J. Appl. Math. & Informatics Vol. **30**(2012), No. 1 - 2, pp. 347 - 352
Website: http://www.kcam.biz

SOME ISOMORPHIC PROPERTIES OF NEAR-RINGS

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Abstract. In this paper, we denote that R is a near-ring and G an R-group. We initiate the study of faithful R-group, the substructures of R and G, also quotients of substructure relations between them. Next, we investigate some isomorphic properties of near-rings and R-groups.

AMS Mathematics Subject Classification : 16Y30.
Key words and phrases : zero symmetric part, constant part, representation, annihilator, monogenic R-group and homomorphism.

1. Introduction

A near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $a(b + c) = ab + ac$ for all a, b, c in R. We note that obviously, $a0 = 0$ and $a(-b) = -ab$ for all a, b in R, but in general, $0a \neq 0$ and $(-a)b \neq -ab$.

If R has a unity 1, then R is called *unitary*. An element d in R is called *distributive* if $(a + b)d = ad + bd$ for all a and b in R.

An *ideal* of R is a subset I of R such that (i) $(I, +)$ is a normal subgroup of $(R, +)$, (ii) $aI \subset I$ for all $a \in R$, (iii) $(I + a)b - ab \subset I$ for all $a, b \in R$. If I satisfies (i) and (ii), then it is called a *left ideal* of R. If I satisfies (i) and (iii), then it is called a *right ideal* of R.

On the other hand, an *R-subgroup* of R is a subset H of R such that (i) $(H, +)$ is a subgroup of $(R, +)$, (ii) $RH \subset H$ and (iii) $HR \subset H$. If H satisfies (i) and (ii), then it is called a *left R-subgroup* of R. If H satisfies (i) and (iii), then it is called a *right R-subgroup* of R. In case that $(H, +)$ is normal in above, we say that *normal R-subgroup*, *normal left R-subgroup* and *normal right R-subgroup* instead of R-subgroup, left R-subgroup and right R-subgroup, respectively. Note that the normal left R-subgroups of R are equivalent to the left ideals of R.

Received May 13, 2011. Revised September 7, 2011. Accepted September 17, 2011.

We consider the following substructures of near-rings: Given a near-ring R, $R_0 = \{a \in R \mid 0a = 0\}$ which is called the *zero symmetric part* of R,

$$R_c = \{a \in R \mid 0a = a\} = \{a \in R \mid ra = a,\ for\ all\ r \in R\} = \{0a \in R \mid a \in R\}$$

which is called the *constant part* of R, and $R_d = \{a \in R \mid a\ is\ distributive\}$ which is called the *distributive part* of R.

A non-empty subset S of a near-ring R is said to be a *subnear-ring* of R, if S is a near-ring under the operations of R, equivalently, for all a, b in S, $a - b \in S$ and $ab \in S$. Sometimes, we denote it by $S < R$.

We note that R_0 and R_c are subnear-rings of R, but R_d is not a subnear-ring of R. A near-ring R with the extra axiom $0a = 0$ for all $a \in R$, that is, $R = R_0$ is said to be *zero symmetric,* also, in case $R = R_c$, R is called a *constant* near-ring, and in case $R = R_d$, R is called a *distributive* near-ring.

Moreover, we note that R_0 is a right ideal of R, but not generally ideal of R, also R_c is an R-subgroup of R, but in general neither a right nor a left ideal of of R.

Let $(G, +)$ be a group (not necessarily abelian). We may obtain some examples of near-rings as following:

First, if we define multiplication on G as $xy = y$ for all x, y in G, then $(G, +, \cdot)$ is a near-ring, because $(xy)z = z = x(yz)$ and $x(y+z) = y+z = xy+xz$, for all x, y, z in G, but in general, $0x = 0$ and $(x+y)z = xz+yz$ are not true. These kinds of near-rings are constant near-rings.

Next, in the set

$$M(G) = \{f \mid f : G \longrightarrow G\}$$

of all the self maps of G, if we define the sum $f + g$ of any two mappings f, g in $M(G)$ by the rule $x(f+g) = xf + xg$ for all $x \in G$ and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G), +, \cdot)$ becomes a near-ring. It is called the *self map near-ring* on the group G. Also, we can define the substructures of $(M(G), +, \cdot)$ as following: $M_0(G) = \{f \in M(G) \mid 0f = 0\}$ and $M_c(G) = \{f \in M(G) \mid f\ is\ constant\}$.

For the remainder concepts and results on near-rings, we refer to G. Pilz [5].

2. Some isomorphic properties of near-rings and R-Groups

Let R and S be two near-rings. Then a mapping f from R to S is called a near-ring homomorphism [5] if (i) $(a+b)f = af + bf$, (ii) $(ab)f = afbf$, for all $a,\ b \in R$. Obviously, $Rf < S$ and $Tf^{-1} = \{a \in R \mid\ af \in T\} < R$ for any $T < S$. As in ring theory, Rf is called the image of f which is denoted by Imf, also, $\{0\}f^{-1} = \{a \in R \mid\ af = 0\}$ is called the kernel of f which is denoted by $Kerf$.

We can replace homomorphism by monomorphism, epimorphism, isomorphism, endomorphism and automorphism, if these terms have their usual meanings as in ring theory [1].

Let R be any near-ring and G an additive group. Then G is called an *R-group* if there exists a near-ring homomorphism

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Such a homomorphism θ is called a *representation* of R on G. We denote it by G_R.

We may write that xr (as a scalar product in G) for $x(r\theta)$ for all $x \in G$ and $r \in R$. If R is unitary, then R-group G is called *unitary*. Thus an R-group is an additive group G satisfying (i) $x(a+b) = xa + xb$, (ii) $x(ab) = (xa)b$ and (iii) $x1 = x$ (if R has a unity 1), for all $x \in G$ and $a,\ b \in R$.

Naturally, we can define a new concept of R-group: An R-group G is called *distributive*, in case $(x+y)a = xa + ya$, for all $x, y \in G$ and for each $a \in R$. For example, every distributive near-ring R is a distributive R-group.

Evidently, every near-ring R can be given the structure of an R-group (unitary, if R is unitary) by right multiplication in R. Moreover, every group G has a natural $M(G)$-group structure, from the representation of $M(G)$ on G by applying the $f \in M(G)$ to the $x \in G$ as a scalar multiplication xf.

A representation θ of R on G is called $faithful$ if $Ker\theta = \{0\}$, that is, θ is a monomorphism, equivalently, $xr = 0$ implies $r = 0$, for all $x \in G$ and $r \in R$. In this case, we say that G is called a *faithful R-group*.

For an R-group G, a subgroup T of G such that $TR \subset T$ is called an *R-subgroup* of G, a normal subgroup N of G such that $NR \subset N$ is called a *normal R-subgroup* of G, and an *R-ideal* of G is a normal subgroup N of G such that $(N+x)a - xa \subset N$ for all $x \in G,\ a \in R$. Also, note that the R-ideals of R_R are equivalent to the right ideals of R.

Let R be a near-ring and let G be an R-group. If there exists x in G such that $G = xR$, that is, $G = \{xr \mid r \in R\}$, then G is called a *monogenic R-group* and the element x is called a *generator* of G ([5]).

Also, for R-groups G and T, a mapping f from G to T is called an R-group homomorphism [5] if (i) $(x+y)f = xf + yf$, (ii) $(xa)f = xfa$, for all $x,\ y \in G$ and $a \in R$. Analogously, we can consider monomorphism, epimorphism, isomorphism, endomorphism and automorphism for R-groups.

Now, we consider that the substructures of R and G, also quotients of substructure relations between them.

Let G be an R-group and $K,\ K_1$ and K_2 be nonempty subsets of G. Define

$$(K_1 : K_2) := \{a \in R | K_1 a \subset K_2\}.$$

We abbreviate that for $x \in G$

$$(\{x\} : K) =: (x : K).$$

Similarly, $(K : x)$ is defined. $(K : o)$ is called the *annihilator* of K in R, denoted it by $Ann(K)$. We say that G is a *faithful R-group* or that R *acts faithfully* on G if $Ann(G) = \{0\}$, that is, $(G : o) = \{0\}$.

Lemma 2.1 ([3]). *Let G be an R-group and K_1 and K_2 subsets of G. Then we have the following conditions:*

(1) *If K_2 is a (normal) subgroup of G, then $(K_1 : K_2)$ is a (normal) subgroup of R.*
(2) *If K_2 is an R-subgroup of G, then $(K_1 : K_2)$ is a right R-subgroup of R.*
(3) *If K_2 is an R-ideal of G and K_1 is an R-subgroup of G, then $(K_1 : K_2)$ is a two-sided ideal of R.*

Corollary 2.2 ([3]). *Let R be a near-ring and G an R-group.*

(1) *For any $x \in G$, $(x : o)$ is a right ideal of R.*
(2) *For any R-subgroup K of G, $(K : o)$ is a two-sided ideal of R.*
(3) *For any subset K of G, $(K : o) = \bigcap_{x\in K}(x : o)$.*
(4) *If G is faithful, then $\bigcap_{x\in G}(x : o) = \{0\}$.*

From now on, we will consider the isomorphism theorem in near-rings (or, R-groups) which is only mentioned already in [5], we can reprove it more concretely as following.

Let $f : R \longrightarrow S$ be a near-ring homomorphism. Then certainly, $f : R^+ \longrightarrow S^+$ be a group homomorphism, where $R^+ = (R, +)$, and so as group

$$R^+/Kerf \cong R^+f.$$

Putting $K := Kerf$, $(K, +)$ is a normal subgroup of $(R, +)$ and $R/K = \{a + K|a \in R\}$. The addition in R defines an addition in R/K by

$$(a + K) + (b + K) = (a + b) + K.$$

This addition is well defined in group theory.

Would it make

$$(a + K)(b + K) = ab + K$$

a well defined binary operation? It is affirmative in the following statement:

Lemma 2.3 (Isomorphism Theorem). *Let K be the kernel of a near-ring homomorphism $f : R \longrightarrow S$. Then $(R/K, +, \cdot) \cong Imf$.*

Proof. If $(a+K)(b+K) = ab+K$ is a well defined binary operation, then easily, $(R/K, +, \cdot)$ is a near-ring.

Suppose that $a + K = a' + K$ and $b + K = b' + K$. Then there exist $x, y \in K$ such that $a = a' + x$ and $b = b' + y$. We need to show that $ab + K = a'b' + K$ or equivalently, $ab - a'b' \in K$.

Now, $ab = (a' + x)(b' + y) = (a' + x)b' + (a' + x)y$. Since $(a' + x)y$ is in K, putting $(a'+x)y = k$ in K, $ab = (a'+x)b'+k$ and $ab-a'b' = (a'+x)b'+k-a'b' = (a' + x)b' - a'b' + k' \in K$, for some $k' \in K$. Hence, multiplication is well defined.

As groups, $(R/K, +) \cong (Rf, +)$, where a mapping $F : R/K \longrightarrow Rf$ which is defined by $(a + K)F = af$ is the group isomorphism. Now, we have

$$((a + K)(b + K))F = (ab + K)F = abf = afbf = (a + K)F(b + K)F.$$

Consequently, F is a near-ring isomorphism. □

We can obtain the following fundamental theorem in near-ring homomorphism:

Theorem 2.4. *Let $f : R \longrightarrow S$ be a near-ring epimorphism with the kernel K of f, and let $\pi : R \longrightarrow R/K$ defined by $a\pi = a + K$ be the natural epimorphism. Then the isomorphism $F : R/K \longrightarrow S$ which is defined by $(a + K)F = af$ is unique such that $\pi F = f$.*

Proof. By Lemma 2.3, there exists a near-ring isomorphism $f : R \longrightarrow S$.

Next, to show that $\pi F = f$, let $a \in R$, and we get $a(\pi F) = (a\pi)F = (a + K)F = af$. Hence, $\pi F = f$.

Finally, to show that the "uniqueness", if $F' : R/K \longrightarrow S$ is a near-ring isomorphism such that $\pi F' = f$, then for all $a + K \in R/K$, we have

$$(a + K)F' = (a\pi)F' = a(\pi F') = af = (a\pi)F = (a + K)F.$$

□

Analogously, we can prove the isomorphism theorem and fundamental theorem for R-groups.

Theorem 2.5. *If R is a near-ring and G an R-group, then $R/Ann(G)$ is isomorphic to a subnear-ring of $M(G)$.*

Proof. Let $a \in R$. We define $\tau_a : G \longrightarrow G$ by $x\tau_a = xa$ for each $x \in G$. Then τ_a is in $M(G)$. Consider the mapping $\phi : R \longrightarrow M(G)$ defined by $a\phi = \tau_a$. Then obviously, we see that $(a + b)\phi = a\phi + b\phi$ and $(ab)\phi = a\phi b\phi$, that is, ϕ is a near-ring homomorphism from R to $M(G)$.

Next, we must show that $Ker\phi = Ann(G)$. Indeed, if $a \in Ker\phi$, then $\tau_a = 0$, which implies that $Ga = G\tau_a = 0$, that is, $a \in Ann(G)$. On the other hand, if $a \in Ann(G)$, then by the definition of $Ann(G)$, $Ga = 0$ hence $0 = \tau_a = a\phi$, this implies that $a \in Ker\phi$. Therefore from the isomorphism theorem on R-groups, the image of R is a near-ring isomorphic to $R/Ann(G)$. Consequently, $R/Ann(G)$ is isomorphic to a subnear-ring of $M(G)$. □

Thus we can obtain the following important statement as in ring theory.

Corollary 2.6. *If G is a faithful R-group, then R is embedded in $M(G)$.*

Theorem 2.7. *Let R be a near-ring and G an R-group. Then we have the following statements:*

(1) *$Ann(G)$ is a two-sided ideal of R. Moreover G is a faithful $R/Ann(G)$-group.*

(2) *For any $x \in G$, we get $xR \cong R/(x : o)$ as R-groups.*

Proof. (1) By Corollary 2.2 and Lemma 2.1, $Ann(G)$ is a two-sided ideal of R. We now make G an $R/Ann(G)$-group by defining, for $r \in R, r + Ann(G) \in R/Ann(G)$, the action $x(r + Ann(G)) = xr$. If $r + Ann(G) = r' + Ann(G)$, then

$-r' + r \in Ann(G)$ hence $x(-r' + r) = 0$ for all x in G, that is to say, $xr = xr'$. This tells us that

$$x(r + Ann(G)) = xr = xr' = x(r' + Ann(G));$$

thus the action of $R/Ann(G)$ on G has been shown to be well defined. The verification of the structure of an $R/Ann(G)$-group is a routine triviality. Finally, to see that G is a faithful $R/Ann(G)$-group, we note that if $x(r + Ann(G)) = 0$ for all $x \in G$, then by the definition of $R/Ann(G)$-group structure, we have $xr = 0$. Hence $r \in Ann(G)$. This says that only the zero element of $R/Ann(G)$ annihilates all of G. Thus G is a faithful $R/Ann(G)$-group.

(2) For any $x \in G$, clearly xR is an R-subgroup of G. The map $\phi : R \longrightarrow xR$ defined by $\phi(r) = xr$ is an R-epimorphism, so that from the isomorphism theorem in near-ring theory and the kernel of ϕ is $(x : o)$, we deduce that

$$xR \cong R/(x : o)$$

as R-groups. □

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