\mu(x+y), 0.5\} < t \leq \min\{\mu(x), \mu(y)\}$, then $t \in A$, $x, y \in \mathscr{F}(t)$, but $x + y \overline{\in} \mathscr{F}(t)$, this is a contradiction, so $\max\{\mu(x+y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in H$. In the same way, we can prove that $\max\{\mu(xy), 0.5\} \geq \mu(y)$ and $\max\{\mu(x), 0.5\} \geq \min\{\mu(a), \mu(b)\}$ with $x + a + z = b + z$ for all $a, b, x, z \in H$. Therefore μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left h-ideal of H, so μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left h-ideal of H.

Conversely, suppose that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left h-ideal of H. Then for all $a, b, x, y, z \in H$, $\max\{\mu(x+y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$, $\max\{\mu(xy), 0.5\} \geq \mu(y)$ and $x + a + z = b + z$ implies $\max\{\mu(x), 0.5\} \geq \min\{\mu(a), \mu(b)\}$. Let $t \in A$, $x, y \in \mathscr{F}(t)$, then $\mu(x) \geq t > 0.5$, $\mu(y) \geq t > 0.5$, hence, $\max\{\mu(x+y), 0.5\} \geq \min\{\mu(x), \mu(y)\} \geq t$, i.e., $\mu(x+y) \geq t$, so $x + y \in \mathscr{F}(t)$. If $z \in H$, $x \in \mathscr{F}(t)$, $\max\{\mu(zx), 0.5\} \geq \mu(x) \geq t$, i.e., $\mu(zx) \geq t$, then $zx \in \mathscr{F}(t)$, $\mathscr{F}(t)$ is a left ideal of H. It is easy to prove $\mathscr{F}(t)$ is a left h-ideal of H, so $(\mathscr{F}, A)$ is a left h-idealistic soft hemiring over H. □

Theorem 3.1.11. *Let μ be a fuzzy set of H and $A = (0, 0.5]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is a left (right) h-idealistic soft hemiring over H if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) h-ideal of H.*

Proof. We only prove the case of left ideals.

Assume that $(\mathscr{F}_q, A)$ is a left h-idealistic soft hemiring over H, then $\mathscr{F}_q(t)$ is a left h-ideal for all $t \in A$. If there exist $x, y \in H$ such that $\max\{\mu(x+y), 0.5\} <$

$\min\{\mu(x), \mu(y)\}$, then we can select $t \in A$, such that $\max\{\mu(x+y), 0.5\} + t \leq 1 < \min\{\mu(x), \mu(y)\} + t$, then $x, y \in \mathscr{F}_q(t)$, $x + y \overline{\in} \mathscr{F}_q(t)$, but $x + y \in \mathscr{F}_q(t)$, it is a contradiction, so $\max\{\mu(x+y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$. In the same way, we can prove that $\max\{\mu(xy), 0.5\} \geq \mu(y)$ and $x + a + z = b + z$ implies $\max\{\mu(x), 0.5\} \geq \min\{\mu(a), \mu(b)\}$ for all $a, b, x, y, z \in H$, so μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left h-ideal of H.

Conversely, suppose that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left h-ideal of H. Then $\forall a, b, x, y, z \in H$, $\max\{\mu(x+y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$, $\max\{\mu(xy), 0.5\} \geq \mu(y)$ and $x + a + z = b + z$ implies $\max\{\mu(x), 0.5\} \geq \min\{\mu(a), \mu(b)\}$. Let $t \in A$, $x, y \in \mathscr{F}_q(t)$, then $\mu(x) + t > 1$, $\mu(y) + t > 1$, hence, $\max\{\mu(x+y), 0.5\} + t \geq \min\{\mu(x), \mu(y)\} + t > 1$, i.e., $\mu(x+y) + t > 1$, so $x + y \in \mathscr{F}_q(t)$. If $z \in H$, $x \in \mathscr{F}_q(t)$, $\max\{\mu(zx), 0.5\} + t \geq \mu(x) + t > 1$, i.e., $\mu(zx) + t > 1$, $zx \in \mathscr{F}_q(t)$, $\mathscr{F}_q(t)$ is a left ideal of H. It is easy to prove $\mathscr{F}_q(t)$ is a left h-ideal of H, so $(\mathscr{F}_q, A)$ is a left h-idealistic soft hemiring over H. □

Theorem 3.1.12. *Let μ be a fuzzy set of H and $A = (0.5, 1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is a left (right) h-idealistic soft hemiring over H if and only if μ is an $(\in, \in \vee q)$-fuzzy left (right) h-ideal of H.*

Proof. The proof is similar to the proof of Theorem 3.1.11. □

Definition 3.1.13([17]). Let $\alpha, \beta \in (0, 1]$ and $\alpha < \beta$. Then a fuzzy set μ of H is called an (α, β)-fuzzy h-ideal of H if for all $a, b, x, y, z \in H$, the following conditions are satisfied:

(E1) $\max\{\mu(x+y), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$;

(E2) $\max\{\mu(xy), \alpha\} \geq \min\{\mu(y), \beta\}(\max\{\mu(xy), \alpha\} \geq \min\{\mu(x), \beta\})$;

(E3) $x + a + z = b + z$ implies $\max\{\mu(x), \alpha\} \geq \min\{\mu(a), \mu(b), \beta\}$.

Theorem 3.1.14. *Let μ be a fuzzy set of H and $A = (\alpha, \beta]$. Then $(\mathscr{F}, A)$ is a left (right) h-idealistic soft hemiring over H if and only if μ is an (α, β)-fuzzy left (right) h-ideal of H.*

Proof. We only prove the case of left ideals.

Assume that $(\mathscr{F}, A)$ is a left h-idealistic soft hemiring over H, then $\mathscr{F}(t)$ is an h-ideal of H for all $t \in A$. If there exist $x, y \in H$ such that $\max\{\mu(x+y), \alpha\} < \min\{\mu(x), \mu(y), \beta\}$, then we can select $t \in (\alpha, \beta]$, such that $\max\{\mu(x+y), \alpha\} < t \leq \min\{\mu(x), \mu(y), \beta\}$, thus $\mu(x) \geq t$, $\mu(y) \geq t$, and $\mu(x+y) < t$, i.e., $x, y \in \mathscr{F}(t)$, $x + y \overline{\in} \mathscr{F}(t)$, but $x + y \in \mathscr{F}(t)$, this is a contradiction, so $\max\{\mu(x+y), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$. In the same way, we can prove that $\max\{\mu(xy), \alpha\} \geq \min\{\mu(y), \beta\}$ and $\forall a, b, x, z \in H$, $x + a + z = b + z$ implies $\max\{\mu(x), \alpha\} \geq \min\{\mu(a), \mu(b), \beta\}$. Hence, μ is an (α, β)-fuzzy h-ideal of H.

Conversely, suppose that μ is an (α, β)-fuzzy h-ideal of H. Let $t \in A$, $x, y \in \mathscr{F}(t)$, then $\mu(x) \geq t$, $\mu(y) \geq t$, hence $\max\{\mu(x+y), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\} \geq t, \max\{\mu(xy), \alpha\} \geq \min\{\mu(y), \beta\}$. Thus, $\mu(x+y) \geq t$, i.e., $x+y \in \mathscr{F}(t)$. Let $z \in H$, $x \in \mathscr{F}(t)$, $\max\{\mu(zx), \alpha\} \geq \min\{\mu(x), \beta\} \geq t$, i.e., $\mu(zx) \geq t$, then $zx \in \mathscr{F}(t)$, so $\mathscr{F}(t)$ is a left ideal of H. It is easy to check that $\mathscr{F}(t)$ is a left h-ideal of H, so $(\mathscr{F}, A)$ is a left h-idealistic soft hemiring over H.

3.2. h-quasi-idealistic soft hemirings

In this subsection, we define the notion of h-quasi-idealistic soft hemirings and investigate some properties of them.

Definition 3.2.1. Let (F, A) be a soft set over H. Then (F, A) is said to be an h-quasi-idealistic soft hemiring over H if and only if $F(x)$ is an h-quasi-ideal of H for all $x \in A$.

Example 3.2.2. Let $\mathbb{N}$ and $\mathbb{P}$ be the set of all non-negative integers and non-negative real numbers. Let H be the set of all 2×2 matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $(a_{ij} \in \mathbb{N})$. Then H is a hemiring. Let Q be the set of all 2×2 matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ $(a \in \mathbb{N})$, $A = \mathbb{P}$. $\forall x \in A$, let $F(x) = \{y | y\rho x \Leftrightarrow y = xq, q \in Q\}$. Let $y_1, y_2 \in F(x)$, then there exist q_1, $q_2 \in Q$ such that $y_1 = xq_1$, $y_2 = xq_2$, $y_1 + y_2 = xq_1 + xq_2 = x(q_1 + q_2) \in F(x)$, so $F(x)$ is closed under addition. It is easy to check that $HF(x) \cap F(x)H \subseteq F(x)$, $\overline{HF(x)} \cap \overline{F(x)H} \subseteq F(x)$ and $x + a + z = b + z$ implies $x \in F(x)$ for all $x, z \in H$ and $a, b \in F(x)$. So (F, A) is an h-quasi-idealistic soft hemiring.

Since any left (right) h-ideal is an h-quasi-ideal of H and any h-quasi-ideal of H is an h-bi-ideal of H. We can easily deduce the following proposition.

Proposition 3.2.3. *Any left (right) h-idealistic soft hemiring over H is an h-quasi-idealistic soft hemiring over H.*

Proposition 3.2.4. *Any h-quasi-idealistic soft hemiring over H is an h-bi-idealistic soft hemiring over H.*

Proposition 3.2.5. *Let μ be a fuzzy set of H and $A = [0, 1]$. Then $(\mathscr{F}, A)$ is an h-quasi-idealistic soft hemiring over H if and only if μ is a fuzzy h-quasi-ideal of H.*

Proof. The proof is from Theorem 2.5 and Theorem 2.6. □

Proposition 3.2.6([17]). *Let μ be a fuzzy set of H and $A = [0, 1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is an h-quasi-idealistic soft hemiring over H if and only if μ is a fuzzy h-quasi-ideal of H.*

Proof. The proof is similar to the proof of Theorem 3.1.4. □

Definition 3.2.7([17]). A fuzzy set μ of H is said to be an $(\in, \in \vee q)$-*fuzzy h-quasi-ideal* of H if for all $t, r \in (0, 1]$ and $a, b, x, y, z \in H$, the following conditions hold:

(J1) $x_t \in \mu$ and $y_r \in \mu$ imply $(x + y)_{\min\{t,r\}} \in \vee q\mu$;
(J2) $a_t \in \mu$ and $b_r \in \mu$ imply $x_{\min\{t,r\}} \in \vee q\mu$ with $x + a + z = b + z$;
(J3) $\forall t \in (0, 1]$ and $x \in H$, $x_t \in (\mu \odot_h \chi_H) \cap (\chi_H \odot_h \mu)$ implies $x_t \in \vee q\mu$.

Lemma 3.2.8([17]). *A fuzzy set μ of H is an $(\in, \in\vee q)$-fuzzy h-quasi-ideal of H if and only if it satisfies:*

(K1) $\forall x, y \in H,\ \mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\}$;

(K2) $\forall a, b, x, z \in H, x+a+z = b+z$ *implies* $\mu(x) \geq \min\{\mu(a), \mu(b), 0.5\}$;

(K3) $\mu(x) \geq \min\{((\mu \odot_h \chi_H) \cap (\chi_H \odot_h \mu))(x), 0.5\}$ for all $x \in H$.

Definition 3.2.9([17]). A fuzzy set μ of H is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-quasi-ideal of H if it satisfies:

(L1) $(x+y)_{\min\{t,r\}} \overline{\in} \mu$, implies $x_t \overline{\in} \vee \overline{q} \mu$ or $y_r \overline{\in} \vee \overline{q} \mu$;

(L2) $x_{\min\{t,r\}} \overline{\in} \mu$ implies $a_t \overline{\in} \vee \overline{q} \mu$ or $b_r \overline{\in} \vee \overline{q} \mu$ for all $a, b, x, z \in H$ with $x+a+z = b+z$;

(L3) $\forall t \in (0,1]$ and $x \in H$, $x_t \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q} (\mu \odot_h \chi_H) \cap (\chi_H \odot_h \mu)$.

Lemma 3.2.10([17]). *A fuzzy set μ of H is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-quasi-ideal of H if and only if for all $x, y, a, b, z \in H$, the following conditions hold:*

(M1) $\max\{\mu(x+y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$;

(M2) $x+a+z = b+z$ *implies* $\max\{\mu(x), 0.5\} \geq \min\{\mu(a), \mu(b)\}$;

(M3) $\max\{\mu(x), 0.5\} \geq ((\mu \odot_h \chi_H) \cap (\chi_H \odot_h \mu))(x)$.

Definition 3.2.11([17]) Let $\alpha, \beta \in (0,1]$ and $\alpha < \beta$. Then a fuzzy set μ of H is called an (α, β)-*fuzzy h-quasi-ideal* of H if for all $x, y, z, a, b \in H$, the following conditions hold :

(N1) $\max\{\mu(x+y), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$;

(N2) $x+a+z = b+z$ implies $\max\{\mu(x), \alpha\} \geq \min\{\mu(a), \mu(b), \beta\}$;

(N2) $\max\{\mu(x), \alpha\} \geq \min\{((\mu \odot_h \chi_H) \cap (\chi_H \odot_h \mu))(x), \beta\}$.

Theorem 3.2.12. *Let μ be a fuzzy set of H and $A = (0, 0.5]$. Then $(\mathscr{F}, A)$ is an h-quasi-idealistic soft hemiring over H if and only if μ is an $(\in, \in\vee q)$-fuzzy h-quasi-ideal of H.*

Proof. The proof is similar to the proof of Theorem 3.1.7. □

Theorem 3.2.13. *Let μ be a fuzzy set of H and $A = (0.5, 1]$. Then $(\mathscr{F}, A)$ is an h-quasi-idealistic soft hemiring over H if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-quasi-ideal of H.*

Proof. The proof is similar to the proof of Theorem 3.1.10. □

Theorem 3.2.14. *Let μ be a fuzzy set of H and $A = (0, 0.5]$. Then $(\mathscr{F}_q, A)$ is an h-quasi-idealistic soft hemiring over H if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-quasi-ideal of H.*

Proof. The proof is similar to the proof of Theorem 3.1.11. □

Theorem 3.2.15. *Let μ be a fuzzy set of H and $A = (0.5, 1]$. Then $(\mathscr{F}_q, A)$ is an h-quasi-idealistic soft hemiring over H if and only if μ is an $(\in, \in\vee q)$-fuzzy h-quasi-deal of H.*

Proof. The proof is similar to the proof of Theorem 3.1.12. □

Theorem 3.2.16. *Let μ be a fuzzy set of H and $A = (\alpha, \beta]$. Then $(\mathscr{F}, A)$ is an h-quasi-idealistic soft hemiring over H if and only if μ is an (α, β)-fuzzy h-quasi-ideal of H.*

Proof. The proof is similar to the proof of Theorem 3.1.14. □

Theorem 3.2.17. *Let (F, A) and (G, B) be a right h-idealistic soft hemiring and a left h-idealistic soft hemiring over H, respectively. Then $(F, A) \wedge (G, B)$ is an h-quasi-idealistic soft hemiring over H.*

Proof. Let (F, A) and (G, B) be a right h-idealistic soft hemiring and a left h-idealistic soft hemiring over H, respectively. $\forall x \in A$, $y \in B$, $F(x)$ and $G(y)$ are a right h-ideal and a left h-ideal over H, respectively, then $F(x) \cap G(y)$ is an h-quasi-ideal over H (see[19]), so $(F, A) \wedge (G, B)$ is an h-quasi-idealistic soft hemiring over H. □

3.3. h-bi-idealistic soft hemirings

In this subsection, we define the notion of h-bi-idealistic soft hemirings and investigate some properties of them.

Definition 3.3.1. Let (F, A) be a soft set over H. Then (F, A) is said to be an h-bi-idealistic soft hemiring over H if and only if $F(x)$ is an h-bi-ideal of H for all $x \in A$.

Example 3.3.2. We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of all positive integers and positive real numbers. Let H be the set of all 2×2 matrices of the form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ $(a, b \in \mathbb{P}, c \in \mathbb{N})$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, H is a hemiring with respect to the usual addition and multiplication of matrices. Let

$R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b \in \mathbb{P}, a < b, c \in \mathbb{N} \right\} \cup \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $A = \mathbb{P}, \forall x \in A, F(x) = \left\{ y \mid x\rho y \Leftrightarrow y = r + \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, r \in R \right\} \cup \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

It is easy to check that $F(x)$ is an h-bi-ideal, so (F, A) is an h-bi-idealistic soft hemiring.

Proposition 3.3.3. *Let μ be a fuzzy set of H and $A = [0, 1]$. Then $(\mathscr{F}, A)$ is an h-bi-idealistic soft hemiring over H if and only if μ is a fuzzy h-bi-ideal of H.*

Proof. The proof is straight from Theorem 2.6. □

Proposition 3.3.4. *Let μ be a fuzzy set of H and $A = [0, 1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is an h-bi-idealistic soft hemiring over H if and only if μ is a fuzzy h-bi-ideal of H.*

Proof. The proof is similar to the proof of Proposition 3.1.4. □

Definition 3.3.5 [17]. A fuzzy set μ of H is said to be an $(\in, \in \vee q)$-*fuzzy h-bi-ideal* of H if for all $t, r \in (0,1]$ and $a, b, x, y, z \in H$, the following conditions hold:

(E1) $x_t \in \mu$ and $y_r \in \mu$ imply $(x+y)_{\min\{t,r\}} \in \vee q\mu$;
(E2) $x_t \in \mu$ and $y_r \in \mu$ imply $(xy)_{\min\{t,r\}} \in \vee q\mu$;
(E3) $a_t \in \mu$ and $b_r \in \mu$ imply $x_{\min\{t,r\}} \in \vee q\mu$ with $x+a+z = b+z$;
(E4) $x_t \in \mu$ and $z_r \in \mu$ imply $(xyz)_{\min\{t,r\}} \in \vee q\mu$.

Lemma 3.3.6([17]). *A fuzzy set μ of H is an $(\in, \in \vee q)$-fuzzy h-bi-ideal of H if and only if for all $a, b, x, y, z \in H$, the following conditions hold:*

(F1) $\mu(x+y) \geq \min\{\mu(x), \mu(y), 0.5\}$;
(F2) $\mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$;
(F3) $x+a+z = b+z$ *implies* $\mu(x) \geq \min\{\mu(a), \mu(b), 0.5\}$;
(F4) $\mu(xyz) \geq \min\{\mu(x), \mu(z), 0.5\}$.

Definition 3.3.7([17]). A fuzzy set μ of H is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-bi-ideal of H if for all $t, r \in (0,1]$ and $a, b, x, y, z \in H$, the following conditions hold:

(G1) $(x+y)_{\min\{t,r\}} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q}\mu$ or $y_r \overline{\in} \vee \overline{q}\mu$;
(G2) $(xy)_{\min\{t,r\}} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q}\mu$ or $y_r \overline{\in} \vee \overline{q}\mu$;
(G3) $x_{\min\{t,r\}} \overline{\in} \mu$ implies $a_t \overline{\in} \vee \overline{q}\mu$ or $b_r \overline{\in} \vee \overline{q}\mu$ with $x+a+z = b+z$;
(G4) $(xyz)_{\min\{t,r\}} \overline{\in} \mu$ implies $x_t \overline{\in} \vee \overline{q}\mu$ or $z_r \overline{\in} \vee \overline{q}\mu$.

Lemma 3.3.8([17]). *A fuzzy set μ of H is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-bi-ideal of H if and only if for all $a, b, x, y, z \in H$, the following conditions hold:*

(H1) $\max\{\mu(x+y), 0.5\} \geq \min\{\mu(x), \mu(y)\}$;
(H2) $\max\{\mu(xy), 0.5\} \geq \min\{\mu(x), \mu(y)\}$;
(H3) $x+a+z = b+z$ *implies* $\max\{\mu(x), 0.5\} \geq \min\{\mu(a), \mu(b)\}$;
(H4) $\max\{\mu(xyz), 0.5\} \geq \min\{\mu(x), \mu(z)\}$.

Theorem 3.3.9. *Let μ be a fuzzy set of H and $A = (0, 0.5]$. Then $(\mathscr{F}, A)$ is an h-bi-idealistic soft hemiring over H if and only if μ is an $(\in, \in \vee q)$-fuzzy h-bi-ideal of H.*

Proof. The proof is similar to the proof of Theorem 3.1.7 □

Theorem 3.3.10. *Let μ be a fuzzy set of H and $A = (0.5, 1]$. Then $(\mathscr{F}, A)$ is an h-bi-idealistic soft hemiring over H if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-bi-ideal of H.*

Proof. The proof is similar to the proof of Theorem 3.1.10. □

Theorem 3.3.11. *Let μ be a fuzzy set of H and $A = (0, 0.5]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is an h-bi-idealistic soft hemiring over H if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy h-bi-ideal of H.*

Proof. The proof is similar to the proof of Theorem 3.1.11. □

Theorem 3.3.12. *Let μ be a fuzzy set of H and $A = (0.5, 1]$. Then $(\mathscr{F}_{\mathrm{q}}, A)$ is an*

h-bi-idealistic soft hemiring over H if and only if μ is an $(\in, \in \vee q)$-fuzzy h-bi-deal of H.

Proof. The proof is similar to the proof of Theorem 3.1.13. □

Definition 3.3.13([17]). Let $\alpha, \beta \in (0,1]$ with $\alpha < \beta$. Then a fuzzy set μ of H is called a fuzzy h-bi-ideal with thresholds $(\alpha, \beta]$ of H if for all $a, b, x, y, z \in H$, the following conditions are satisfied:

(I1) $\max\{\mu(x+y), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$;
(I2) $\max\{\mu(xy), \alpha\} \geq \min\{\mu(x), \mu(y), \beta\}$;
(I3) $x + a + z = b + z$ implies $\max\{\mu(x), \alpha\} \geq \min\{\mu(a), \mu(b), \beta\}$;
(I4) $\max\{\mu(xyz), \alpha\} \geq \min\{\mu(x), \mu(z), \beta\}$.

Theorem 3.3.14. *Let μ be a fuzzy set of H and $A = (\alpha, \beta]$. Then $(\mathscr{F}, A)$ is an h-bi-idealistic soft hemiring over H if and only if μ is an (α, β)-fuzzy h-bi-ideal of H.*

Proof. The proof is similar to the proof of Theorem 3.1.14. □

Theorem 3.3.15. *For any fuzzy subset μ of H. If μ is a fuzzy left h-ideal (right h-ideal, h-quasi-ideal, h-bi-ideal) of H, then soft set (F, A) over H is a left h-idealistic (right h-idealistic, h-quasi-idealistic, h-bi-idealistic) soft hemiring over H.*

Proof. We only prove the case of left h-ideal, the others are similar. Since any μ is a fuzzy left h-ideal, the characteristic function $\chi_{F(x)}$ of $F(x)$ for all $x \in A$ is a fuzzy left h-ideal, so $F(x)$ is a left h-ideal, therefore (F, A) is a left h-idealistic soft hemiring over H. □

Theorem 3.3.16. *Let (F, A) and (G, B) be a right h-idealistic soft hemiring and a left h-idealistic soft hemiring over H, respectively. Then $(F, A) * (G, B)$ is an h-bi-idealistic soft hemiring over H.*

Proof. Let (F, A) and (G, B) be a right h-idealistic soft hemiring and a left h-idealistic soft hemiring over H, respectively. $\forall x \in A$, $y \in B$, $F(x)$ and $G(y)$ are a right h-ideal and a left h-ideal over H, respectively, then $F(x)G(y)$ is an h-bi-ideal (see[19]) over H, so $(F, A) * (G, B)$ is an h-bi-idealistic soft hemiring over H. □

4. h-hemiregular hemirings

The notion of h-hemiregularity of a hemiring was first introduced by Zhan et al. in [32]. In this section, we investigate the characterizations of h-hemiregular hemirings by soft set theory.

Definition 4.1([32]). A hemiring H is said to be h-hemiregular if for each $x \in H$, there exist $a, a', z \in H$ such that $x + xax + z = xa'x + z$.

The h-closure $\overline{A}$ of A in a hemiring H is defined as $\overline{A}=\{x \in H | x + a_1 + z = a_2 + z$ for some $a_1, a_2 \in A, z \in H\}$.

Lemma 4.2([32]). *For a hemiring H, we have*

(1) $A \subseteq \overline{A}$, $\forall \{0\} \subseteq A \subseteq H$;
(2) If $A \subseteq B \subseteq H$, then $\overline{A} \subseteq \overline{B}$;
(3) $\overline{\overline{A}} = \overline{A}$, $\forall A \subseteq H$;
(4) $\overline{AB} = \overline{\overline{A}\,\overline{B}}$, $\forall A, B \subseteq H$.

It is clear that if A is a left (right) ideal of H, then $\overline{A}$ is the smallest left (right) h-ideal of H containing A.

Let (F, A) and (G, A) be soft sets over H. (G, A) is called a soft h-closure of (F, A) if $G(x)$ is an h-closure of $F(x)$ for all $x \in A$, denoted by $\overline{(F, A)}$. Similarly, if (F, A) is a left (right) idealistic soft hemiring over H, then $\overline{(F, A)}$ is the smallest left (right) h-idealistic soft hemiring over H containing (F, A), $\overline{\overline{(F, A)}} = (F, A)$ for every soft set over H.

Lemma 4.3([32]). *Let A and B be any right h-ideal and left h-ideal of H, then $\overline{AB} \subseteq A \cap B$.*

Lemma 4.4([32]). *H is h-hemiregular if and only if for every right h-ideal A and for every left h-ideal B satisfying $\overline{AB} = A \cap B$.*

Theorem 4.5. *H is h-hemiregular if and only if $\overline{(F, A) * (G, B)} = (F, A) \wedge (G, B)$ for every right h-idealistic soft hemiring (F, A) over H and every left h-idealistic soft hemiring (G, B) over H.*

Proof. Assume that H is h-hemiregular. Let (F, A) and (G, B) be a right h-idealistic soft hemiring and a left h-idealistic soft hemiring, respectively. $\forall x \in A$, $y \in B$, $F(x)$ is a left h-ideal and $G(y)$ is a right h-ideal over H, respectively. Then $\overline{F(x)G(y)} = F(x) \cap G(y)$, so $\overline{(F, A) * (G, B)} = (F, A) \wedge (G, B)$.

Conversely, let M be a right h-ideal and N a left h-ideal of H, respectively. Let $M = F(x)$ for all $x \in A$, $N = G(y)$ for all $y \in B$, then (F, A) is a right h-idealistic soft hemiring and (G, B) is a left h-idealistic soft hemiring, then $\overline{F(x)G(y)} = F(x) \cap G(y)$ because of $\overline{(F, A) * (G, B)} = (F, A) \wedge (G, B)$, so $\overline{MN} = M \cap N$. Then H is h-hemiregular. □

Theorem 4.6. *H is h-hemiregular if and only if $\overline{(F, A) * (G, B)} = \overline{(F, A)} \wedge \overline{(G, B)}$ for every right idealistic soft hemiring (F, A) over H and every left idealistic soft hemiring (G, B) over H.* *Proof.* Since (F, A) is a right idealistic soft hemiring, $F(x)$ is a right ideal of H for all $x \in A$. In similarly, $G(y)$ is a left ideal of H for all $y \in B$, so $\overline{F(x)}$ and $\overline{G(y)}$ are a right h-ideal over H and a left h-ideal over H, respectively. That H is h-hemiregular implies $\overline{\overline{F(x)}\ \overline{G(y)}} = \overline{F(x)} \cap \overline{G(y)} = \overline{F(x)G(y)}$, so $\overline{(F, A) * (G, B)} = \overline{(F, A)} \wedge \overline{(G, B)}$.

Conversely, let M, N be a right h-ideal and a left h-ideal over H, respectively. Since (F, A) and (G, B) are a right idealistic soft hemiring and a left idealistic soft hemiring, respectively. $F(x)$ and $G(y)$ are a right ideal and a left ideal over H, respectively. so $\overline{F(x)}$ and $\overline{G(y)}$ are a right h-ideal and a left h-ideal over H, respectively. Let $M = \overline{F(x)}$ for all $x \in A$, $N = \overline{G(y)}$ for all $y \in B$. Since $\overline{(F, A) * (G, B)} = \overline{(F, A)} \wedge \overline{(G, B)}$, $\overline{F(x)G(y)} = \overline{F(x)} \cap \overline{G(y)}$, in addition,

$\overline{F(x)G(y)} = \overline{\overline{F(x)}\ \overline{G(y)}}$, we have $\overline{\overline{F(x)}\ \overline{G(y)}} = \overline{F(x)} \cap \overline{G(y)}$, and $\overline{MN} = M \cap N$, so H is h-hemiregular. □

Lemma 4.7([27]). *Let S be a hemiring. Then the following statements are equivalent:*

(1) *S is h-hemiregular;*

(2) *$B \cap A = \overline{BAB}$ for every h-bi-ideal B and every h-ideal A of S;*

(3) *$Q \cap A = \overline{QSQ}$ for every h-quasi-ideal Q and every h-ideal A of S.*

Theorem 4.8. *If $(F, A) \wedge (G, B) \wedge (F, A) = \overline{(F, A) * (G, B) * (F, A)}$ for every h-bi-idealistic soft hemiring (F, A) and every h-idealistic soft hemiring (G, B) over H. Then H is h-hemiregular.*

Proof. Let Q and T be any h-bi-ideal and h-ideal of H, $F(x) = Q$ for all $x \in A$, $G(y) = T$ for all $y \in B$, then (F, A) is an h-bi-idealistic soft hemiring and (G, B) is an h-idealistic soft hemiring, then $(F, A) \wedge (G, B) \wedge (F, A) = \overline{(F, A) * (G, B) * (F, A)}$, i.e., $Q \cap T \cap Q = Q \cap T = \overline{QTQ}$. So H is h-hemiregular. □

Corollary 4.9. *If $(F, A) \wedge (G, B) \wedge (F, A) = \overline{(F, A) * (G, B) * (F, A)}$ for every h-quasi-idealistic soft hemiring (F, A) and every h-idealistic soft hemiring (G, B) over H. Then H is h-hemiregular.*

Proof. The proof is similar to the proof of Theorem 4.8. □

Definition 4.10. Let (F, A) be a soft set over H. Then (F, A) is said to be an absolute soft hemiring over H if and only if $F(x) = H$ for all $x \in A$.

Corollary 4.11. *If $(F, A) \wedge (G, B) \wedge (F, A) = \overline{(F, A) * (G, B) * (F, A)}$ for every h-quasi-idealistic soft hemiring (F, A) and every absolute soft hemiring (G, B) over H. Then H is h-hemiregular.*

Proof. The proof is similar to the proof of Theorem 4.8. □

Corollary 4.12. *If $(F, A) \wedge (G, B) \wedge (F, A) = \overline{(F, A) * (G, B) * (F, A)}$ for every h-bi-idealistic soft hemiring (F, A) and every absolute soft hemiring (G, B) over H. Then H is h-hemiregular.*

Proof. The proof is similar to the proof of Theorem 4.8. □

Theorem 4.13. *H is h-hemiregular if and only if $(F, A) \wedge (G, B) \wedge (N, C) = \overline{(F, A) * (G, B) * (N, C)}$ for every right h-idealistic soft hemiring (F, A), every h-idealistic soft hemiring (G, B) and every left h-idealistic soft hemiring (N, C) over H.*

Proof. Since H is h-hemiregular, for all $a \in F(x) \cap G(y) \cap N(z)$, $a + ax_1a + r = ax_2a + r, x_1, x_2 \in H$, and $ax_1a + ax_1ay_1ax_1a + r' = ax_1ay_2ax_1a + r'$, $y_1, y_2, r' \in H$. It is clear that $ax_1 \in F(x)$, $ay_1a \in G(y)$, $x_1a \in N(y)$, so $ax_1ay_1ax_1a \in F(x)G(y)N(z)$, similarly, $ax_1ay_2ax_1a \in F(x)G(y)N(z)$. Thus $ax_1a \in \overline{F(x)G(y)N(z)}$, and $a \in \overline{\overline{F(x)G(y)N(z)}} = \overline{F(x)G(y)N(z)}$, so $F(x) \cap G(y) \cap N(z) \subseteq \overline{F(x)G(y)N(z)}$. On the other hand, for all $x \in A$, $y \in B$, $z \in C$, $F(x)$ is a right h-ideal, $G(y)$ is an h-ideal and $N(z)$ is a left h-ideal over H. $\overline{F(x)G(y)N(z)} \subseteq \overline{F(x)HH} \subseteq \overline{F(x)} = F(x)$. Similarly, $\overline{F(x)G(y)N(z)} \subseteq G(y)$, $\overline{F(x)G(y)N(z)} \subseteq N(z)$, then $\overline{F(x)G(y)N(z)} \subseteq F(x) \cap G(x) \cap N(z)$. Hence, $(F,A) \wedge (G,B) \wedge (N,C) = \overline{(F,A) * (G,B) * (N,C)}$.

Conversely, let (F,A), (G,B) and (N,C) be any right h-idealistic soft hemiring, any h-idealistic soft hemiring and any left h-idealistic soft hemiring over H, then $(F,A) \wedge (G,B) \wedge (F,A) = \overline{(F,A) * (G,B) * (F,A)}$. Let $x \in A$, $z \in C$, we have $F(x) \cap N(z) = F(x) \cap H \cap N(z) = \overline{F(x)HN(z)} \subseteq \overline{F(x)N(z)}$, by Lemma 4.2, $\overline{F(x)N(z)} \subseteq F(x) \cap N(z)$. So $\overline{F(x)N(z)} = F(x) \cap N(z)$ and $\overline{(F,A) * (N,C)} = (F,A) \wedge (N,C)$. It follows that H is h-hemiregular. □

Corollary 4.14. *Let H be a hemiring, then the following conditions are equivalent:*

(1) *H is h-hemiregular;*

(2) $(F,A) \wedge (G,B) \widetilde{\subset} \overline{(F,A) * (G,B)}$ *for every h-bi-idealistic soft hemiring (F,A) and every left h-idealistic soft hemiring (G,B) of H;*

(3) $(F,A) \wedge (G,B) \widetilde{\subset} \overline{(F,A) * (G,B)}$ *for every h-quasi-idealistic soft hemiring (F,A) and every left h-idealistic soft hemiring (G,B) of H;*

(4) $(F,A) \wedge (G,B) \widetilde{\subset} \overline{(F,A) * (G,B)}$ *for every right h-idealistic soft hemiring (F,A) and every h-bi-idealistic soft hemiring (G,B) of H;*

(5) $(F,A) \wedge (G,B) \widetilde{\subset} \overline{(F,A) * (G,B)}$ *for every right h-idealistic soft hemiring (F,A) and every h-quasi-idealistic soft hemiring (G,B) of H;*

(6) $(F,A) \wedge (G,B) \wedge (N,C) \widetilde{\subset} \overline{(F,A) * (G,B) * (N,C)}$ *for every right h-idealistic soft hemiring (F,A), every h-bi-idealistic soft hemiring (G,B) and every left h-idealistic soft hemiring (N,C) of H;*

(7) $(F,A) \wedge (G,B) \wedge (N,C) \widetilde{\subset} \overline{(F,A) * (G,B) * (N,C)}$ *for every right h-idealistic soft hemiring (F,A), every h-quasi-idealistic soft hemiring (G,B) and every left h-idealistic soft hemiring (N,C) of H.*

Proof. (1)⇒(2). Assume that (1) holds. Let $x \in A$, $y \in B$, then $F(x)$ is an h-bi-ideal, $G(y)$ is a left h-ideal. Since H is h-hemiregular, $\forall a \in F(x) \cap G(y)$, there exist x_1, x_2, $z \in H$ such that $a + ax_1a + z = ax_2a + z$. It is clear that $ax_ia \in F(x)G(y) (i = 1,2)$, so $a \in \overline{F(x)G(y)}$. Then $(F,A) \wedge (G,B) \widetilde{\subset} \overline{(F,A) * (G,B)}$.

(2)⇒(1). Assume that (2) holds. Let (F,A) and (G,B) be any right h-idealistic soft hemiring and any left h-idealistic soft hemiring. Then $\forall x \in A$, $y \in B$, $F(x)$ and $G(y)$ are a right and left h-ideal of H, respectively, by Lemma 4.2, $\overline{F(x)G(y)} \subseteq F(x) \cap G(y)$, hence $\overline{(F,A) * (G,B)} \widetilde{\subset} (F,A) \wedge (G,B)$. It is easy to check that (F,A) is an h-bi-idealistic soft hemiring, so $(F,A) \wedge (G,B) \widetilde{\subset} \overline{(F,A) * (G,B)}$, we have $(F,A) \wedge (G,B) = \overline{(F,A) * (G,B)}$. It follows that H is h-hemiregular by Theorem 4.4. Similarly, we can show that (1)⇔(3), (1)⇔ (4), (1)⇔(5).

(1)⇔(6) and (7)⇒(1). The proof is similar to the proof of Theorem 4.13.

It is clear that (6) implies (7). □

5. h-intra-hemiregular hemirings

In this section, we investigate the characterizations of h-intra-hemiregular hemirings by soft set theory.

Definition 5.1([27]). H is h-intra-hemiregular if and only if for each $x \in H$, there exist a_i, a_i', b_j, b_j', $z \in H$ such that $x + \sum_{i=1}^{m} a_i x^2 a_i' + z = \sum_{j=1}^{n} b_j x^2 b_j' + z$.
Equivalent definitions: (1) $x \in \overline{Sx^2S}$, $\forall x \in S$, (2) $A \subseteq \overline{SA^2S}$, $\forall A \subseteq S$.

Lemma 5.2([27]). *Let H be a hemiring. Then the following conditions are equivalent:*

(1) *H is h-intra-hemiregular*;
(2) *$L \cap R \subseteq \overline{LR}$ for every left h-ideal L and every right h-ideal R of H.*

Theorem 5.3. *H is h-intra-hemiregular if and only if for any left h-idealistic soft hemiring (F, A) and any right h-idealistic soft hemiring (G, B) over H, we have $(F, A) \wedge (G, B) \widetilde{\subset} \overline{(F, A) * (G, B)}$.*

Proof. Assume that H is an h-intra-hemiregular hemiring. Let (F, A) and (G, B) be any left h-idealistic soft hemiring and right h-idealistic soft hemiring over H, respectively. Then $\forall x \in A$, $y \in B$, $F(x)$ is a left h-ideal of H and $G(y)$ is a right h-ideal of H, and $F(x) \cap G(y) \subseteq \overline{F(x)G(y)}$, thus $(F, A) \wedge (G, B) \widetilde{\subset} \overline{(F, A)(G, B)}$.

Conversely, suppose that L and R are any left h-ideal of H and any right h-ideal of H, respectively. Let $F(x) = L$ for all $x \in A$, and $G(y) = R$ for all $y \in B$, by the assumption, we have $F(x) \cap G(y) \subseteq \overline{F(x)G(y)}$, then $L \cap R \subseteq \overline{LR}$. Therefore H is h-intra-hemiregular. □

Lemma 5.4([27]). *Let H be a hemiring. Then the following conditions are equivalent:*

(1) *H is both h-hemiregular and h-intra-hemiregular*;
(2) *$B = \overline{B^2}$ for every h-bi-ideal B of H*;
(3) *$Q = \overline{Q^2}$ for every h-quasi-ideal Q of H.*

Theorem 5.5. *For a hemiring H. If $(F, A) = \overline{(F, A) * (F, A)}$ for every h-bi-idealistic soft hemiring (F, A) over H. Then H is h-hemiregular and h-intra-hemiregular.*

Proof. Assume that B is any h-bi-ideal of H, let $F(x) = B$ for all $x \in A$, then (F, A) is an h-bi-idealistic soft hemiring over H. Since $(F, A) = \overline{(F, A) * (F, A)}$, $B = \overline{B^2}$, it follows that H is h-hemiregular and h-intra-hemiregular. □

Corollary 5.6. *For a hemiring H. If $(F, A) = \overline{(F, A) * (F, A)}$ for every h-quasi-idealistic soft hemiring (F, A) over H.* Then H is h-hemiregular and h-intra-hemiregular.

Proof. The proof is obvious. □

Theorem 5.7. *Let H be a hemiring. Then the following conditions are equivalent*:

(1) *H is both h-hemiregular and h-intra-hemiregular*;
(2) *$(F, A) \wedge (G, B) \widetilde{\subset} \overline{(F, A) * (G, B)}$ for any h-bi-idealistic soft hemirings (F, A)*

and (G, B) *over* H;

(3) $(F, A) \wedge (G, B) \widetilde{\subset} \overline{(F, A) * (G, B)}$ *for every h-bi-idealistic soft hemiring* (F, A) *and every h-quasi-idealistic soft hemiring* (G, B) *over* H;

(4) $(F, A) \wedge (G, B) \widetilde{\subset} \overline{(F, A) * (G, B)}$ *for every h-quasi-idealistic soft hemiring* (F, A) *and every h-bi-idealistic soft hemiring* (G, B) *over* H;

(5) $(F, A) \wedge (G, B) \widetilde{\subset} \overline{(F, A) * (G, B)}$ *for any h-quasi-idealistic soft hemirings* (F, A) *and* (G, B) *over* H.

Proof. (1)⇒(2). Let (F, A) and (G, B) be any h-bi-idealistic soft hemirings over H, then $\forall x \in A$ and $y \in B$, $F(x)$ and $G(y)$ both are h-bi-ideals over H. We show that $F(x) \cap G(y)$ is an h-bi-ideal. It is clear that $F(x) \cap G(y)$ is closed under addition and multiplication. $(F(x) \cap G(y))H(F(x) \cap G(y)) \subseteq F(x)HF(x) \subseteq F(x)$, $(F(x) \cap G(y))H(F(x) \cap G(y)) \subseteq G(y)HG(y) \subseteq G(y)$, so $(F(x) \cap G(y))H(F(x) \cap G(y)) \subseteq F(x) \cap G(y)$, $\forall a, b \in F(x) \cap G(y)$ and $s, z \in H$, it is easy to check that $s + a + z = b + z$ implies $s \in F(x) \cap G(y)$, so $F(x) \cap G(y)$ is an h-bi-ideal. If (1) holds, by Lemma 5.4, $F(x) \cap G(y) \subseteq \overline{(F(x) \cap G(y))(F(x) \cap G(y))} \subseteq \overline{F(x)G(y)}$ and so $(F, A) \wedge (G, B) \widetilde{\subset} \overline{(F, A) * (G, B)}$.

The proof of (2)⇒(3), (2)⇒(4), (3)⇒(5) and (4)⇒(5) is obvious.

(5)⇒(1). Assume Q is any h-quasi-ideal over H. Let $F(x) = Q$ for all $x \in A$, then (F, A) is an h-quasi-idealistic soft hemiring over H and $(F, A) \wedge (F, A) \widetilde{\subset} \overline{(F, A) * (F, A)}$, so $Q = \overline{Q^2}$, by Lemma 5.4, (1) holds.

Finally, we can get that (3)⇔(4) by the above proof.

Acknowledgements The authors are extremely grateful to the referees for giving them many valuable comments and helpful suggestions which helps to improve the presentation of this paper.

This research is partially supported by a grant of the Innovation Term of Education Committee of Hubei Province, China (T201109), the Natural Science Foundation of Hubei Province, China (2009CDB340) and the Foundation of Hubei University for Nationalities(MY2010T002).

References

[1] H. Aktas, N. Cağman, *Soft sets and soft groups*, Inform. Sci., **177**(2007), 2726-2735.

[2] M.I. Ali, M. Shabir, K.P. Shum, *On soft ideals over semigroups*, Southeast Asian Bull. Math., **34**(2010), 595-611.

[3] A.O. Atagun, A. Sezgin, *Soft substructures of rings, fields and modules*, Comput. Math. Appl., **61**(3)(2011), 592-601.

[4] S.K. Bhakat, P. Das, $(\in, \in \vee q)$-*fuzzy subgroups*, Fuzzy Sets Syst., **80**(1996), 359-368.

[5] D. Chen, E. C. C. Tsang, D. S. Yeung, X. Wang, *The parametrization reduction of soft sets and its applications*, Comput. Math. Appl., **49**(2005), 757-763.

[6] F. Feng, C. Li, B. Davvaz, M.I. Ali, *Soft sets combined with fuzzy sets and rough sets: a tentative approach*, Soft Comput., **14**(2010), 899-911.

[7] F. Feng, X.Y. Liu, v. Leoreanu-Fotea, Y.B. Jun, *Soft sets and soft rough sets*, Inform. Sci., **181**(6)(2011), 1125-1137.

[8] F. Feng, Y.B. Jun, X. Zhao, *Soft semirings*, Comput. Math. Appl., **56**(2008), 2621-2628.

[9] K. Gong, Z. Xiao, X. Zhang, *The bijective soft set with its operations*, Comput. Math. Appl., **60**(2010), 2270-2278.

[10] M. Henriksen, *Ideals in semirings with commutative addtion*, Amer. Math. Soc. Notices, **6**(1958), 321.

[11] K. Iizuka, *On the Jacobson radial of a semiring*, Tohoku Math. J., **11**(1959), 409-421.

[12] Y.B. Jun, *Soft BCK/BCI-algebras*, Comput. Math. Appl., **56**(2008), 1408-1413.

[13] Y.B. Jun, M.A. Ozturk, S.Z. Song, *On fuzzy h-ideals in hemirings*, Inform. Sci., **162**(2004), 211-226.

[14] Y.B. Jun, C.H. Park, *Applications of soft sets in ideal theory of BCK/BCI-algebras*, Inform. Sci., **178**(2008), 2466-2475.

[15] X. Liu, *Normal soft groups*, Journal of Hubei University for Nationalities(Natural Science Edition), **27**(2)(2009), 168-170.

[16] X. Liu, D. Xiang, J. Zhan, *Soft rings related to fuzzy set theory*, submitted.

[17] X. Ma, J. Zhan, *Generalized fuzzy h-bi-ideals and fuzzy h-quasi-ideals of hemirings*, Inform. Sci., **179**(2009), 1249-1268.

[18] X. Ma, J. Zhan, *On fuzzy h-ideals of hemirings*, J. Syst. Sci. Complexity, **20**(2007), 470-478.

[19] P.K. Maji, R. Biswas, A.R. Roy, *Soft set theory*, Comput. Math. Appl., **45**(2003), 555-562.

[20] P.K. Maji, A.R. Roy, R. Biswas, *An application of soft sets in a decision making problem*, Comput. Math. Appl., **44**(2002), 1077-1083.

[21] P. Majumdar, S.K. Samanta, *On soft mappings*, Comput. Math. Appl., **60**(9)(2010), 2666-2672.

[22] D. Molodtsov, *Soft set theory-first results*, Comput. Math. Appl., **37**(1999), 19-31.

[23] A. Sezgin, A.O. Atagun, *On operations of soft sets*, Comput. Math. Appl., **61**(5)(2011), 1457-1467.

[24] M. Shabir, M.Naz, *On soft topological spaces*, Comput. Math. Appl., **61**(2011), 1786-1799.

[25] D. Xiang, X. Liu, *Summary report of soft set theory*, Journal of Hubei University for Nationalities (Natural Science Edition), **27**(4)(2009), 361-365.

[26] Y. Yin, X. Huang, D. Xu, H. Li, *The characterizations of h-semisimple hemirings*, Int. J. Fuzzy Syst., **11**(2009), 116-122.

[27] Y. Yin, H. Li, *The characterizations of h-hemiregular hemirings and h-intra-hemiregular hemirings*, Inform. Sci., **178**(2008), 3451-3464.

[28] L.A. Zadeh, *Fuzzy sets*, Inform. Control, **8**(1965), 338-353.

[29] L.A. Zadeh, *Toward a generalized theory of uncertainty (GTU)-an outline*, Inform. Sci., **172**(2005), 1-40.

[30] J. Zhan, B. Davvaz, *L-fuzzy h-ideals with operators in hemirings*, Northeast. Math. J., **23**(2007), 1-14.

[31] J. Zhan, *On properties of fuzzy left h-ideals in hemirings*, Int. J. Math. Math. Sci., **19**(2005), 3127-3144.

[32] J. Zhan, W.A. Dudek, *Fuzzy h-ideals of hemirings*, Inform. Sci., **177**(2007), 876-886.

[33] J. Zhan, K.P. Shum, *Intuitionistic M-fuzzy h-ideals in M-hemirings*, J. Fuzzy Math., **14**(2006), 929-945.

[34] J. Zhan, Z. Tan, *T-fuzzy k-ideals of semirings*, Sci. Math. Japon., **58**(2003), 597-601.