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Four-dimensional Naturally Reductive Pseudo-Riemannian Homogeneous Spaces

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Abstract. Our attention is turned to four-dimensional pseudo-Riemannian naturally reductive homogeneous spaces. In particular, our study leads to a complete classification of them.

1. Introduction

Naturally reductive homogeneous Riemannian spaces are the simplest kind of homogeneous spaces. As is well-known all symmetric spaces are naturally reductive, but there are also many other examples. See for example [17].

The complete classification of the simply connected 3-dimensional naturally reductive spaces is given in [17]. One obtains the symmetric spaces and further the Lie groups $SU(2)$, $SL(2,\mathbb{R})$ and the Heisenberg group with some left-invariant metric. Moreover, it is proved in [11] that the simply connected, complete, three-dimensional spaces with volume-preserving geodesic symmetries are homogeneous and naturally reductive.

Three-dimensional naturally reductive Lorentzian spaces have been investigated by Cordero and Parker in [3], in order to determine the possible forms and the symmetry groups of their curvature tensor. Afterward, G. Calvaruso and R.A. Marinosci classified in [2] all three-dimensional Lorentzian g.o. spaces and naturally reductive spaces.

The next step is to consider the four-dimensional manifolds. In [13], four-dimensional naturally reductive homogeneous Riemannian spaces were considered, and a full classification was given. In particular, one obtains the symmetric spaces or the Riemannian product $M^3 \times \mathbb{R}$, where M^3 is naturally reductive and isometric to the Lie groups $SU(2)$, $\widetilde{SL(2,\mathbb{R})}$ and the Heisenberg group with some left-invariant metric.

In this work, we want to consider four-dimensional naturally reductive mani-

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folds. Our study leads to a complete classification of the pseudo-Riemannian naturally reductive ones.

A (connected) pseudo-Riemannian manifold (M, g) is *homogeneous* if there exists a connected Lie group G of isometries acting transitively and effectively on it [16]. We recall here few results concerning homogeneous manifolds, in the Riemannian and pseudo-Riemannian case. Denote by H the isotropy group at a fixed point $o \in M$ (the *origin*). Then (M, g) can be identified with $(G/H, g)$. In general, there exists more than one such group $G \subset I(M)$ (= full group of isometries of (M, g)). For any fixed choice $M = G/H$, the Lie group G acts transitively and effectively on G/H from the left. The pseudo-Riemmanian metric g of M can be considered as a G-invariant metric on G/H. The pair $(G/H, g)$ is called *pseudo-Riemannian homogeneous space*. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively and by $\mathfrak{m}$ a complement of $\mathfrak{h}$ in $\mathfrak{g}$. If $\mathfrak{m}$ is stable under the action of $\mathfrak{h}$, the $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is called a *reductive split*, and $(\mathfrak{g}, \mathfrak{h})$ a *reductive pair*. Contrary to the Riemannian case, for a pseudo-Riemannian homogeneous space $(M = G/H, g)$ the Lie algebra $\mathfrak{g}$ of G needs not to admit a reductive decomposition.

A pseudo-Riemannian reductive homogeneous space $(M = K/H, g)$ is called *naturally reductive* if there exists at least one reductive split $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ such that

$$g([X, Y]_{\mathfrak{m}}, Z) + g([X, Z]_{\mathfrak{m}}, Y) = 0 \tag{1.1}$$

for all $X, Y, Z \in \mathfrak{m}$. It is not always easy to decide whether a homogeneous (reductive) pseudo-Riemannian manifold is or is not naturally reductive, because condition (1.1) must be checked for all groups of isometries acting transitively on M. It is also well known that (1.1) holds if and only if the Levi-Civita connection of (M, g) and the canonical connection (of the reductive split $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$) have exactly the same geodesics [17].

Up to our knowledge, extensive studies have been made on the geometry of a naturally reductive and homogeneous Riemannian and Lorentzian manifolds by several authors (see [6], [1], [2], [4], [5], [7], [8], [9], [11], [12], [17], [18]).

In [7], P. M. Gadea and J. A. Oubiña introduced *homogeneous pseudo-Riemannian structures* in order to obtain a characterization of reductive homogeneous pseudo-Riemannian manifolds similar to the one given for homogeneous Riemannian manifolds by Ambrose and Singer [1]. More specifically, by using the representation theory, they determined eight classes of homogeneous structures which are defined by the invariant subspaces of a certain space $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$. In [8], the same authors obtained a classification of homogeneous pseudo-Riemannian structures into eight primitive classes as Tricerri and Vanhecke made for the Riemannian case [17]. Clearly, homogeneous spaces of type $\{0\}$ are just symmetric ones and for the case at hand, it is worth knowing that the homogeneous spaces with a $\mathcal{T}_3$ structure are naturally reductive spaces.

The paper is organized in the following way. In Section 2, we shall recall the basic definitions and properties of naturally reductive spaces. In Section 3, we shall report the classification of four-dimensional naturally reductive pseudo-Riemannian manifolds.

2. Preliminaries on naturally reductive pseudo-Riemannian spaces

Let (M, g) be a (connected) homogeneous pseudo-Riemannian manifold. We say that M is *naturally reductive* if there exists at least one reductive split $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ such that

$$< [X, Y]_{\mathfrak{m}}, Z > + < [X, Z]_{\mathfrak{m}}, Y >= 0 \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{m}$. Here $<, >$ denotes the induced pseudo-Riemannian metric from g on $\mathfrak{m}$.
Let $\widetilde{\nabla}$ be the *canonical connection* of the homogeneous space $M = G/H$ with respect to a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Then it holds

$$[X, Y]_{\mathfrak{m}} = -\widetilde{T}(X, Y), \quad [X, Y]_{\mathfrak{h}} = -\widetilde{R}(X, Y) \tag{2.2}$$

for $X, Y \in \mathfrak{m}$ (under the canonical identification of $\mathfrak{m}$ with the tangent space T_oM), where $\widetilde{T}, \widetilde{R}$ denote the torsion tensor and the curvature tensor of $\widetilde{\nabla}$, respectively. We remind that any G-invariant tensor field on M is parallel with respect to the connection $\widetilde{\nabla}$.

We denote by ∇ the Levi-Civita connection of (M, g) and by R its curvature tensor. Moreover, let us denote by

$$D = \nabla - \widetilde{\nabla}. \tag{2.3}$$

Then $\widetilde{\nabla}$ is the unique linear connection on M which is complete and has parallel curvature $\widetilde{R}$ and parallel torsion $\widetilde{T}$, that is,

$$\widetilde{\nabla}\widetilde{R} = \widetilde{\nabla}\widetilde{T} = 0, \quad \widetilde{\nabla}g = \widetilde{\nabla}D = 0. \tag{2.4}$$

Further, because $R, \nabla R, \nabla^2 R, \ldots$, are $G-$invariant, we get

$$\widetilde{\nabla}R = \widetilde{\nabla}(\nabla R) = \ldots = \widetilde{\nabla}(\nabla^k R) = \ldots = 0, \quad k = 1, 2, \ldots. \tag{2.5}$$

By means (2.4) and (2.5), we get that the endomorphism $\widetilde{R}_x(X, Y)$, at any fixed point $x \in M$, acts as a derivation on the tensor algebra $\mathfrak{T}(T_o(M))$:

$$\begin{aligned} &\widetilde{R}(X, Y) \cdot g = \widetilde{R}(X, Y) \cdot D = \widetilde{R}(X, Y) \cdot \widetilde{T} = \widetilde{R}(X, Y) \cdot \widetilde{R} = 0, \\ &\widetilde{R}(X, Y) \cdot R = \widetilde{R}(X, Y) \cdot (\nabla^k R) = 0, \quad k = 1, 2, \ldots. \end{aligned} \tag{2.6}$$

The tensors $\widetilde{R}$ and $\widetilde{T}$ are skew-symmetric and the two identities of Bianchi hold, i.e.,

$$\mathfrak{S}_{X,Y,Z}\widetilde{R}(X, Y)Z = \mathfrak{S}_{X,Y,Z}\widetilde{T}(\widetilde{T}(X, Y), Z) \tag{2.7}$$

$$\mathfrak{S}_{X,Y,Z}\widetilde{R}(\widetilde{T}(X,Y),Z)=0. \tag{2.8}$$

here $\mathfrak{S}$ denotes the cyclic sum.

As concerns the Riemannian curvature tensor R, we have

$$R(X,Y)=\widetilde{R}(X,Y)+[D_X,D_Y]+D_{\widetilde{T}(X,Y)}. \tag{2.9}$$

By (2.1) and by standard arguments we get

$$D_XY=-\frac{1}{2}\widetilde{T}(X,Y). \tag{2.10}$$

By using the first of Bianchi identities, the formula (2.9) becomes

$$R(X,Y)Z=\widetilde{R}(X,Y)Z-\frac{1}{4}\mathfrak{S}_{X,Y,Z}\widetilde{R}(X,Y)Z+\frac{1}{4}\widetilde{T}(Z,\widetilde{T}(X,Y)). \tag{2.11}$$

In [13], the authors gave the full classification of four-dimensional naturally reductive Riemannian manifolds. More specifically, they proved

Theorem 2.1([13]). *Let (M,g) be a four-dimensional simply connected naturally reductive Riemannian manifold. Then M is either symmetric or it is a Riemannian product of the form $M=M^3\times\mathbb{R}$, where M^3 is again naturally reductive and isometric to one of the following spaces:*

a) $SU(2)$,

b) $\widetilde{SL(2,\mathbb{R})}$,

c) H_3,

equipped with a special left-invariant metric.

Recall that H. Wu ([19], [20], [21]) extended De Rham decomposition theorem to the pseudo-Riemannian context in which the decomposition of the manifold reflects the decomposition of the tangent space into invariants subspaces of the holonomy group that are non degenerate but may have degenerate invariant subspaces, that is, may be reducible. By using the above results of Wu, the following criterions hold, the standard proof of which is analogous to the Riemannian case (see [10], [13]).

Proposition 2.2. *Let (M,g) be a simply connected naturally reductive pseudo-Riemannian space. Let the tangent space T_oM at the origin admit an orthogonal decomposition $T_oM=V_1\oplus V_2$, such that the metric is nondegenerate on V_1, and*

$$\begin{aligned}\pi_i\widetilde{T}(X,Y)&=\widetilde{T}(\pi_iX,\pi_iY)\\ \pi_i\widetilde{R}(X,Y)Z&=\widetilde{R}(\pi_iX,\pi_iY)\pi_iZ\end{aligned} \tag{2.12}$$

for $i=1,2$ and $X,Y,Z\in T_oM$, where π_i denotes the canonical projection on V_i. Then M is a pseudo-Riemannian direct product $(M,g)=(M_1,g_1)\times(M_2,g_2)$, with

$dimM_i = dimV_i$.

Proposition 2.3. *Let $(M, g) = (M_1, g_1) \times (M_2, g_2)$ be a homogeneous pseudo-Riemannian manifold which is a direct product. If (M, g) is a naturally reductive then the factors (M_i, g_i), $i = 1, 2$ are naturally reductive.*

3. The classification in dimension four

Let (M, g) be a simply connected naturally reductive pseudo-Riemannian space of dimension four, and let $0 \in M$ be its origin.

With respect to a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of T_oM the torsion $\widetilde{T}$ takes the form

$$\widetilde{T}(e_i, e_j) = \Sigma_k \varepsilon_k t_{ij}^k e_k, \qquad i, j, k = 1, \ldots, 4,$$

where $\varepsilon_i = \pm 1$. By applying (2.1) and (2.2), we have $t_{ij}^k + t_{ik}^j = 0$ and because of the skew-symmetry of $\widetilde{T}$, we also get $t_{ij}^k + t_{ji}^k = 0$; thus, it follows

$$t_{ii}^k = t_{ik}^i = t_{ki}^i = 0, \qquad i, k = 1, 2, 3, 4.$$

We can express $\widetilde{T}$ in the following form:

$$\begin{array}{ll} \widetilde{T}(e_1, e_2) = \varepsilon_3 a e_3 + \varepsilon_4 b e_4 & \widetilde{T}(e_2, e_3) = \varepsilon_1 a e_1 + \varepsilon_4 d e_4 \\ \widetilde{T}(e_1, e_3) = -\varepsilon_2 a e_2 + \varepsilon_4 c e_4 & \widetilde{T}(e_2, e_4) = \varepsilon_1 b e_1 - \varepsilon_3 d e_3 \\ \widetilde{T}(e_1, e_4) = -\varepsilon_2 b e_2 - \varepsilon_3 c e_3 & \widetilde{T}(e_3, e_4) = \varepsilon_1 c e_1 + \varepsilon_2 d e_2 \end{array} \tag{3.1}$$

with respect to a suitable pseudo-orthonormal basis $\{e_1, \ldots, e_4\}$.

The Lie algebra of the isotropy subgroup of $I(M)$ at the origin 0 can be identified, by the linear isotropy representation, with the algebra $\mathfrak{h}$ of all the endomorphisms $A : T_0(M) \to T_0(M)$ which, as derivations of the tensor algebra $\mathcal{T}(T_0(M))$, satisfy the conditions

$$A(g) = A(R) = A(\nabla R) = \ldots = A(\nabla^k R) = 0, \quad k = 1, 2, \ldots. \tag{3.2}$$

From (2.6), we get that all the curvature transformations $\widetilde{R}(X, Y)$ of the canonical connection belong to the algebra $\mathfrak{h}$.

Let us suppose $\widetilde{R}(X, Y) \neq 0$ (otherwise M is proved to be symmetric) and put $h(U, V) = g(\widetilde{R}(X, Y)U, V)$. Thus, h is an exterior 2-form on $T_0(M)$; then there exists a orthonormal basis of $(T_0M)^*$, $\{\xi^1, \ldots, \xi^4\}$ such that

$$h = \lambda\, \xi^1 \wedge \xi^2 + \mu\, \xi^3 \wedge \xi^4, \qquad \lambda\mu \neq 0. \tag{3.3}$$

Let consider the pseudo-orthonormal dual basis $\{E_1, \ldots, E_4\}$ of $T_0(M)$. The matrix of h has components

$$h_{ij} = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & -\mu & 0 \end{pmatrix} \tag{3.4}$$

and $det\ (h_{ij}) = \lambda^2\mu^2$.

Let us calculate $\widetilde{R}(X,Y)E_i = \sum_k \varepsilon_k h_{ik} E_k$. We get

Proposition 3.1. *There exists a pseudo-orthonormal basis $\{E_1,\dots,E_4\}$ with respect to which the endomorphism $\widetilde{R}(X,Y)$ takes the form*

$$\widetilde{R}(X,Y) = \lambda\ A + \mu\ B, \quad \lambda\mu \neq 0,$$

where $A, B \in End(T_0(M))$ such that

$$\begin{array}{l} AE_1 = \varepsilon_2 E_2, \quad AE_2 = -\varepsilon_1 E_1, \quad AE_3 = AE_4 = 0 \\ BE_1 = BE_2 = 0, \quad BE_3 = \varepsilon_4 E_4, \quad BE_4 = -\varepsilon_3 E_3. \end{array}$$

Proof. It holds $\widetilde{R}(X,Y)E_i = \sum_k \varepsilon_k R_i^k E_k$, where $R_i^k = g(\widetilde{R}(X,Y)E_i, E_k) = h_{ik}$. Thus, we get

$$\begin{array}{ll} \widetilde{R}(X,Y)E_1 = \varepsilon_2\lambda E_2 & \widetilde{R}(X,Y)E_3 = \varepsilon_4\mu E_4 \\ \widetilde{R}(X,Y)E_2 = -\varepsilon_1\lambda E_1 & \widetilde{R}(X,Y)E_4 = -\varepsilon_3\mu E_3. \end{array}$$

This ends the proof. □

Let us study the rank of the matrix of h, by distinguishing the signature of the metric.

Case A: signature $(2,2)$.

1. **rk $\boldsymbol{h = 4}$:** this means $det\ h_{ij} \neq 0$, that is $\lambda\mu \neq 0$.

   In this case, there exists a suitable pseudo-orthonormal basis $\{E_1,\dots,E_4\}$ such as

   $< E_i, E_j >= \varepsilon_i\delta_{ij}$, with $\varepsilon_1 = \varepsilon_2 = +1$ and $\varepsilon_3 = \varepsilon_4 = -1$, and

   $$\begin{array}{ll} \widetilde{R}(X,Y)E_1 = \lambda E_2 & \widetilde{R}(X,Y)E_2 = -\lambda E_1 \\ \widetilde{R}(X,Y)E_3 = -\mu E_4 & \widetilde{R}(X,Y)E_4 = \mu E_3. \end{array} \tag{3.5}$$

   By applying $(\lambda A+\mu B)(\widetilde{T}) = 0$ to (3.1), and taking into account the signature, we get

   $$\begin{array}{l} (\lambda A + \mu B)(\widetilde{T}(E_1,E_2)) = \widetilde{T}((\lambda A + \mu B)E_1, E_2) + \widetilde{T}(E_1, (\lambda A + \mu B)E_2) \\ \qquad \Leftrightarrow \mu a E_4 - \mu b E_3 = 0, \\ (\lambda A + \mu B)(\widetilde{T}(E_1,E_3)) = \widetilde{T}((\lambda A + \mu B)E_1, E_3) + \widetilde{T}(E_1, (\lambda A + \mu B)E_3) \\ \qquad \Leftrightarrow \lambda d E_4 - \mu b E_2 = 0, \\ (\lambda A + \mu B)(\widetilde{T}(E_2,E_3)) = \widetilde{T}((\lambda A + \mu B)E_2, E_3) + \widetilde{T}(E_2, (\lambda A + \mu B)E_3) \\ \qquad \Leftrightarrow \lambda c E_4 - \mu b E_1 = 0. \end{array}$$

   Thus, we get the conditions: $\mu a = \mu b = \lambda c = \lambda d = 0$, that is, $a = b = c = d = 0$, and so $\widetilde{T} = 0$. This means that M is symmetric.

2. **rk $\boldsymbol{h} < 4$:** in this case $det\ h_{ij} = 0$, that is $\lambda\mu = 0$. Then, $\lambda \neq 0$ and $h = \lambda\xi^1 \wedge \xi^2$.

   With respect to a suitable pseudo-orthonormal basis $\{E_1, \ldots, E_4\}$ such as $< E_i, E_j >= \varepsilon_i \delta_{ij}$, with $\varepsilon_1 = \varepsilon_2 = +1$ and $\varepsilon_3 = \varepsilon_4 = -1$, it holds

$$\begin{array}{ll} \widetilde{R}(X,Y)E_1 = \lambda E_2 & \widetilde{R}(X,Y)E_2 = -\lambda E_1 \\ \widetilde{R}(X,Y)E_3 = 0 & \widetilde{R}(X,Y)E_4 = 0. \end{array} \tag{3.6}$$

   By applying $A(\widetilde{T}) = 0$ to (3.1), with $A = \widetilde{R}(X,Y)$, and taking into account the signature, we get

$$\begin{aligned} A(\widetilde{T}(E_1,E_3)) &= \widetilde{T}(AE_1,E_3) + \widetilde{T}(E_1,AE_3) \\ &\Leftrightarrow -\lambda d E_4 = 0, \\ A(\widetilde{T}(E_2,E_3)) &= \widetilde{T}(AE_2,E_3) + \widetilde{T}(E_2,AE_3) \\ &\Leftrightarrow \lambda c E_4 = 0. \end{aligned}$$

   Thus, we get the conditions: $c = d = 0$. Rewriting (3.1), we get

$$\begin{aligned} \widetilde{T}(E_1,E_2) &= -aE_3 - bE_4 \\ \widetilde{T}(E_1,E_3) &= -aE_2 \\ \widetilde{T}(E_1,E_4) &= -bE_2 \\ \widetilde{T}(E_2,E_3) &= aE_1 \\ \widetilde{T}(E_2,E_4) &= bE_1. \end{aligned} \tag{3.7}$$

   Then, if $a = b = 0$, $\widetilde{T} = 0$ and M is symmetric. Let us suppose $(a,b) \neq (0,0)$, $(\Leftrightarrow a^2 + b^2 \neq 0)$. We introduce a new orthogonal basis

$$e_1 = \frac{E_1}{\varrho} \quad e_2 = \frac{E_2}{\varrho} \quad e_3 = \frac{aE_3 + bE_4}{\varrho^2} \quad e_4 = \frac{bE_3 - aE_4}{\varrho^2} \tag{3.8}$$

   with $\varrho = \sqrt{a^2 + b^2} > 0$. Then (3.7) takes the form,

$$\begin{array}{ll} \widetilde{T}(e_1,e_2) = -e_3 & \widetilde{T}(e_1,e_4) = 0 \\ \widetilde{T}(e_1,e_3) = -e_2 & \widetilde{T}(e_2,e_4) = 0 \\ \widetilde{T}(e_2,e_3) = e_1 & \widetilde{T}(e_3,e_4) = 0. \end{array} \tag{3.9}$$

   By the second identity of Bianchi, we get

$$\begin{array}{l} \widetilde{R}(e_2,e_4) = \widetilde{R}(e_1,e_4) = \widetilde{R}(e_3,e_4) = 0 \\ \widetilde{R}(e_i,e_j)e_4 = 0, \quad i,j = 1,2,3. \end{array} \tag{3.10}$$

We can conclude that, if (M,g) is a four-dimensional naturally reductive pseudo-Riemannian manifold of neutral signature $(2,2)$, the decomposition $T_oM = \{e_1,e_2,e_3\} \oplus \{e_4\}$ satisfies the hypothesis of Proposition 2.2. Hence M is a direct product $M = M^3 \times \mathbb{R}$, and M^3 is a Lorentzian naturally reductive manifold, according to Proposition 2.3. But the complete classification of all three-dimensional non-symmetric naturally reductive Lorentzian spaces is known (see [2], [9]).

We are now able to get the following explicit classification:

Theorem 3.2. *Let (M, g) be a four-dimensional connected, simply connected naturally reductive pseudo-Riemannian manifold with neutral signature $(2, 2)$. Then M is symmetric or it is a product of the form $M = M^3 \times \mathbb{R}$, where M^3 is isometric to one of the following spaces:*

a) $SL(2, \mathbb{R})$,

b) $SU(2)$,

c) H_3,

equipped with a suitable left-invariant Lorentzian metric.

Case B: Lorentzian signature.

1. **rk $\boldsymbol{h} = 4$:** this means $det\ h_{ij} \neq 0$, that is $\lambda\mu \neq 0$.

   In this case, there exists a suitable pseudo-orthonormal basis $\{E_1, \ldots, E_4\}$ such as $< E_i, E_j >= \varepsilon_i \delta_{ij}$, with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +1$, $\varepsilon_4 = -1$, and

$$\begin{array}{ll} \widetilde{R}(X,Y)E_1 = \lambda E_2 & \widetilde{R}(X,Y)E_2 = -\lambda E_1 \\ \widetilde{R}(X,Y)E_3 = -\mu E_4 & \widetilde{R}(X,Y)E_4 = -\mu E_3. \end{array} \tag{3.11}$$

   By applying $(\lambda A + \mu B)(\widetilde{T}) = 0$ to (3.1), and taking into account the signature, we get the conditions: $\mu a = \mu b = \lambda c = \lambda d = 0$, that is, $a = b = c = d = 0$, and so $\widetilde{T} = 0$. This means that M is symmetric.

2. **rk $\boldsymbol{h} < 4$:** in this case $det\ h_{ij} = 0$, that is $\lambda\mu = 0$. Then, $\mu \neq 0$ and $h = \mu \xi^3 \wedge \xi^4$.

   With respect to a suitable pseudo-orthonormal basis $\{E_1, \ldots, E_4\}$ such as $< E_i, E_j >= \varepsilon_i \delta_{ij}$, with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ and $= \varepsilon_4 = -1$, it holds

$$\begin{array}{ll} \widetilde{R}(X,Y)E_1 = 0 & \widetilde{R}(X,Y)E_2 = 0. \\ \widetilde{R}(X,Y)E_3 = -\mu E_4 & \widetilde{R}(X,Y)E_4 = -\mu E_3. \end{array} \tag{3.12}$$

   By applying $A(\widetilde{T}) = 0$ to (3.1), with $A = \widetilde{R}(X, Y)$, and taking into account the signature, we obtain

$$\begin{gathered} A(\widetilde{T}(E_1, E_2)) = \widetilde{T}(AE_1, E_2) + \widetilde{T}(E_1, AE_2) \\ \Leftrightarrow -\mu a E_4 + \mu b E_3 = 0. \end{gathered}$$

Thus, since $\mu \neq 0$, we get the conditions: $a = b = 0$. Rewriting (3.1), following formulas hold:

$$\begin{array}{ll} \widetilde{T}(E_1,E_2)=0 & \widetilde{T}(E_2,E_3)=-dE_4 \\ \widetilde{T}(E_1,E_3)=-cE_4 & \widetilde{T}(E_2,E_4)=-dE_3 \\ \widetilde{T}(E_1,E_4)=-cE_3 & \widetilde{T}(E_3,E_4)=cE_1+dE_2. \end{array} \tag{3.13}$$

Then, if $c = d = 0$, $\widetilde{T} = 0$ and M is symmetric. Let be $(c,d) \neq (0,0)$, $(\Leftrightarrow c^2 + d^2 \neq 0)$. We can introduce a new orthogonal basis

$$e_1 = \frac{cE_1 + dE_2}{\varrho^2} \quad e_2 = \frac{dE_1 - cE_2}{\varrho^2} \quad e_3 = \frac{E_3}{\varrho} \quad e_4 = \frac{E_4}{\varrho} \tag{3.14}$$

with $\varrho = \sqrt{c^2 + d^2} > 0$. Then (3.13) takes the form,

$$\begin{array}{ll} \widetilde{T}(e_1,e_3)=-e_4 & \widetilde{T}(e_1,e_2)=0 \\ \widetilde{T}(e_1,e_4)=-e_3 & \widetilde{T}(e_3,e_2)=0 \\ \widetilde{T}(e_3,e_4)=e_2 & \widetilde{T}(e_4,e_2)=0. \end{array} \tag{3.15}$$

By the second identity of Bianchi, we get

$$\begin{array}{l} \widetilde{R}(e_1,e_2)=\widetilde{R}(e_3,e_2)=\widetilde{R}(e_4,e_2)=0 \\ \widetilde{R}(e_i,e_j)e_2=0, \quad i,j=1,3,4. \end{array} \tag{3.16}$$

We can conclude that, if (M,g) is a four-dimensional naturally reductive Lorentzian manifold, the decomposition $T_oM = \{e_1,e_3,e_4\} \oplus \{e_2\}$ satisfies the hypothesis of Proposition 2.2. Hence M is a direct product $M = M^3 \times \mathbb{R}$, and M^3 is a Lorentzian naturally reductive manifold, according to Proposition 2.3. But the complete classification of all three-dimensional non-symmetric naturally reductive Lorentzian spaces is known (see [2], [9]).

We are now able to get the following explicit classification:

Theorem 3.3. *Let (M,g) be a four-dimensional connected, simply connected naturally reductive Lorentzian manifold. Then M is symmetric or it is a product of the form $M = M^3 \times \mathbb{R}$, where M^3 is isometric to one of the following spaces:*

a) $SL(2,\mathbb{R})$,

b) $SU(2)$,

c) H_3,

equipped with a suitable left-invariant Lorentzian metric.

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