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Five Cycles are Highly Ramsey Infinite

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ABSTRACT. In a previous paper, the author proved that all odd cycles, except five cycles, are highly Ramsey-infinite. In this paper, we fill in the missing case, and show that five cycles are highly Ramsey-infinite.

1. Introduction

All graphs in this paper are simple, finite and undirected. For graphs G, and H, and an integer r, G is *r-Ramsey* for H, if any arbitrary colouring of the edges of G with r colours, yields a copy of H all edges of which are the same colour. A graph G is *r-Ramsey-minimal* for H if it is r-Ramsey for H but no proper subgraph of G is. H is *r-Ramsey-infinite* if there are infinitely many graphs G that are r-Ramsey-minimal for H. In [4, 5] Nesětřil and Rödl started to characterise which graphs are 2-Ramsey-infinite. The full characterisation proceeded in many steps, but was completed in the 1990s in [3] and [6]. The non-symmetric version of the problem is still open, and significant progress was made relatively recently in [1]. For a more thorough list of references see [1] and [7].

In [2], a stronger version of 'Ramsey-infinite' was introduced. They showed that for any 3-connected graph H, there is a constant c such that for large enough n, there are at least $2^{cn \log n}$ graphs on at most n vertices that are 2-Ramsey-minimal for H. In [7] we took this a step further. A graph H is *highly r-Ramsey-infinite* if for some constant c, and large enough n, there are at least 2^{cn^2} non-isomorphic graphs on at most n vertices that are r-Ramsey-minimal for H.

In [7] it was shown that for $k \geq 3$ and $r \geq 2$ the clique K_k is highly r-Ramsey-infinite. In [8] it was shown that for odd $g \geq 7$ and $r \geq 2$ the cycle C_g is highly r-Ramsey-infinite. In this paper, we fill in the missing case and prove the following.

Theorem 1.1. *For all integers $r \geq 2$, C_5 is highly r-Ramsey-infinite.*

We remark that the main construction shares an underlying idea with the main

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constuctions in [7] and [8], but is considerably simpler, and with only small changes can be made to replace them both.

2. Notation and definitions

We identify a graph G with its edgeset $E(G)$. We let $[r]$ denote the set $\{1,\ldots,r\}$. Given a function ϕ defined on a set S we let $\phi(S)$ denote the set $\{\phi(s) \mid s \in S\}$. An r-colouring of a graph G is a mapping from the edges to the set $[r]$. An r-colouring of a graph G is C_5-free if there is no monochromatic copy of C_5 in G, that is, there is no copy of C_5 all of whose edges get the same colour. We will frequently index vertices 'modulo m', for some integer m; when we do this, we use the symbols $1,\ldots,m$, instead of $0,\ldots,m-1$.

The following alternate definition of *highly r-Ramsey-infinite* was shown implicitly in both [7] and [8], and is easier to work with.

Lemma 2.1. *A graph H is* highly r-Ramsey-infinite *if there is some constant c such that for all odd $m \geq 3$ there are 2^{m^2} different labelled graphs on at most $c \cdot m$ vertices that are r-Ramsey-minimal for H.*

Proof. Indeed let H be as in the statement of the lemma. Let $c' = 1/8c^2$, and $n_0 > 3c$ be large enough that $2^{2c'n_0^2}/n_0! > 2^{c'n_0^2}$. Given $n > n_0$, let m be the maximum odd integer for which $c \cdot m \leq n$. So $m \geq \frac{n}{2c}$.

By assumption, there are at least 2^{m^2} different labelled graphs on at most $c \cdot m \leq n$ vertices that are r-Ramsey-minimal for H. So there are at least

$$\frac{2^{m^2}}{n!} \geq \frac{2^{(n/2c)^2}}{n!} = \frac{2^{2c'n^2}}{n!} > 2^{c'n^2}$$

non-isomorphic such graphs. Thus H is r-Ramsey-minimal. □

3. Gadgets

We will use the following graphs whose existence was proved in [8].

Definition 3.1. For $r \geq 2$, a *negative signal sender $S = S_r^-$* is a graph containing *signal edges* e and f, and satisfying the following properties.

i. S has a C_5-free r-colouring.

ii. Under any C_5-free r-colouring of S, e and f get different colours.

iii. S has girth 5 and the distance between e and f in S is 6.

A *positive signal sender $S = S_r^+$* is defined similarly, but we replace the word 'different' in (ii) with 'the same'.

We will often use these senders in constructions in the following way. Given a graph G with edges e_1 and e_2 we will take a copy S of S_r^- (or S_r^+,) disjoint from G, and we will identify the edges e_1 and e_2 with the edges e and f of S, respectively.

When we do this we say that we 'connect the edges e_1 and e_2 with a negative (positive) sender.' We will usually connect several pairs of edges with senders, it is always assumed that these senders are all distinct and disjoint.

The following was proved in [8] as an immediate consequence of property (iii) in Definition 3.1.

Proposition 3.2. *Given a graph G with edges e_1 and e_2, when we connect the edges e_1 and e_2 with a negative or positive sender S, there are no cycles of length five or less, that are not entirely within G or entirely within S.*

In [8], senders were used to construct the following more general gadget. It was constructed for r colours, but we only need it for 2.

Lemma 3.3. *Let $\Gamma \subset \{\nu \mid \nu : W \to [2]\}$ be a set of 2-colourings of a set W, which is closed under permutation of [2]. There exists a graph M with the following properties.*

i. $W \subset M(= E(M))$

ii. *A mapping $\nu : W \to [2]$ can be extended to a C_5-free 2-colouring of M if and only if ν is in Γ.*

iii. *M has girth 5 and the distance between any two edges of W is at least 6*

The following comes from an easy application of Lemma 3.3.

Corollary 3.4. *There exists a graph N containing signal edges e, f and f' and satisfying the following properties.*

i. *A 2-colouring ϕ of $\{e, f, f'\}$ can be extended to a C_5-free 2-colouring of N, if and only if $\phi(\{f, f'\}) \neq \phi(e)$.*

ii. *N has girth 5 and the distance between any two signal edges is at least 6.*

The following follows from property (ii) of Corollary 3.4 just as 3.2 follows from property (iii) of Definition 3.1.

Proposition 3.5. *Given a graph G and the graph N from Corollary 3.4 we introduce no new cycles of length five or less by identifying the edges e, f and f' of N with edges of G.*

Lemma 3.6. *For every odd integer $m \geq 3$, there exists a graph $T = T(m)$ containing signal edges $f_*, f_1, \ldots, f_m$ and satisfying the following properties. (All indices in the lemma and the proof are modulo m.)*

i. *For every C_5-free 2-colouring ϕ of T with $\phi(f_*) = 1$ there is some $\alpha \in [m]$ such that $\phi(f_\alpha) = \phi(f_{\alpha+1}) = 2$.*

ii. *For every $\alpha \in [m]$ there is a C_5-free 2-colouring ϕ of T with $\phi(f_*) = 1$ such that $\phi(f_i) \neq \phi(f_{i+1})$ for all $i \neq \alpha$.*

iii. *There exists some constant c_T, independent of m, such that $|V(T)| \leq c_T m$.*

Proof. For $i = 1, \ldots, m$ let N_i be a copy of the graph N given by Corollary 3.4. Let e_i, f_i and f'_i be the copies of e, f, and f' respectively in N_i. Construct T from the disjoint graphs $N_1, \ldots, N_m$ and a disjoint edge f_* by identifying e_i with f_*, and f_i with f'_{i+1}, for $i = 1, \ldots, m$.

We verify that this graph T satisfies properties (i - iii). Let ϕ be a C_5-free 2-colouring of T with $\phi(f_*) = 1$. For every $i \in [m]$, as $\phi(e_i) = \phi(f_*) = 1$, at least one of the edges f_i and $f_{i+1} = f'_i$ get colour 2. So at least half of the edges $f_1, \ldots, f_m$ get colour 2. As m is odd, this gives property (i).

For property (ii), let $\alpha \in [m]$ be fixed. Define a 2-colouring ϕ of T as follows. Let $\phi(f_*) = 1$, and let $\phi(f_i) = 2$ for $i = \alpha, \alpha+1, \alpha+3, \ldots, \alpha+(m-2)$ (modulo m), and $\phi(f_i) = 1$ otherwise. For each $i \in [m] \setminus \{\alpha\}$, $\phi(f_i)$ and $\phi(f_{i+1})$ are not both 1, so by property (i) of Corollary 3.4 there is an extension of ϕ to a C_5-free 2-colouring of N_i; let ϕ be extended by this extension. By Proposition 3.5, any copy of C_5 in T is entirely within one of the graphs $N_1, \ldots, N_m$. Thus this is a C_5-free colouring of T.

Property (iii) follows from the fact that T is built from m copies of the graph N from Corollary 3.4, which does not depend on m. $\square$

4. Proof of Theorem 1.1

In the first two subsections of this section we construct auxillary graphs G_0 and $\mathcal{G}$. In the third subsection we use them to construct 2^{m^2} different graphs that are 2-Ramsey for C_5. In the final subsection, we prove Theorem 1.1 by induction on r, using the graphs from the earlier subsections for the base case $r = 2$.

4.1. The Graph G_0

Let P be the 3-path p_1xyp_2. We define four colourings $\phi_{11}, \phi_{12}, \phi_{21}$ and ϕ_{22} of P by

$$\phi_{ij}(p_1x) = i \qquad \phi_{ij}(xy) = j \qquad \phi_{ij}(yp_2) = i.$$

Let these colourings be defined similarily on any copy of P.

Let C consist of vertices $\{c_1, c_2, c_3\}$ with c_α and $c_{\alpha+1}$ (modulo 3) connected by a copy P_α of P for each $\alpha \in [3]$. (So C is a 9-cycle.) For $i, j \in [2]$, let ϕ_{ij} be the colouring on C that restricts to ϕ_{ij} on each of P_1, P_2, and P_3. Let E be the set of 6 possible edges between $\{p_1, p_2\}$ and $\{c_1, c_2, c_3\}$. Let $G_0 = P \cup C \cup E$.

Claim 4.1. *The graph G_0 satisfies the following properties.*

i. There is no C_5-free 2-colouring ϕ of G_0 that restricts to ϕ_{11} on P and to ϕ_{22} on C, or vice-versa.

ii. Any 2-colouring ϕ of $P \cup C$ that restricts on P or C to ϕ_{12} or ϕ_{21}, can be extended to a C_5-free 2-colouring of G_0.

iii. For any $e \in E$, the 2-colouring ϕ of $P \cup C$ which restricts to ϕ_{11} on P and ϕ_{22} on C, extends to a C_5-free 2-colouring of $G_e = G_0 \setminus \{e\}$.

Proof. (i) Assume that there is such a C_5-free 2-colouring ϕ of G_0. By considering, for $\alpha \in \{1, 2, 3\}$, the subgraph of G_0 induced by the vertices of C and P_α, it is not hard to check that ϕ must have different colours on p_1c_α and $p_1c_{\alpha+1}$. So

$$\phi(p_1c_1) \neq \phi(p_1c_2) \neq \phi(p_1c_3) \neq \phi(p_1c_1).$$

But, ϕ being a 2-colouring, this means that $\phi(p_1c_1) \neq \phi(p_1c_1)$, which is impossible.

(ii) Let ϕ restrict on C to either ϕ_{12} or ϕ_{21}. If ϕ restricts on P to ϕ_{11} let $\phi(E) = 2$, otherwise, let $\phi(E) = 1$. It is easy to verify that this ϕ is C_5-free. Similarly, let ϕ restrict on P to either ϕ_{12} or ϕ_{21}. If ϕ restricts on C to ϕ_{11} let $\phi(E) = 2$, otherwise, let $\phi(E) = 1$.

(iii) Assume, without loss of generality, that $e = p_1c_1$. Extend ϕ to $E \setminus \{e\}$ as follows. Let ϕ have colour 1 on p_1c_3 and p_2c_2 and colour 2 on all other edges in $E \setminus \{e\}$. One can check that this is a C_5-free 2-colouring of G_e. □

4.2. The Graph $\mathcal{G}^*$

For any copy C' of C and P' of P, refer to the edges that get colour 1 under the colouring ϕ_{12} as '1-edges', and the edges that get colour 2 under ϕ_{12} as '2-edges'.

Let odd $m \geq 3$ be fixed. Let T^C and T^P be copies of the graph $T(m)$ from Lemma 3.6. For $i = 0, \ldots, m$, let f_i^C and f_i^P be the copies of f_i in T^C and T^P respectively. For $i = 1, \ldots, m$, let C^i be a copy of C, and let P^i and Q^i be copies of P.

To construct $\mathcal{G}^*$ from the disjoint graphs T^P, T^C, C^i, P^i and Q^i, join f_0^C and f_0^P with a negative sender, and for $i = 1, \ldots, m$, do the following (indices modulo m).

- Join the 1-edges in C^i to f_i^C, and the 2-edges in C^i to f_{i+1}^C with positive senders.
- Join the 1-edges in P^i and Q^i to f_i^P, and the 2-edges in P^i and Q^i to f_{i+1}^P with positive senders.

We now observe some properties of $\mathcal{G}^*$ which are almost immediate from the construction, and the corresponding properties of T listed in Lemma 3.6.

Claim 4.2. *$\mathcal{G}^*$ has the following properties.*

i. For any C_5-free 2-colouring ϕ of $\mathcal{G}^*$ with $\phi(f_0^C) = 1$ there exist $\alpha, \beta \in [m]$ such that ϕ restricts on C^α to ϕ_{22} and on P^β and Q^β to ϕ_{11}.

ii. For any choice of $\alpha, \beta \in [m]$ there is a C_5-free 2-colouring ϕ of $\mathcal{G}^*$, with $\phi(f_0^C) = 1$, that restricts on C^i, P^j and Q^j to ϕ_{12} or ϕ_{21} for all $i \neq \alpha$ and $j \neq \beta$.

iii. There exists some constant c independent of m, such that $|V(\mathcal{G}^*)| < cm$.

Proof. For item (i), let ϕ be a C_5-free 2-colouring of $\mathcal{G}^*$ with $\phi(f_0^C) = 1$. By Lemma 3.6 (i), there exists $\alpha \in [m]$ such that $\phi(f_\alpha^C) = \phi(f_{\alpha+1}^C) = 2$. As ϕ is C_5-free on the positive senders connecting these edges to C^α, ϕ restricts on C^α to ϕ_{22}. The sender from f_0^C to f_0^P ensures that $\phi(f_0^P) = 2$, and so we can argue similarly that for some $\beta \in [m]$, ϕ restricts on P^β and Q^β to ϕ_{11}.

Item (ii) follows from item (ii) of Lemma 3.6 just as (i) followed from (i) of Lemma 3.6.

Item (iii) follows from property (iii) of Lemma 3.6 and the fact that $\mathcal{G}^*$ consists of two copies of T and $15 \cdot m + 1$ senders. Indeed, $c = 2c_T + 16s$ is sufficient, where s is number of vertices of the largest sender used. □

4.3. The Graph $\mathcal{G}(\mathcal{I})$

For copies C' of C and P' of P, we say we 'complete C' and P' to a copy of G_0' to mean we add all edges between the copies of c_1, c_2, and c_3 in C' and the copies of p_1, and p_2 in P'.

For any set $\mathcal{I} = (I_1, \ldots I_m)$ of subsets of $[m]$ construct $\mathcal{G} = \mathcal{G}(\mathcal{I})$ from $\mathcal{G}^*$, by adding only edges, as follows.

For each $i, j \in [m]$

- complete C^i and P^j to a copy of G_0 if $i \in I_j$, and
- complete C^i and Q^j to a copy of G_0 otherwise.

Let E^{ij} be the edges added between C^i and P^j or Q^j. Let $E^{\mathcal{I}} = \mathcal{G} \setminus \mathcal{G}^*$ be the union of all the E^{ij}.

Claim 4.3. *$\mathcal{G}$ is 2-Ramsey for C_5.*

Proof. Towards contradiction, assume that there is a C_5-free 2-colouring ϕ of $\mathcal{G}$. By item (i) of Claim 4.2, there are $\alpha, \beta \in [m]$ such that ϕ restricts on C^α to ϕ_{22} and on P^β and Q^β to ϕ_{11}, (or vice versa). By construction C^α and either P^β or Q^β induce a copy of G_0, and so ϕ restricted to this copy of G_0 contradicts item (i) of Claim 4.1. □

Claim 4.4. *For any edge e of $E^{\mathcal{I}}$, $\mathcal{G} \setminus \{e\}$ has a C_5-free 2-colouring.*

Proof. Assume, without loss of generality, that e is in E^{11}. We define a C_5-free 2-colouring ϕ of $\mathcal{G} \setminus \{e\}$.

By item (ii) of Claim 4.2 there is a 2-colouring of $\mathcal{G}^*$ that restricts on C^1 to ϕ_{11}, on P^1 and Q^1 to ϕ_{22}, and on all other C^i, P^j and Q^j to ϕ_{12} or ϕ_{21}. Define ϕ to restrict to such a colouring on $\mathcal{G}^*$.

For every $i, j \in [m]$ with not both $i, j = 1$, there is, by item (ii) of Claim 4.1, a C_5-free 2-colouring of the copy of G_0 in $\mathcal{G}^*$ induced by the vertices of $C^i \cup P^j \cup Q^j$, which agrees with ϕ on C^i, P^j and Q^j. Define ϕ on E^{ij} to agree with this colouring.

By item (iii) of Claim 4.1 there is a C_5-free 2-colouring of the graph induced by $C^1 \cup P^1 \cup Q^1$ (a copy of G_0 less an edge of E) , which agrees with ϕ on C^1, P^1 and Q^1. Define ϕ on E^{11} to agree with this colouring.

We now show that this 2-colouring ϕ of $\mathcal{G} \setminus \{e\}$ is C_5-free. By construction it is C_5-free on $\mathcal{G}^*$ and on the (partial) copies of G_0 induced by any C^i and any P^j

or Q^j. So we show that the only copies of C_5 in $\mathcal{G}$ are entirely within one of these graphs. Let C_0 be a copy of C_5 in $\mathcal{G}$ not entirely within $\mathcal{G}^*$. As $E^{\mathcal{I}}$ is bipartite, C_0 must contain edges of $\mathcal{G}^*$. As the vertices of $\mathcal{G}^*$ that are incident to edges of $E^{\mathcal{I}}$ are distance at least 6 apart, unless they are the endpoints in a copy of the 3-path P in one of C^i, P^j or Q^j, C_0 must intesect $\mathcal{G}^*$ in one of these paths. Thus it is entirely within the copy of G_0 induced by some C^i and some P^j or Q^j. □

4.4. The proof of Theorem 1.1

The proof is by induction on r. The most difficult part, the base case $r = 2$ is almost done. Indeed, let c_2 be the constant c from Claim 4.2 (iii). By Lemma 2.1 it is enough to show that for odd $m \geq 3$ there are 2^{m^2} different labelled graph on at most $c_2 m$ vertices that are 2-Ramsey minimal for C_5. For each of the 2^{m^2} choices of $\mathcal{I}$ of m subsets of $[m]$, the graph $\mathcal{G}(\mathcal{I})$ is 2-Ramsey by Claim 4.3, and any 2-Ramsey-minimal subgraph of it contains all of $E^{\mathcal{I}}$ by Claim 4.4. Since $E^{\mathcal{I}}$ is different for different choices of $\mathcal{I}$, this gives us 2^{m^2} different 2-Ramsey-minimal graphs on at most cm vertices. This is enough.

For the induction on r we use the following construction. Let $\mathcal{G}_{r-1}$ be some graph on at most $c_{r-1}m$ vertices that is $(r-1)$-Ramsey minimal for H. Construct $\mathcal{G}_r$ from $\mathcal{G}_{r-1}$ as follows.

i. Add a new vertex v_0.

ii. For each vertex $v \in V(\mathcal{G}_{r-1})$ add a new vertex v' and the edges $v_0 v'$ and $v'v$.

iii. Add a new edge e_0.

iv. Connect every edge added in step (ii) to e_0 with a positive sender.

Clearly $\mathcal{G}_r$ has less than $2s|V(\mathcal{G}_{r-1})|$ vertices where s is the number of vertices in a positive sender. So $\mathcal{G}_r$ has less then $c_r m$ vertices where $c_r = 2sc_{r-1}$.

Claim 4.5. *$\mathcal{G}_r$ is r-Ramsey for C_5.*

Proof. Assume, towards contradiction, that $\mathcal{G}_r$ has a C_5-free r-colouring ϕ. Then ϕ gets the same colour on all edges added in steps (ii) as they are all joined to e_0 with positive senders. Let this colour be r. Every edge in $\mathcal{G}_{r-1}$ completes a C_5 with such edges, so must get some colour other than r, so ϕ restricted to $\mathcal{G}_{r-1}$ is a C_5-free $(r-1)$-colouring. As this is impossible, $\mathcal{G}_r$ is r-Ramsey for C_5. □

Claim 4.6. *For any edge $e \in \mathcal{G}_{r-1}$, $\mathcal{G}_r \setminus \{e\}$ has a C_5-free $(r-1)$-colouring.*

Proof. Let e be an edge of $\mathcal{G}_{r-1}$. As $\mathcal{G}_{r-1}$ is $(r-1)$-Ramsey-minimal there is a C_5-free $(r-1)$-colouring ϕ of $\mathcal{G}_{r-1} \setminus \{e\}$. Extend ϕ to a r-colouring of $\mathcal{G}_r$ by setting $\phi(f) = r$ on all edges f introduced in step (ii) of the construction, and on the edge e_0. As these edges form a forest, this introduces no monochromatic copies of C_5. As the edges e_0 and f have the same colour for any f introduded in step (ii), ϕ can be extended to a C_5-free colouring of sender between them which was added in step (iv) of the construction. By Proposition 3.2, this ϕ is a C_5-free r-colouring of $\mathcal{G}_r \setminus \{e\}$. □

Now assume that the theorem has been proved for $r-1$, that is, that there are 2^{m^2} different labelled graphs on at most $c_{r-1}m$ vertices that are $(r-1)$-Ramsey minimal for H. From each such graph $\mathcal{G}_{r-1}$ the above construction gives a graph $\mathcal{G}_r$ on at most $c_r m$ vertices, which by Claim 4.5, is r-Ramsey for C_5.

By Claim 4.6 the r-Ramsey-minimal subgraphs of $\mathcal{G}_r$ and $\mathcal{G}'_r$ constructed from different $\mathcal{G}_{r-1}$ and $\mathcal{G}'_{r-1}$ are different. So we have 2^{m^2} different graphs on at most $c_r m$ vertices that are r-Ramsey-minimal for C_5. The theorem thus holds for r, and so follows by induction.

5. Concluding Remarks

In [8] we observed that no bipartite graph can be highly 2-Ramsey-infinite, but we expect that any graph that is non-bipartite and 2-Ramsey-infinite, is highly 2-Ramsey-infinite.

Apart from C_5 being non-bipartite, the important aspects for our proof that C_5 is highly 2-Ramsey-infinite are the existence of positive and negative signal senders for C_5, and the fact that C_5 has a vertex of degree 2 (in the construction of G_0).

It was proved in [2] that senders exist for all 3-connected graphs H. However, such graphs cannot have vertices of degree 2. It would be interesting to extend the construction of the graphs $\mathcal{G}(\mathcal{I})$ from this paper work for other 3-connected graphs. I cannot see how to do this though. Similarily, it would be interesting to construct senders for more 2-connected graphs. This also seems to be difficult.

References

[1] B. Bollobás, J. Donadelli, Y. Kohayakawa, R.H. Schelp, *Ramsey minimal graphs,* J. of the Brazilian Computer Society, **7**(3)(2002), 27-37.

[2] S. Burr, J. Nešetřil,V. Rödl, *On the use of senders in generalized Ramsey theory for graphs,* Discrete Math., **54**(1)(1985), 1-13.

[3] R. Faudree, *Ramsey minimal graphs for forests,* Ars Combin., **31**(1991), 117-124.

[4] J. Nešetřil, V. Rödl, *Partitions of vertices,* Comment. Math. Univ. Carolinae, **17**(1)(1976), 85-95.

[5] J. Nešetřil, V. Rödl, *The structure of critical Ramsey graphs,* Acta Math. Acad. Sci. Hungar., **32**(3-4)(1978), 295-300.

[6] V. Rödl, A. Ruciński, *Threshold functions for Ramsey properties,* J. Amer. Math. Soc., **8**(4)(1995), 917-942.

[7] V. Rödl, M. Siggers, *On Ramsey Minimal Graphs,* SIAM J. Discrete Math., **22**(2)(2008), 467-488.

[8] M. Siggers, *On Highly Ramsey Infinite Graphs,* J. Graph Theory, **59**(2)(2008), 97-114.