KYUNGPOOK Math. J. 52(2012), 39-47
http://dx.doi.org/10.5666/KMJ.2012.52.1.39

Further Results about the Normal Family of Meromorphic Functions and Shared Sets

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Abstract. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let k, $n(\geq 2)$ be two positive integers, and let $S = \{a_1, a_2, ..., a_n\}$, where $a_1, a_2, ..., a_n$ are distinct finite complex numbers. If for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least $k+1$, f and $G(f)$ share the set S in D, where $G(f) = P(f^{(k)}) + H(f)$ is a differential polynomial of f, then $\mathcal{F}$ is normal in D.

1. Introduction

Let $\mathcal{C}$ be the whole complex domain. Let D be a domain in $\mathcal{C}$ and $\mathcal{F}$ a family of meromorphic functions defined in D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence $\{f_{n_j}\}$ which converse spherically locally uniformly in D, to a meromorphic function or ∞.(see [3]).

Let f and g be meromorphic functions on a domain D, and let a and b be two complex numbers. If $g(z) = b$ whenever $f(z) = a$, we write

$$f(z) = a \Rightarrow g(z) = b.$$

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* Corresponding Author.
Received March 3, 2010; revised December 11, 2010; accepted September 23, 2011.
2010 Mathematics Subject Classification: 30D35.
Key words and phrases: Meromorphic functions, Nevanlinna theory, Normal family, Share value.

If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write

$$f(z) = a \Leftrightarrow g(z) = b.$$

If $f(z) = a \Leftrightarrow g(z) = a$, we say that f and g share a on D.

Let $a_i(z), (i = 1, 2, \ldots, q-1), b_j(z), (j = 1, 2, \ldots, n)$ be analytic in D, $n_0, n_1, \ldots, n_k$ be non-negative integers. q be a positive integer. Set

$$P(\omega) = \omega^q + a_{q-1}(z)\omega^{q-1} + \ldots + a_1(z)\omega,$$

$$M(f, f', \ldots, f^{(k)}) = f^{n_0}(f')^{n_1} \ldots (f^{(k)})^{n_k},$$

$$\gamma_M = n_0 + n_1 + \ldots + n_k,$$

$$\Gamma_M = n_0 + 2n_1 + \ldots + (k+1)n_k.$$

$M(f, f', \ldots, f^{(k)})$ is called the differential monomial of F, γ_M the degree of $M(f, f', \ldots, f^{(k)})$ and Γ_M the weight of $M(f, f', \ldots, f^{(k)})$.

Let $M_i(f, f', \ldots, f^{(k)}), (i = 1, 2, \ldots, n)$ be differential monomials of f. Set

$$H(f, f', \ldots, f^{(k)}) = b_1(z)M_1(f, f', \ldots, f^{(k)}) + \ldots + b_n(z)M_n(f, f', \ldots, f^{(k)}),$$

$$\gamma_H = \max\{\gamma_{M_1}, \gamma_{M_2}, \ldots, \gamma_{M_n}\},$$

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$H(f, f', \ldots, f^{(k)})$ is called the differential polynomial of f, γ_H the degree of $H(f, f', \ldots, f^{(k)})$ and Γ_H the weight of $H(f, f', \ldots, f^{(k)})$.

Set

$$\frac{\Gamma}{\gamma}|_H = \max\{\frac{\Gamma_{M_1}}{\gamma_{M_1}}, \frac{\Gamma_{M_2}}{\gamma_{M_2}}, \ldots, \frac{\Gamma_{M_n}}{\gamma_{M_n}}\},$$

$$G(f) = P(f^{(k)}) + H(f, f', \ldots, f^{(k)}).$$

Schwick [7] discovered a connection between normality criteria and shared values and proved

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and a, b, c be distinct complex numbers, If for each $f \in \mathcal{F}$, f and f' share a, b, c, then $\mathcal{F}$ is normal in D.*

Pang and Zalcman [6] improved Theorem A as follows.

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let a, b be two distinct complex numbers. If, for each $f \in \mathcal{F}$, f and f' share a and b in D, then $\mathcal{F}$ is normal in D.*

In 2002, Fang and Zalcman [2] proved

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let a be nonzero finite complex number, and let k be a positive integer. If, for each $f \in \mathcal{F}$,* *all zeros of f have multiplicity at least $k+1$, and f and $f^{(k)}$ share a in D, then $\mathcal{F}$ is normal in D.*

Let S be a set of complex numbers. If $f(z) \in S$ if and only if $g(z) \in S$ in a domain D, then we say f and g share the set S in D.

It is natural to ask that that whether Theorem C is valid or not if f and $f^{(k)}$ share a value a was replaced by f and $f^{(k)}$ share a set S?

Recently, Lei, Fang and Yang [4] proved:

Theorem D. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let $n(\geq 2)$, k be two positive integers, and let $S = \{a_1, a_2, ..., a_n\}$, where $a_1, a_2, ..., a_n$ are distinct finite complex numbers. If for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least $k+1$, and f and $f^{(k)}$ share the set S in D, then $\mathcal{F}$ is normal in D.*

Theorem E. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let n, m, k be three positive integers, and let $S = \{a_1, a_2, ..., a_n\}$, $S_2 = \{b_1, b_2, ..., b_m\}$ where $a_1, a_2, ..., a_n, b_1, b_2, ...b_m$ are distinct finite complex numbers. If for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least k, and f and $f^{(k)}$ share the sets S_1 and S_2 in D, then $\mathcal{F}$ is normal in D.*

In [4], Lei, Fang and Yang give the examples to show all zeros of f have multiplicity are best possible in the above theorems.

In this paper, we extend Theorem D and Theorem E as follows.

Theorem 1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let $n(\geq 2)$, k be two positive integers, and let $S = \{a_1, a_2, ..., a_n\}$, where $a_1, a_2, ..., a_n$ are distinct finite complex numbers. If for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least $k+1$, and f and $G(f)$ share the set S in D, where $G(f) = P(f^{(k)}) + H(f)$ be a differential polynomial of f satisfying $q \geq \gamma_H$, and $\frac{\Gamma}{\gamma}|_H < k+1$, then $\mathcal{F}$ is normal in D.*

Theorem 2. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let n, m, k be three positive integers, and let $S_1 = \{a_1, a_2, ..., a_n\}$, $S_2 = \{b_1, b_2, ..., b_m\}$ where $a_1, a_2, ..., a_n, b_1, b_2, ...b_m$ are distinct finite complex numbers. If for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least k, and f and $P(f^{(k)})$ share the sets S_1 and S_2 in D, where $a_i(z)$ in $P(f)$ are constants, then $\mathcal{F}$ is normal in D.*

By Theorem 1, we immediately deduce

Corollary 1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let $n(\geq 2)$, k be two positive integers, and let $S = \{a_1, a_2, ..., a_n\}$, where $a_1, a_2, ..., a_n$ are distinct finite complex numbers. If for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least $k+1$, and f and $L(f)$ share the set S in D, where $L(f) = f^{(k)} + a_1(z)f^{(k-1)} + a_2(z)f^{(k-2)} +, ..., +a_{k-1}(z)f' + a_k(z)f$ and $a_i(z), (i = 1, 2, \ldots, k)$ are analytic in D, then $\mathcal{F}$ is normal in D.*

By Theorem 2, we immediately deduce

Corollary 2. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let n, m, k be three positive integers, and let $S_1 = \{a_1, a_2, ..., a_n\}$, $S_2 = \{b_1, b_2, ..., b_m\}$ where $a_1, a_2, ..., a_n, b_1, b_2, ...b_m$ are distinct finite complex numbers. If for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least k, and f and $(f^{(k)})^q$ share the set S_1 and S_2 in D, where q is a positive integer, then $\mathcal{F}$ is normal in D.*

2. Preliminaries and lemmas

In order to prove our results, we need the following lemmas.

Lemma 2.1[6,9]. *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc $\triangle$ with the property that for each $f \in \mathcal{F}$, all zeros of multiplicity at least k. Suppose that there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f \in \mathcal{F}$ and $f = 0$. If $\mathcal{F}$ is not normal in Δ, then for $0 \leq \alpha \leq k$, there exist*

1. *a number $r \in (0, 1)$;*
2. *a sequence of complex numbers z_n, $|z_n| < r$;*
3. *a sequence of functions $f_n \in \mathcal{F}$;*
4. *a sequence of positive numbers $\rho_n \to 0^+$;*

such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ locally uniformly (with respect to the spherical metric) to a nonconstant meromorphic function $g(\xi)$ on $\mathbb{C}$, and moreover, the zeros of $g(\xi)$ are of multiplicity at least k, $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$. In particular, g has order at most 2.

Here, as usual, $g^\sharp(\xi) = \frac{|g'(\xi)|}{1+|g(\xi)|^2}$ is the spherical derivative.

Lemma 2.2([1]). *Let $f(z)$ be a transcendental meromorphic function of finite order and k be a positive integer, let a be a non-zero finite complex number. If all zeros of $f(z)$ are of multiplicity at least $k+1$, then $f^{(k)}(z)$ assume a infintely often.*

Lemma 2.3([8]). *Let $f(z) = a_n z^n + a_{n-1} z^{n-1} +, ..., +a_0 + \frac{q(z)}{p(z)}$, where $a_0, a_1, ..., a_n$ are constants with $a_n \neq 0$, and $q(z)$ and $p(z)$ are co-prime polynomials with $\deg q(z) < \deg p(z)$; and let k be a positive integer. If $f^{(k)}(z) \neq 1$, then*

$$f(z) = \frac{z^k}{k!} +, ..., +a_0 + \frac{1}{az + b},$$

where $a (\neq 0), b, a_0, ...$ are constants.

3. Proof of Theorem 1 and Theorem 2

Proof of Theorem 1. We may assume that $S = \{a, b\}$, where a and b are two distinct constants and $D = \Delta = \{|z| < 1\}$, the unit disk. Now we consider two cases

Case 1. $ab \neq 0$. Suppose that $\mathcal{F}$ is not normal in $D = \Delta$. Withoutloss of generality, we assume that $\mathcal{F}$ is not normal at $z_0 = 0$. Then, by Lemma 2.1, there exist

1. a number $r \in (0,1)$;
2. a sequence of complex numbers z_j, $|z_j| < r$;
3. a sequence of functions $f_j \in \mathcal{F}$;
4. a sequence of positive numbers $\rho_j \to 0^+$

such that $g_j(\xi) = \frac{f_j(z_j+\rho_j\xi)}{\rho_j^k}$ converges uniformly with respect to the spherical metric to a non-constant mermorphic functions $g(\xi)$ in C. Moreover, $g(\xi)$ is of order at most 2, all of whose zeros have multiplicity at least $k+1$. Set $Q(w) = w^q + a_{q-1}(0)w^{q-1}+, ..., +a_1(0)w$.

We claim that

(i) $Q(g^{(k)}) \neq a$,
(ii) $Q(g^{(k)}) \neq b$.

Suppose now that $Q(g^{(k)}(\xi_0)) = a$. We claim that $Q(g^{(k)}) \not\equiv a$. Otherwise, from the definition of $Q(w)$, there exist a nonzero constant h such that $g^{(k)}(\xi) \equiv h$, g must be a polynomial of at most degree k, which contradicts the fact that each zero of $g(\xi)$ are of multiplicity at least $k+1$. Since $Q(g^{(k)}(\xi_0)) = a$. Obviously, $g(\xi_0) \neq \infty$. Hence there exists $\delta > 0$ such that $g(\xi)$ is analytic on $G_{2\delta} = \{\xi : |\xi - \xi_0| < 2\delta\}$. Thus $g_j^{(i)}(\xi)(i = 0, 1, 2, ..., k)$ are analytic on $G_\delta = \{\xi : |\xi - \xi_0| < \delta\}$ for large j and $g_j^{(i)}(\xi)$ converges uniformly to $g^{(i)}(\xi)$ $(i = 0, 1, 2, ..., k)$ on $\overline{G}_\delta = \{\xi : |\xi - \xi_0| \leq \delta\}$.

As

$$G(f_j)(z_j + \rho_j\xi) - a = P(f_j^{(k)}(z_j + \rho_j\xi)) + H(f_j, f_j', \ldots, f_j^{(k)})(z_j + \rho_j\xi) - a,$$

and

$$H(f_j, f_j', \ldots, f_j^{(k)})(z_j + \rho_j\xi) = \sum_{i=1}^{n} b_i(z_j + \rho_j\xi)\rho_j^{(k+1)\gamma_{M_i} - \Gamma_{M_i}} M_i(g_j, g_j', \ldots, g_j^{(k)})(\xi).$$

Considering $b_i(z)$ are analytic on D $(i = 1, 2, \ldots, n)$, we have

$$|b_i(z_j + \rho_j\xi)| \leq M\left(\frac{1+r}{2}, b_i(z)\right) < \infty, (i = 1, 2, \ldots, n)$$

for sufficiently large j.

Hence we deduce from $\frac{\Gamma}{\gamma}|_H < k+1$ that

$$\sum_{i=1}^{n} b_i(z_j + \rho_j\xi)\rho_j^{(k+1)\gamma_{M_i} - \Gamma_{M_i}} M_i(g_j, g_j', \ldots, g_j^{(k)})(\xi)$$

converges uniformly to 0 on $D_{\frac{\delta}{2}} = \{\xi : |\xi - \xi_0| < \frac{\delta}{2}\}$.

Thus we know that $G(f_j)(z_j + \rho_j\xi) - a$ converges uniformly to $Q(g^{(k)}) - a$ on $D_{\frac{\delta}{2}} = \{\xi : |\xi - \xi_0| < \frac{\delta}{2}\}$.

Hence, by Hurwitz's theorem we deduce that there exist ξ_j, $\xi_j \to \xi_0$ such that, for large j,

$$P(g_j^{(k)}(\xi_j)) + \sum_{i=1}^{m} b_i(z_j + \rho_j \xi_j)\rho_j^{(k-1)n_1+,...,+n_{k-1}} M_i(g, g_j', ..., g_j^{(k)})(\xi_j) = a,$$

thus

$$P(f_j^k(z_j + \rho_j \xi_j)) + H(f_j, f_j', ..., f_j^{(k)})(z_j + \rho_j \xi_j) = a.$$

It follows from $G(f) = a \Rightarrow f = a$ or b that $f_j(z_j + \rho_j \xi_j) = a$ or b.

Thus we have $g(\xi_0) = \lim_{j\to\infty} g_j(\xi_j) = \infty$, which contradicts $Q(g^{(k)}(\xi_0)) = a$.

In a similar fashion, we can prove that $Q(g^{(k)})(\xi_0) \neq b$. This completes the proof of (i) and (ii).

By (ii), it follows $Q(g^{(k)}) \neq b$ and the definition of $Q(w)$ that there exist a non-zero constant d satisfying $g^{(k)} \neq d$. By Lemma 2.2, we know g is not a transcendental meromorphic function. Since g has zeros of multiplicity at least $k+1$ and $g^{(k)} \neq d$, it follows that g is not a polynomial. Hence by Lemma 2.2, we obtain that $g(\xi) = \frac{d\xi^k}{k!} + \cdots + a_0 + \frac{1}{A\xi + B}$, where $B, A(\neq 0)$, $a_0, ...$ are constants. Thus

$$g^{(k)}(\xi) = d + \frac{(-1)^k k! A^k}{(A\xi + B)^{k+1}}.$$

It follows that $g^{(k)}(\xi) = h$ has solutions, So $Q(g^{(k)}) = a$ has solutions, which contradicts (i). Hence $\mathcal{F}$ is normal in D.

Case 2. $ab = 0$. We may assume that $b = 0$. Suppose that $\mathcal{F}$ is not normal in $D = \Delta$. Withoutloss of generality, we assume that $\mathcal{F}$ is not normal at $z_0 = 0$. Then, by Lemma 2.1, there exist

1. a number $r \in (0, 1)$;
2. a sequence of complex numbers z_j, $|z_j| < r$;
3. a sequence of functions $f_j \in \mathcal{F}$;
4. a sequence of positive numbers $\rho_j \to 0^+$

such that $g_j(\xi) = \frac{f_j(z_j + \rho_j \xi)}{\rho_j^k}$ converges uniformly with respect to the spherical metric to a non-constant mermorphic functions $g(\xi)$ in C. Moreover, $g(\xi)$ is of order at most 2, all of whose zeros have multiplicity at least $k+1$. Set $Q(w) = w^q + a_{q-1}(0)w^{q-1}+, ..., +a_1(0)w$.

We claim that

(iii) $Q(g^{(k)}) \neq a$,

(iv) $Q(g^{(k)}) = 0 \Leftrightarrow g = 0$.

Now we prove (iii). Suppose now that $Q(g^{(k)}(\xi_0)) = a$. We claim that $Q(g^{(k)}) \not\equiv a$. Otherwise, from the definition of $Q(w)$, there exist a nonzero constant h such that $g^{(k)}(\xi) \equiv h$, g must be a polynomial of at most degree k, which contradicts the fact that each zero of $g(\xi)$ are of multiplicity at least $k+1$. Since $Q(g^{(k)}(\xi_0)) = a$.

Suppose $Q(g^{(k)}(\xi_0)) = a$, by Hurwitz's theorem, we know that there exists $\{\xi_j\}$, $\xi_j \to \xi_0$, such that, for j sufficiently large, and the similarly proof about case 1, we obtain $G(f_j)(z_j + \rho_j\xi_j)$ converges uniformly to $Q(g^{(k)})$ on $D_{\frac{\delta}{2}} = \{\xi : |\xi - \xi_0| < \frac{\delta}{2}\}$. So

$$G(f_j(z_j + \rho_j\xi_j)) = a.$$

It now follows from $G(f) = a \Rightarrow f = a$ or 0 that $f_j(z_j + \rho_j\xi_j) = a$ or 0. If $f_j(z_j + \rho_j\xi_j) = a$, then

$$g_j(\xi_j) = \frac{f_j(z_j + \rho_j\xi_j)}{\rho_j^k} = \frac{a}{\rho_j^k}.$$

So $g(\xi_0) = \lim_{j\to\infty} g_j(\xi_j) = \infty$, which contradicts $Q(g^{(k)}(\xi_0)) = a$. If $f_j(z_j+\rho_j\xi_j) = 0$, then $Q(g^{(k)}(\xi)) = a \Rightarrow g = 0$. From the definition of $Q(w)$, similarly the proof of case 1, there exists a nonzero constant h such that $g^{(k)}(\xi) = h$. So all zeros of $g^{(k)}(\xi) = h$ are the zeros of $g = 0$. By the all zeros of g have multiplicity at least $k + 1$, we know all zeros of g are zeros of $g^{(k)}$. So all zeros of $g^{(k)}(\xi) = h$ are the zeros of $g^{(k)}(\xi) = 0$, we deduce $h = 0$, a contradiction.

Next, we prove(iv). Suppose now that $Q(g^{(k)}(\xi_0)) = 0$. We claim that $Q(g^{(k)}) \not\equiv 0$. Otherwise, from the definition of $Q(w)$, we get $g^{(k)}(\xi) \equiv 0$, g must be a polynomial of at most degree $k - 1$, which contradicts the fact that each zero of $g(\xi)$ are of multiplicity at least $k + 1$. Since $Q(g^{(k)}(\xi_0)) = 0$. By Hurwitz's theorem, we know that there exists $\{\xi_j\}$, $\xi_j \to \xi_0$, such that, for j sufficiently large, and the similarly above proof, we obtain $G(f_j(z_j + \rho_j\xi_j)) = 0$. It now follows from $G(f) = 0 \Rightarrow f = a$ or 0 that $f_j(z_j + \rho_j\xi_j) = a$ or 0. If $f_j(z_j + \rho_j\xi_j) = a$, then $g(\xi_0) = \lim_{j\to\infty} g_j(\xi_j) = \infty$, which contradicts $Q(g^{(k)}(\xi_0)) = 0$.

Hence $f_j(z_j + \rho_j\xi_j) = 0$, so that $g(\xi_0) = \lim_{j\to\infty} g_j(\xi_j) = 0$. Thus we deduce that $Q(g^{(k)}) = 0 \Rightarrow g = 0$. Obviously, $g = 0 \Rightarrow g^{(k)} = 0 \Rightarrow Q(g^{(k)}) = 0$. This proves (iv).

By (iii), it follows $Q(g^{(k)}) \neq a$ and the definition of $Q(w)$ that there exist a non-zero constant h satisfying $g^{(k)} \neq h$. By Lemma 2.2, We know g is not a transcendental meromorphic function. Since g has zeros of multiplicity at least $k + 1$ and $g^{(k)} \neq h$, it follows that g is not a polynomial. Hence by Lemma 2.3, we obtain that $g(\xi) = \frac{h\xi^k}{k!} + \cdots + a_0 + \frac{1}{A\xi+B}$, where $B, A(\neq 0)$, $a_0, ...$ are constants. By the condition that g has zeros of multiplicity at least $k + 1$, thus $g(\xi)$ has only a zero. On the other hand, $g^{(k)}(\xi) = h + \frac{(-1)^k k! A^k}{(A\xi+B)^{k+1}}$. Obviously, $g^{(k)} = 0$ has $k + 1$ distinct solutions, which contradicts $g^{(k)} = 0 \Rightarrow Q(g^{(k)}) = 0 \Rightarrow g = 0$. Hence $\mathcal{F}$ is normal in D. The proof of Theorem 1.1 is complete. □

Proof of theorem 2. We may assume that $S_1 = \{a, b\}$, $S_2 = \{c\}$ where a, b, c are three distinct constants and $D = \Delta = \{|z| < 1\}$. Suppose that $\mathcal{F}$ is not normal in Δ. Then by Lemma 2.1, we can find $f_j \in \mathcal{F}$, $z_j \in \Delta$ and $\rho_j \to 0^+$ such that $g_j(\xi) = f_j(z_j + \rho_j\xi)$ converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic function g on C, all of whose zeros have multiplicity at least k.

We claim that

(i) $g^{(k)} \neq 0$,

(ii) $g \neq a, g \neq b$, and $g \neq c$.

Suppose now that $g^{(k)}((\xi_0)) = 0$. Obviously, $g(\xi_0) \neq \infty$. We claim that $g^{(k)} \not\equiv 0$. Otherwise, g must be a polynomial of degree less than k, which contradicts the fact that each zero of $g(\xi)$ are of multiplicity at least k. Similarly above the proof of theorem 1, there exist a constant h' satisfying $g^{(k)} = h' \Rightarrow P(g^{(k)}) = a$, and $g_j^{(k)}(\xi) - \rho_j^k h' \to g^{(k)}(\xi)$ on a neighborhood of ξ_0; also there exists ξ_j, $\xi_j \to \xi_0$, such that (for j sufficiently large)

$$0 = g^{(k)}(\xi_0) = g_j^{(k)}(\xi_j) - \rho_j^k h' = \rho_j^k (f_j^{(k)}(z_j + \rho_j \xi_j) - h').$$

Thus $f_j^{(k)}(z_j + \rho_j \xi_j) = h'$. So we obtain $P(f_j^{(k)}(\xi_j)) = a$.

It follows from $P(f^{(k)}) = a \Rightarrow f = a$ or b that $f_j(z_j + \rho_j \xi_j) = a$ or b, and so $g(\xi_0) = \lim_{j\to\infty} g_j(\xi_j) = a$ or b. In a similar fashion, there exist a constant p satisfying $g^{(k)} = p \Rightarrow P(g^{(k)}) = c$. Using $g_j^{(k)}(\xi) - \rho_j^k p$, and the similarly above the proof, we obtain that $g(\xi_0) = c$, which contradicts $a \neq c$ and $b \neq c$. This completes the proof of (i).

Next, we prove (ii). Suppose that $g(\xi_0) = a$. Then there exists $\xi_j \to \xi_0$, such that(for j sufficiently large)

$$a = g(\xi_0) = g_j(\xi_j) = f_j(z_j + \rho_j \xi_j).$$

Thus $P(f_j^{(k)}) = a$ or b. So there exist two constants h' and d such that $g_j^{(k)}(\xi_j) = h'$ or d. Therefore, $g^{(k)}(\xi_0) = \lim_{j\to\infty} \rho_j^k g_j^{(k)}(\xi_j) = 0$, which contradicts (i).

In a similar fashion, we obtain that $g \neq b$ and $g \neq c$.

Now by Picard's Theorem and (ii), g is a constant, a contradiction. Thus $\mathcal{F}$ is normal in D. Theorem is proved. □

Acknowledgements This paper is supported by project 10XKJ01 from Leading Academic Discipline Project of Shanghai Dianji University, and also supported by the Key Development Project 12C104 from Shanghai Dianji University.

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