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ON CONVERGENCES FOR ARRAYS OF ROWWISE PAIRWISE NEGATIVELY QUADRANT DEPENDENT RANDOM VARIABLES†

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Abstract. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise and pairwise negatively quadrant dependent random variables with mean zero, $\{a_{ni}, i \geq 1, n \geq 1\}$ an array of weights and $\{b_n, n \geq 1\}$ an increasing sequence of positive integers. In this paper we consider some results concerning complete convergence of $\sum_{i=1}^{b_n} a_{ni}X_{ni}$.

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1. Introduction

A sequence $\{X_n, n \geq 1\}$ of random variables is said to converge completely to a constant c if $\sum_{n=1}^{\infty} P\{|X_n - c| \geq \epsilon\} < \infty$ for all $\epsilon > 0$. This concept of complete convergence was introduced by Hsu and Robbins(1947). Moreover, they proved that the sequence of arithmetic means of independent and identically distributed(i.i.d.) random variables converges completely to the expected value if the variance of summands is finite. This result has been generalized and extended in several directions(see Baum and Katz(1965), Chow(1973), Gut(1992, 1993), Li et al.(1992, 1995), Liang and Su(1999), Liang(2000), Hu et al.(2001), and Ahmed et al.(2002)). In particular, Kuczmaszewska(2009) obtained the following result:

Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise negatively associated random variables and $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers. Let $\{b_n, n \geq 1\}$ be an increasing sequence of positive integers and $\{c_n, n \geq 1\}$ be a sequence positive numbers. If for some $q > 2$, $0 < t < 2$ and any $\epsilon > 0$ the following conditions are fulfilled

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(a) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\{|a_{ni}X_{ni}| \geq \epsilon b_n^{\frac{1}{t}}\} < \infty$,
(b) $\sum_{n=1}^{\infty} c_n b_n^{-\frac{q}{t}} \sum_{i=1}^{b_n} |a_{ni}|^q E|X_{ni}|^q I[|a_{ni}X_{ni}| < \epsilon b_n^{\frac{1}{t}}] < \infty$,
(c) $\sum_{n=1}^{\infty} c_n b_n^{-\frac{q}{t}} (\sum_{i=1}^{b_n} a_{ni}^2 EX_{ni}^2 I[|a_{ni}X_{ni}| < \epsilon b_n^{\frac{1}{t}}])^{\frac{q}{2}} < \infty$,

then $\sum_{n=1}^{\infty} c_n P[\max_{1\leq i\leq b_n} |\sum_{j=1}^{i}(a_{nj}X_{nj} - a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}])|$
$\geq \epsilon b_n^{\frac{1}{t}}] < \infty$.

Next, we turn our attention to dependence structures for random variables. Lehmann(1966) introduced a simple and natural definition of bivariate dependence: A sequence $\{X_n, n \geq 1\}$ of random variables is said to be pairwise negatively quadrant dependent if for any r_i, r_j and $i \neq j$, $P(X_i > r_i, X_j > r_j) \leq P(X_i > r_i)P(X_j > r_j)$.

Negative quadrant dependence is shown to be weaker than negative association which is a key concept of negative dependence studied by Joag-Dev and Proschan(1983). Matula(1992) proved a strong law of large numbers, Wu(2006) showed the maximal inequality and Meng and Lin(2009) obtained the weak law of large numbers for negatively quadrant dependent random variables. But there are few literature on the complete convergence for negatively quadrant dependent random variables.

Inspired by Kuczmaszewska(2009) we investigate some results concerning complete convergence of weighted sums $\sum_{i=1}^{b_n} a_{ni}X_{ni}$, where $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of constants(weights), $\{X_{ni}, i \geq 1, n \geq 1\}$ is an array of rowwise pairwise negatively quadrant dependent random variables and $\{b_n, n \geq 1\}$ is an increasing sequence of positive integers.

2. Preliminaries

Lemma 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise negatively quadrant dependent random variables and $\{f_n, n \geq 1\}$ be a sequence of nondecreasing functions. Then $\{f_n(X_n), n \geq 1\}$ is still a sequence of pairwise negatively quadrant dependent random variables.*

Definition 2.2. A real valued function $h(x)$, positive and measurable on $[a, \infty)$ for some $a > 0$, is said to be slowly varying if

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = 1 \text{ for each } \lambda > 0.$$

Lemma 2.3 (Bai and Su(1985)**).** *If $h(x) > 0$ is slowly varying function as $x \to \infty$, then*
(a) $\lim_{x\to\infty} \frac{h(x+u)}{h(x)} = 1$ for each $u > 0$,
(b) $\lim_{k\to\infty} \sup_{2^k\leq x<2^{k+1}} \frac{h(x)}{h(2^k)} = 1$,
(c) $c_1 2^{kr}h(\epsilon 2^k) \leq \sum_{j=1}^{k} 2^{jr}h(\epsilon 2^j) \leq c_2 2^{kr}h(\epsilon 2^k)$ for every $r > 0, \epsilon > 0$, positive integer k and some positive constants c_1 and c_2,
(d) $c_3 2^{kr}h(\epsilon 2^k) \leq \sum_{j=k}^{\infty} 2^{jr}h(\epsilon 2^j) \leq c_4 2^{kr}h(\epsilon 2^k)$ for every $r < 0, \epsilon > 0$,

positive integer k and some positive constants c_3 and c_4.

Definition 2.4. An array $\{X_{nj}, j \geq, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant D such that $P\{|X_{nj}| > x\} \leq DP\{D|X| > x\}$ for all $x \geq 0, j \geq 1$ and $n \geq 1$.

3. Main results

Theorem 3.1. *Let $\{X_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise pairwise negatively quadrant dependent random variables with mean zero and $\{a_{nj}, j \geq 1, n \geq 1\}$ be an array of positive numbers. Let $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive integers and $\{c_n, n \geq 1\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} c_n = \infty$. Assume that for some $0 < t < 2$ and any $\epsilon > 0$*

$$\sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} P\{|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}\} < \infty \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} \sum_{j=1}^{b_n} a_{nj}^2 E(X_{nj})^2 I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}] < \infty. \tag{3.2}$$

Then

$$\sum_{n=1}^{\infty} c_n P\{|\sum_{j=1}^{b_n}(a_{nj}X_{nj} - a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}])| \geq \epsilon b_n^{\frac{1}{t}}\} < \infty. \tag{3.3}$$

Proof. Let

$$\tilde{X_{nj}} = X_{nj}I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}] + \frac{\epsilon b_n^{\frac{1}{t}}}{a_{nj}} I[a_{nj}X_{nj} \geq \epsilon b_n^{\frac{1}{t}}] - \frac{\epsilon b_n^{\frac{1}{t}}}{a_{nj}} I[a_{nj}X_{nj} < -\epsilon b_n^{\frac{1}{t}}],$$

$Y_{nj} = \tilde{X_{nj}} - E\tilde{X_{nj}}$.

Since $a_{nj}EX_{nj}I[|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}] = -a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}]$, it follows from (3.1) that, for sufficient large n

$$\begin{aligned} &P\{|\sum_{j=1}^{b_n}(a_{nj}X_{nj} - a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}])| \geq \epsilon b_n^{\frac{1}{t}}\} \\ &\leq \sum_{j=1}^{b_n} P\{|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}\} + \epsilon^{-2} b_n^{-\frac{2}{t}} E(\sum_{j=1}^{b_n} Y_{nj})^2. \end{aligned} \tag{3.4}$$

We estimate

$$EY_{nj}^2 \leq EX_{nj}^2 I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}] + \frac{\epsilon^2 b_n^{\frac{2}{t}}}{a_{nj}^2} P\{|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}\}. \tag{3.5}$$

Thus by (3.4) and (3.5) we get

$$
\begin{aligned}
(3.6)\qquad & P\{|\sum_{j=1}^{b_n} a_{nj}X_{nj} - a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}]| \geq \epsilon b_n^{\frac{1}{t}}\} \\
\leq \quad & 2\sum_{j=1}^{b_n} P\{|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}\} \\
& + \epsilon^{-2} b_n^{-\frac{2}{t}} \{\sum_{j=1}^{b_n} a_{nj}^2 EX_{nj}^2 I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}]\} = I + II,
\end{aligned}
$$

which yields (3.3) since $\sum_{n=1}^{\infty} c_n I < \infty$ by (3.1) and $\sum_{n=1}^{\infty} c_n II < \infty$ by (3.2). □

Theorem 3.2. *Let $\{X_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise pairwise negatively quadrant dependent random variables with $EX_{nj} = 0$ for all $j \geq 1, n \geq 1$, $\{a_{nj}, j \geq 1, n \geq 1\}$ be an array of positive real numbers and $\{b_n, n \geq 1\}$ be an increasing sequence of positive integers. Assume that for some sequence $\{\lambda_n, n \geq 1\}$ with $0 < \lambda_n \leq 1$ we have $E|X_{nj}|^{1+\lambda_n} < \infty$ for $1 \leq j \leq b_n, n \geq 1$. If for some sequence $\{c_n, n \geq 1\}$ of positive real numbers with $\sum_{n=1}^{\infty} c_n = \infty$ and $0 < t < 2$*

$$
(3.7)\qquad \sum_{n=1}^{\infty} c_n (b_n^{\frac{1}{t}})^{-1-\lambda_n} \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{nj}|^{1+\lambda_n} < \infty,
$$

then for any $\epsilon > 0$

$$
(3.8)\qquad \sum_{n=1}^{\infty} c_n P\{|\sum_{j=1}^{b_n} a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}\} < \infty.
$$

Proof. By assumption $\sum_{n=1}^{\infty} c_n = \infty$ and (3.7) we have

$$
(3.9)\qquad (b_n^{\frac{1}{t}})^{-1-\lambda_n} \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{nj}|^{1+\lambda_n} < 1.
$$

By the assumption (3.7) we estimate

$$
\begin{aligned}
& \sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} P\{|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}\} \\
< \quad & \sum_{n=1}^{\infty} c_n (\epsilon b_n^{\frac{1}{t}})^{-1-\lambda_n} \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{nj}|^{1+\lambda_n} < \infty,
\end{aligned}
$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} \sum_{j=1}^{b_n} a_{nj}^2 E|X_{ni}|^2 I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}] \\
\leq\ & \sum_{n=1}^{\infty} c_n (b_n^{\frac{1}{t}})^{-1-\lambda_n} \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{nj}|^{1+\lambda_n} < \infty,
\end{aligned}$$

from which (3.1) and (3.2) of Theorem 3.1 are fulfilled.

To complete the proof, it remains to show that

$$I_0 = b_n^{-\frac{1}{t}} |\sum_{j=1}^{b_n} a_{nj} EX_{nj} I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}]| \to 0 \ as\ n \to \infty.$$

We estimate

$$\begin{aligned}
I_0 &\leq b_n^{-\frac{1}{t}} \sum_{j=1}^{b_n} |a_{nj} EX_{nj} I[|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}]| \\
&\leq b_n^{-\frac{1}{t}} \sum_{j=1}^{b_n} |a_{nj}| E|X_{nj}| I[|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}] \\
&\leq b_n^{-\frac{1}{t}} \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{nj}|^{1+\lambda_n} (\epsilon b_n^{\frac{1}{t}})^{-\lambda_n} \\
&\leq \epsilon^{-\lambda_n} (b_n^{\frac{1}{t}})^{-1-\lambda_n} \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{nj}|^{1+\lambda_n} \to 0,\ as\ n \to \infty
\end{aligned}$$

by assumption (3.7). □

Lemma 3.3. *Let $\{X_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise pairwise negatively quadrant dependent dependent random variables and $\{a_{nj}, j \geq 1, n \geq 1\}$ be an array of positive numbers. Let $h(x) > 0$ be a slowly varying function as $x \to \infty$, $\alpha > \frac{1}{2}$ and $\alpha r \geq 1$. If for $0 < t < 2$ the following conditions hold for any $\epsilon > 0$*

$$\sum_{n=1}^{\infty} n^{\alpha r-2} h(n) \sum_{j=1}^{n} P\{|a_{nj}X_{nj}| \geq \epsilon n^{\frac{1}{t}}\} < \infty, \tag{3.10}$$

$$\sum_{n=1}^{\infty} n^{\alpha r-2-\frac{2}{t}} h(n) \sum_{j=1}^{n} a_{nj}^2 EX_{nj}^2 I[|a_{nj}X_{nj}| < \epsilon n^{\frac{1}{t}}] < \infty, \tag{3.11}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha r-2} h(n) P\{|\sum_{j=1}^{b_n} a_{nj}X_{nj} - a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon n^{\frac{1}{t}}]| \geq \epsilon n^{\frac{1}{t}}\}. \tag{3.12}$$

Proof. Let $c_n = n^{\alpha r-2}h(n)$ and $b_n = n$. Then by Theorem 3.1 (3.12) follows. □

Theorem 3.4. *Let $\{X_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise and identically distributed pairwise negatively quadrant dependent random variables with $EX_{11} = 0$ and let $h(x) > 0$ be a slowly varying function as $x \to \infty$. If for $\alpha > \frac{1}{2}$, $\alpha r \geq 1$ and $0 < t < 2$, $E|X_{11}|^{\alpha rt}h(|X_{11}|^t) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\alpha r-2}h(n)P\{|\sum_{j=1}^{n} X_{nj}| \geq \epsilon n^{\frac{1}{t}}\} < \infty. \tag{3.13}$$

Proof. It is enough to show that under the assumptions of Theorem 3.4, for $a_{nj} = 1, j \geq 1, n \geq 1$ the conditions (3.10) and (3.11) of Lemma 3.3 hold. Indeed, by Lemma 2.3 we obtain

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{\alpha r-1}h(n)P\{|X_{11}| \geq \epsilon n^{\frac{1}{t}}\} \\
\leq\ & C\sum_{k=1}^{\infty}(2^k)^{\alpha r}h(2^k)P\{|X_{11}| \geq \epsilon(2^k)^{\frac{1}{t}}\} \\
\leq\ & C\sum_{m=1}^{\infty} P\{\epsilon(2^m)^{\frac{1}{t}} \leq |X_{11}| < \epsilon(2^{m+1})^{\frac{1}{t}}\}\sum_{j=1}^{m}(2^j)^{\alpha r}h(2^j) \\
\leq\ & C\sum_{m=1}^{\infty}(2^m)^{\alpha r}h(2^m)P\{\epsilon(2^m)^{\frac{1}{t}} \leq |X_{11}| < \epsilon(2^{m+1})^{\frac{1}{t}}\} \\
\leq\ & CE|X_{11}|^{\alpha rt}h(|X_{11}|^t) < \infty,
\end{aligned}$$

from which (3.10) is satisfied.

To prove that (3.11) is fulfilled, we first note that

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{\alpha r-1-\frac{2}{t}}h(n)E|X_{11}|^2 I[|X_{11}| < \epsilon n^{\frac{1}{t}}] \\
\leq\ & C\sum_{k=1}^{\infty}(2^k)^{\alpha r-\frac{2}{t}}h(2^k)\int_0^{(2^k)^{\frac{1}{t}}} x^2 dF(x) \\
\leq\ & C\sum_{k=1}^{\infty}(2^k)^{\alpha r-\frac{2}{t}}h(2^k)\sum_{i=1}^{k}\int_{(2^{i-1})^{\frac{1}{t}}}^{(2^i)^{\frac{1}{t}}} x^2 dF(x) \\
\leq\ & C\sum_{m=1}^{\infty}(2^m)^{\alpha r-\frac{2}{t}}h(2^m)\int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}} x^2 dF(x) \\
=\ & C\sum_{m=1}^{\infty}(2^m)^{\alpha r-\frac{2}{t}}\int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}} \frac{h(2\times 2^{m-1})}{h(|x|^t)}h(|x|^t)x^2 dF(x) \\
=\ & I_1.
\end{aligned}$$

By (b) of Lemma 2.3, we see that for sufficiently large m

$$I_1 \leq C \sum_{m=1}^{\infty} (2^m)^{\alpha r - \frac{2}{t}} \int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}} h(|x|^t) x^2 dF(x).$$

Then,

$$\begin{aligned} & \sum_{m=1}^{\infty} (2^m)^{\alpha r - \frac{2}{t}} \int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}} h(|x|^t) x^2 dF(x) \\ \leq & \sum_{m=1}^{\infty} \int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}} (|x|^t)^{\alpha r} h(|x|^t) x^2 dF(x) \\ = & E|X_{11}|^{\alpha r t} h(|X_{11}|^t) < \infty. \end{aligned}$$

Hence, (3.11) is satisfied. Finally, to complete the proof it remains to show that, for each $1 \leq j \leq n$

$$n^{-\frac{1}{t}} j |EX_{11} I[|X_{11}| < \epsilon n^{\frac{1}{t}}]| \to 0 \ as \ n \to \infty.$$

If $\alpha r t < 1$, then

$$n^{-\frac{1}{t}} j |EX_{11} I[|X_{11}| < \epsilon n^{\frac{1}{t}}] \leq (\epsilon)^{1-\alpha r t} n^{1-\alpha r} E|X_{11}|^{\alpha r t} \to 0 \ as \ n \to \infty.$$

If $\alpha r t \geq 1$, then by $EX_{11} = 0$ we obtain

$$\begin{aligned} & n^{-\frac{1}{t}} j |EX_{11} I[|X_{11}| < \epsilon n^{\frac{1}{t}}] \leq n^{1-\frac{1}{t}} | - EX_{11} I[|X_{11}| \geq \epsilon n^{\frac{1}{t}}]| \\ \leq & (\epsilon)^{1-\alpha r t} n^{1-\alpha r} E|X_{11}|^{\alpha r t} \to 0 \ as \ n \to \infty. \end{aligned}$$

Hence, the proof of Theorem 3.4 is complete. □

Theorem 3.5. *Let $\{X_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise pairwise quadrant dependent random variables with mean zero and $\{a_{nj}, j \geq 1, n \geq 1\}$ be an array of positive numbers. Let $\{c_n, n \geq 1\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} c_n = \infty$. Assume that for any $\epsilon > 0$*

$$\sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} P\{|a_{nj} X_{nj}| \geq \epsilon\} < \infty \tag{3.14}$$

and

$$\sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} a_{nj}^2 E(X_{nj})^2 I[|a_{nj} X_{nj}| < \epsilon] < \infty. \tag{3.15}$$

Then

$$\sum_{n=1}^{\infty} c_n P\{|\sum_{j=1}^{b_n} (a_{nj} X_{nj} - a_{nj} E X_{nj} I[|a_{nj} X_{nj}| < \epsilon])| \geq \epsilon\} < \infty. \tag{3.16}$$

Proof. In Theorem 3.1 take $b_n = 1$. Then the result follows. □

Corollary 3.6. *Let $\{X_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise pairwise quadrant dependent random variables with mean zeros. Let the random variables in each row be stochastically domianted by a random variable X with $EX^2 < \infty$ and let $\{a_{nj}, j \geq 1, n \geq 1\}$ be an array of positive numbers such that $\lim_{n\to\infty} a_{nj} = 0$ for each $j \geq 1$. If for some $\delta > 1$*

$$\sup_{j\geq 1} |a_{nj}| = O(n^{-\delta}) \text{ and } \sum_{j=1}^{\infty} |a_{nj}| \leq C \text{ for all } n \geq 1, \tag{3.17}$$

where C is a positive constant, then for any $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\{|\sum_{j=1}^{n} a_{nj}X_{nj}| \geq \epsilon\} < \infty.$$

Proof. Let $c_n = 1,\ b_n = n$ for $n \geq 1$ in Theorem 3.5. Then, by Definition 2.4 and (3.17) we have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{j=1}^{n} P\{|a_{nj}X_{nj}| \geq \epsilon\} &\leq \sum_{n=1}^{\infty}\sum_{j=1}^{n} P\{|a_{nj}X| \geq \frac{\epsilon}{D}\} \\ \leq \quad \sum_{n=1}^{\infty}\sum_{j=1}^{n} P\{|X| \geq C\epsilon n^{\delta}\} &\leq C\sum_{n=1}^{\infty} n^{1-2\delta}EX^2 < \infty\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{j=1}^{n} a_{nj}^2 EX_{nj}^2 I[|a_{nj}X_{nj}| < \epsilon] &\leq \sum_{n=1}^{\infty}\sum_{j=1}^{n} a_{nj}^2 EX^2 I[|a_{nj}X| < \epsilon] \\ \leq \quad C\sum_{n=1}^{\infty} \sup_{j\geq 1}|a_{nj}|EX^2 \sum_{j=1}^{n}|a_{nj}| &\leq C\sum_{n=1}^{\infty} n^{-\delta}EX^2 < \infty \text{ for } \delta > 1.\end{aligned} \tag{3.19}$$

Hence, (3.14) and (3.15) in Theorem 3.5 are fulfilled. To complete the proof, we need to prove

$$\sum_{j=1}^{n} a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon] \to 0 \ as \ n \to \infty.$$

We have for each $1 \leq j \leq n$

$$\begin{aligned}|\sum_{j=1}^{n} a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon]| &\leq \sum_{j=1}^{n} |a_{nj}||EX_{nj}I[|a_{nj}X_{nj}| < \epsilon]| \\ \leq \quad \sum_{j=1}^{n} |a_{nj}||EX_{nj}I[|a_{nj}X_{nj}| \geq \epsilon]| &\leq Cn^{-\delta}EX^2 \to 0 \ as \ n \to \infty.\end{aligned}$$

Hence the proof is complete by Theorem 3.5. □

Corollary 3.7. *Let $\{X_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise pairwise negatively quadrant dependent random variables with $EX_{nj} = 0$ for all $j \geq 1, n \geq 1$,*

$\{a_{nj}, j \geq 1, n \geq 1\}$ be an array of positive real numbers and $\{b_n, n \geq 1\}$ be an increasing sequence of positive integers. Assume that for some sequence $\{\lambda_n, n \geq 1\}$ of real numbers with $0 < \lambda_n \leq 1$ we have $E|X_{nj}|^{1+\lambda_n} < \infty$ for $1 \leq j \leq b_n, n \geq 1$. If for some sequence $\{c_n, n \geq 1\}$ of positive real numbers with $\sum_{n=1}^{\infty} c_n = \infty$

$$\sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{nj}|^{1+\lambda_n} < \infty, \tag{3.20}$$

then

$$\sum_{n=1}^{\infty} c_n P\{|\sum_{j=1}^{b_n} a_{nj} X_{nj}| > \epsilon\} < \infty. \tag{3.21}$$

Proof. By assumption $\sum_{n=1}^{\infty} c_n = \infty$ we have

$$\sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{nj}|^{1+\lambda_n} < 1. \tag{3.22}$$

By the condition (3.20) we estimate

$$\sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} P\{|a_{nj} X_{nj}| \geq \epsilon\} \leq \sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{nj}|^{1+\lambda_n} < \infty$$

and

$$\begin{aligned} & \sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} a_{nj}^2 E X_{ni}^2 I[|a_{nj} X_{nj}| < \epsilon] \\ \leq \quad & \sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{nj}|^{1+\lambda_n} I[|a_{nj} X_{nj}| \leq \epsilon] < \infty. \end{aligned}$$

Hence (3.14) and (3.15) in Theorem 3.5 are fulfilled. Finally we estimate

$$\begin{aligned} & |\sum_{j=1}^{b_n} a_{nj} E X_{nj} I[|a_{nj} X_{nj}| < \epsilon \log_2 b_n]| \\ \leq \quad & \sum_{j=1}^{b_n} |a_{nj}| E|X_{ni}| I[|a_{nj} X_{nj}| \geq \epsilon \log_2 b_n] \\ \leq \quad & \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{ni}|^{1+\lambda_n} (\epsilon)^{\lambda_n} \leq \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{ni}|^{1+\lambda_n} \\ \leq \quad & \sum_{j=1}^{b_n} |a_{nj}|^{1+\lambda_n} E|X_{ni}|^{1+\lambda_n} < \infty \textit{ as } n \to \infty \end{aligned}$$

by (3.22). Hence, the proof is complete. □

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