J. Appl. Math. & Informatics Vol. **30**(2012), No. 1 - 2, pp. 305 - 320
Website: http://www.kcam.biz

MULTIPLE POSITIVE SOLUTIONS OF INTEGRAL BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS†

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Abstract. In this paper, we study a class of integral boundary value problems for fractional differential equations. By using some fixed point theorems, the results of existence of at least three positive solutions for the boundary value problems are obtained.

AMS Mathematics Subject Classification: 34B15, 26A33.
Key words and phrases : fractional differential equation, Caputo derivative, integral boundary value problem, multiple positive solutions, Fredholm integral equation theory.

1. Introduction

We investigate the the existence of multiple positive solutions for the fractional differential equations with integral boundary conditions

$$\begin{cases} {}^{C}D^{p}u(t)+f(t,u(t),{}^{C}D^{q}u(t))=0, \quad t\in(0,1),\\ u(0)=\int_0^1 g_0(s)u(s)\mathrm{d}s,\\ u(1)+a{}^{C}D^{q}u(1)=\int_0^1 g_1(s)u(s)\mathrm{d}s,\\ u''(0)=u'''(0)=\cdots=u^{(n-1)}(0)=0, \end{cases} \tag{1}$$

where ${}^{C}D^{p}$ and ${}^{C}D^{q}$ are the standard Caputo derivatives, $p>2$, $0<q<1$, $a>0$ are real numbers, $f\in C([0,1]\times[0,+\infty)\times(-\infty,+\infty),[0,+\infty))$, g_0 and g_1 are given functions.

It is well known that fractional differential equations have been applied in various sciences such as physics, mechanics, chemistry, engineering, etc. As a result, fractional differential equations have been intensely studied, see [1], [2] and the references therein.

Received March 7, 2011. Revised June 7, 2011. Accepted June 14, 2011. *Corresponding author. †This work was supported by the Innovation Program of Shanghai Municipal Education Commission(No.10ZZ93), Natural Science Foundation of China(No.11071164) and Anhui Provincial Natural Science Foundation(No.10040606Q50).

Research on boundary value problems of ordinary differential equations of integer order, which involve integer order derivative either in the nonlinear or in the boundary conditions, is much, see [3]-[7]. Recently, there are many papers which deal with the existence of the solutions of two-point, three point, multi-point and integral boundary value problems of fractional differential equations, see [8]-[12]. Some of these papers were done under the assumption that neither the integer order derivative nor the fractional derivative was involved in the nonlinear term or in the boundary value conditions, see [8], [9]. There are some papers considering the existence of the solutions for three points and multi-point boundary value problems with dependence on fractional derivatives, see [10], [11]. Moreover, there are also papers dealing with the existence of the solutions for integral boundary value problems, which involve integer order derivative in the nonlinear term or in the boundary conditions, see [12].

However, research of the existence of at least three positive solutions of integral boundary problems with dependence on fractional derivatives both in the nonlinear term and the boundary conditions is rare. This paper is concerned with the existence of multiple positive solutions for the boundary value problem (BVP) (1). By using the theory of Fredholm integral equations and a fixed point theorem, we obtain the results of existence of at least three positive solutions for the integral boundary value problems, which involve fractional derivative not only in the nonlinear term but also in the integral boundary conditions.

2. Preliminaries

In this section, we will introduce definitions and preliminary facts which are used throughout this paper.

Definition 2.1 ([13]). The fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$I_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s,$$

provided that the right side is point wise defined on $(0, +\infty)$, and Γ denotes the Gamma function.

Definition 2.2 ([13]). The Caputo derivative of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$${}^C D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{x^{(n)}(s)}{(t-s)^{\alpha+1-n}}\mathrm{d}s, \quad n-1 < \alpha < n,$$

provided the right integral converges, where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Throughout the paper, we assume that the following hypothesis holds:

($\mathbf{H}_1$) Let $p > 2$, $0 < q < 1, a > 0$ are real numbers, and $n - 1 = [p] < p < [p] + 1 = n$.

Lemma 2.1. *Suppose that $y \in C[0, 1]$, and (H_1) holds. Then the following* *integral boundary value problem*

$$\begin{cases} {}^C D^p u(t) + y(t) = 0, \quad t \in (0,1), \\ u(0) = \int_0^1 g_0(s)u(s)\mathrm{d}s, \\ u(1) + a {}^C D^q u(1) = \int_0^1 g_1(s)u(s)\mathrm{d}s, \\ u''(0) = u'''(0) = \cdots = u^{(n-1)}(0) = 0 \end{cases} \tag{2}$$

is equivalent to the following fractional integral equation

$$u(t) = \int_0^1 G(t,s)y(s)\mathrm{d}s + \int_0^1 \Phi(t,s)u(s)\mathrm{d}s, \tag{3}$$

where

$$G(t,s) = \begin{cases} \frac{t\Gamma(2-q)\left(a\Gamma(p)(1-s)^{p-q-1} + \Gamma(p-q)(1-s)^{p-1}\right) - (a+\Gamma(2-q))\Gamma(p-q)(t-s)^{p-1}}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)}, \\ \qquad\qquad\qquad\qquad 0 \le s \le t \le 1, \\ \frac{t\Gamma(2-q)\left(a\Gamma(p)(1-s)^{p-q-1} + \Gamma(p-q)(1-s)^{p-1}\right)}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)}, \ 0 \le t \le s \le 1. \end{cases}$$

$$\Phi(t,s) = \frac{\Gamma(2-q)g_1(s)t + [a + \Gamma(2-q)(1-t)]g_0(s)}{a + \Gamma(2-q)}, \ (t,s) \in [0,1] \times [0,1].$$

Proof. By ${}^C D^p u(t) + y(t) = 0$, $t \in (0,1)$ and the boundary conditions $u''(0) = u'''(0) = \cdots = u^{(n-1)}(0) = 0$, we have

$$\begin{aligned} u(t) &= -I_t^p y(t) + u(0) + u'(0)t + \frac{u''(0)}{2!}t^2 + \cdots + \frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1} \\ &= -\frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} y(s)\mathrm{d}s + u(0) + u'(0)t. \end{aligned}$$

According to the properties of Caputo derivative, we get

$$\begin{aligned} {}^C D^q u(t) &= -I_t^{p-q} y(t) + {}^C D^q (u(0) + u'(0)t) \\ &= -\frac{\int_0^t (t-s)^{p-q-1} y(s)\mathrm{d}s}{\Gamma(p-q)} + \frac{u'(0)t^{1-q}}{\Gamma(2-q)}. \end{aligned}$$

Then

$$u(1) = -\frac{1}{\Gamma(p)} \int_0^1 (1-s)^{p-1} y(s)\mathrm{d}s + u(0) + u'(0),$$

and

$${}^C D^q u(1) = -\frac{\int_0^1 (1-s)^{p-q-1} y(s)\mathrm{d}s}{\Gamma(p-q)} + \frac{u'(0)}{\Gamma(2-q)}.$$

By the boundary conditions $u(0) = \int_0^1 g_0(s)u(s)\mathrm{d}s$ and $u(1) + a\, {}^C D^q u(1) = \int_0^1 g_1(s)u(s)\mathrm{d}s$, we have

$$-\frac{1}{\Gamma(p)} \int_0^1 (1-s)^{p-1} y(s)\mathrm{d}s + u(0) + u'(0) - \frac{a}{\Gamma(p-q)} \int_0^1 (1-s)^{p-q-1} y(s)\mathrm{d}s$$

$$+\frac{au'(0)}{\Gamma(2-q)}=\int_0^1 g_1(s)u(s)\mathrm{d}s.$$

Hence,

$$\begin{aligned}u'(0)=&\frac{a\Gamma(2-q)}{(a+\Gamma(2-q))\Gamma(p-q)}\int_0^1(1-s)^{p-q-1}y(s)\mathrm{d}s\\&+\frac{\Gamma(2-q)}{(a+\Gamma(2-q))\Gamma(p)}\int_0^1(1-s)^{p-1}y(s)\mathrm{d}s\\&+\frac{\Gamma(2-q)}{a+\Gamma(2-q)}\int_0^1(g_1(s)-g_0(s))u(s)\mathrm{d}s.\end{aligned}$$

We can easily get that

$$\begin{aligned}u(t)=&-\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}y(s)\mathrm{d}s+\int_0^1 g_0(s)u(s)\mathrm{d}s\\&+\frac{at\Gamma(2-q)}{(a+\Gamma(2-q))\Gamma(p-q)}\int_0^1(1-s)^{p-q-1}y(s)\mathrm{d}s\\&+\frac{t\Gamma(2-q)}{(a+\Gamma(2-q))\Gamma(p)}\int_0^1(1-s)^{p-1}y(s)\mathrm{d}s\\&+\frac{t\Gamma(2-q)}{a+\Gamma(2-q)}(\int_0^1 g_1(s)u(s)\mathrm{d}s-\int_0^1 g_0(s)u(s)\mathrm{d}s)\\=&-\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}y(s)\mathrm{d}s\\&+\frac{at\Gamma(2-q)}{(a+\Gamma(2-q))\Gamma(p-q)}\int_0^1(1-s)^{p-q-1}y(s)\mathrm{d}s\\&+\frac{t\Gamma(2-q)}{(a+\Gamma(2-q))\Gamma(p)}\int_0^1(1-s)^{p-1}y(s)\mathrm{d}s\\&+\frac{t\Gamma(2-q)}{a+\Gamma(2-q)}\int_0^1 g_1(s)u(s)\mathrm{d}s\\&+\frac{a+\Gamma(2-q)(1-t)}{a+\Gamma(2-q)}\int_0^1 g_0(s)u(s)\mathrm{d}s\\=&\int_0^1 G(t,s)y(s)\mathrm{d}s+\int_0^1\Phi(t,s)u(s)\mathrm{d}s.\end{aligned}$$

That is, every solution of (2) is a solution of (3). On the other hand, it is easy to verify that each solution of (3) is a solution of (2). The proof is completed. □

Lemma 2.2. *Suppose (H_1) holds, then the function $G(t,s)$ in Lemma 2.1 satisfies the following conditions:*
(i) $G(t,s)$ is continuous on $[0,1]\times[0,1]$;
(ii) $G(t,s)\geq 0$, for any $(t,s)\in[0,1]\times[0,1]$;
(iii) There exists a constant $r_1>0$ such that $G(t,s)\leq r_1(1-s)^{p-q-1}$, for any

$(t,s)\in[0,1]\times[0,1]$;
(iv) There exists a constant $r_2>0$ such that $G(t,s)\geq r_2(1-s)^{p-q-1}$, for any $(t,s)\in[\xi,1]\times[0,1]$, where $\xi\in(0,1)$;
(v) There exists a constant $r_3>0$ such that $|\frac{\partial G(t,s)}{\partial t}|\leq r_3(1-s)^{p-q-2}$, for any $(t,s)\in[0,1]\times[0,1]$.

Proof. (i) It is easy to check that (i) holds. (ii) Denote for $0\leq s\leq t\leq 1$,

$$G_1(t,s)=-\frac{(t-s)^{p-1}}{\Gamma(p)}+\frac{at\Gamma(p)\Gamma(2-q)(1-s)^{p-q-1}+t\Gamma(2-q)\Gamma(p-q)(1-s)^{p-1}}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)},$$

and for $0\leq t\leq s\leq 1$,

$$G_2(t,s)=\frac{at\Gamma(p)\Gamma(2-q)(1-s)^{p-q-1}+t\Gamma(2-q)\Gamma(p-q)(1-s)^{p-1}}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)}.$$

It is easy to see that $G_2(t,s)\geq 0$, for any $0\leq t\leq s\leq 1$. So we will prove that $G_1(t,s)\geq 0$, for any $0\leq s\leq t\leq 1$. In fact, for $0\leq s<t\leq 1$, we have that

$$\begin{aligned}t(1-s)^{p-1}-(t-s)^{p-1}=&t(1-s)^{p-1}-t^{p-1}(1-\frac{s}{t})^{p-1}\\ \geq&t^{p-1}(1-s)^{p-1}-t^{p-1}(1-\frac{s}{t})^{p-1}\geq 0.\end{aligned}$$

This implies that $(t-s)^{p-1}\leq t(1-s)^{p-1}\leq t(1-s)^{p-q-1}$. Hence,

$$\begin{aligned}&(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)G_1(t,s)\\ =&at\Gamma(2-q)\Gamma(p)(1-s)^{p-q-1}+t\Gamma(2-q)\Gamma(p-q)(1-s)^{p-1}\\ &-(a+\Gamma(2-q))\Gamma(p-q)(t-s)^{p-1}\\ =&a\big(\Gamma(2-q)\Gamma(p)t(1-s)^{p-q-1}-\Gamma(p-q)(t-s)^{p-1}\big)\\ &+\Gamma(2-q)\Gamma(p-q)\big(t(1-s)^{p-1}-(t-s)^{p-1}\big).\end{aligned}$$

Since $\Gamma(p)\Gamma(2-q)>\Gamma(p-q)$, for $p>2, 0<q<1$, then

$$a\big(\Gamma(2-q)\Gamma(p)t(1-s)^{p-q-1}-\Gamma(p-q)(t-s)^{p-1}\big)\geq 0,$$

and

$$\Gamma(2-q)\Gamma(p-q)\big(t(1-s)^{p-1}-(t-s)^{p-1}\big)\geq 0.$$

Hence $G_1(t,s)\geq 0$, for any $0\leq s\leq t\leq 1$. Therefore $G(t,s)\geq 0$, for any $(t,s)\in[0,1]\times[0,1]$.

(iii) For $0\leq t\leq s\leq 1$, we have

$$\begin{aligned}&(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)G_2(t,s)\\ =&at\Gamma(2-q)\Gamma(p)(1-s)^{p-q-1}+t\Gamma(2-q)\Gamma(p-q)(1-s)^{p-1}\\ =&[at\Gamma(2-q)\Gamma(p)+t\Gamma(2-q)\Gamma(p-q)(1-s)^{q}](1-s)^{p-q-1}\\ \leq&[a\Gamma(p)+\Gamma(p-q)]\Gamma(2-q)(1-s)^{p-q-1}.\end{aligned}$$

And for $0\leq s\leq t\leq 1$,

$$(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)G_1(t,s)$$

$$
\begin{aligned}
&\leq at\Gamma(2-q)\Gamma(p)(1-s)^{p-q-1}+t\Gamma(2-q)\Gamma(p-q)(1-s)^{p-1}\\
&\leq [a\Gamma(p)+\Gamma(p-q)]\Gamma(2-q)(1-s)^{p-q-1}.
\end{aligned}
$$

Let

$$r_1=\frac{[a\Gamma(p)+\Gamma(p-q)]\Gamma(2-q)}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)}.$$

Then we have

$$G(t,s)\leq r_1(t-s)^{p-q-1},\ \text{for any } (t,s)\in[0,1]\times[0,1].$$

(iv) We have proved in (ii) that $(t-s)^{p-1}\leq t(1-s)^{p-1}\leq t(1-s)^{p-q-1}$, for $0\leq s<t\leq 1$. Therefore, for any $0\leq s\leq t\leq 1$ with $t\geq\xi$, we have

$$
\begin{aligned}
&(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)G_1(t,s)\\
=&at\Gamma(2-q)\Gamma(p)(1-s)^{p-q-1}+t\Gamma(2-q)\Gamma(p-q)(1-s)^{p-1}\\
&-(a+\Gamma(2-q))\Gamma(p-q)(t-s)^{p-1}.\\
=&a\Gamma(2-q)\Gamma(p)t(1-s)^{p-q-1}-a\Gamma(p-q)(t-s)^{p-1}\\
&+\Gamma(2-q)\Gamma(p-q)t(1-s)^{p-1}\\
&-\Gamma(2-q)\Gamma(p-q)(t-s)^{p-1}\\
\geq&a\Gamma(2-q)\Gamma(p)t(1-s)^{p-q-1}-a\Gamma(p-q)(t-s)^{p-1}\\
\geq&a\Gamma(2-q)\Gamma(p)t(1-s)^{p-q-1}-a\Gamma(p-q)t(1-s)^{p-q-1}\\
\geq&a\xi[\Gamma(p)\Gamma(2-q)-\Gamma(p-q)](1-s)^{p-q-1}.
\end{aligned}
$$

And for $0<\xi\leq t\leq s\leq 1$,

$$
\begin{aligned}
&(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)G_2(t,s)\\
=&at\Gamma(2-q)\Gamma(p)(1-s)^{p-q-1}+t\Gamma(2-q)\Gamma(p-q)(1-s)^{p-1}\\
\geq&a\xi\Gamma(2-q)\Gamma(p)(1-s)^{p-q-1}\\
\geq&a\xi[\Gamma(p)\Gamma(2-q)-\Gamma(p-q)](1-s)^{p-q-1}.
\end{aligned}
$$

Let

$$r_2=\frac{a\xi[\Gamma(p)\Gamma(2-q)-\Gamma(p-q)]}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)},$$

then we have,

$$G(t,s)\geq r_2(1-s)^{p-q-1},\ \text{for any } (t,s)\in[\xi,1]\times[0,1],\ \text{where } \xi\in(0,1).$$

Since $\Gamma(p)\Gamma(2-q)>\Gamma(p-q)$, with $p>2$, $0<q<1$, then $r_2>0$.

(v) In view of the expression of $G(t,s)$, we can easily get that

$$\frac{\partial G(t,s)}{\partial t}=\begin{cases}-\frac{(p-1)(t-s)^{p-2}}{\Gamma(p)}+\frac{a\Gamma(p)\Gamma(2-q)(1-s)^{p-q-1}+\Gamma(2-q)\Gamma(p-q)(1-s)^{p-1}}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)}, & \\ \qquad\qquad\qquad\qquad 0\leq s\leq t\leq 1, & \\ \frac{a\Gamma(p)\Gamma(2-q)(1-s)^{p-q-1}+\Gamma(2-q)\Gamma(p-q)(1-s)^{p-1}}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)}, \quad 0\leq t\leq s\leq 1. & \end{cases}$$

From the expression of $\frac{\partial G(t,s)}{\partial t}$, we obtain that, for any $(t,s) \in [0,1] \times [0,1)$,

$$\begin{aligned}
|\frac{\partial G(t,s)}{\partial t}| \leq & \frac{a\Gamma(p)\Gamma(2-q)(1-s)^{p-q-1} + \Gamma(2-q)\Gamma(p-q)(1-s)^{p-1}}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)} \\
& + \frac{(p-1)(1-s)^{p-2}}{\Gamma(p)} \\
= & [\frac{a\Gamma(p)\Gamma(2-q)(1-s) + \Gamma(2-q)\Gamma(p-q)(1-s)^{q+1}}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)} \\
& + \frac{(p-1)(1-s)^{p-2}}{\Gamma(p)(1-s)^{p-q-2}}](1-s)^{p-q-2} \\
\leq & [\frac{(p-1)(1-s)^{q}}{\Gamma(p)} + \frac{a\Gamma(p)\Gamma(2-q) + \Gamma(2-q)\Gamma(p-q)}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)}](1-s)^{p-q-2} \\
\leq & [\frac{p-1}{\Gamma(p)} + \frac{\Gamma(2-q)(a\Gamma(p) + \Gamma(p-q))}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)}](1-s)^{p-q-2}.
\end{aligned}$$

Let

$$r_3 = \frac{p-1}{\Gamma(p)} + \frac{\Gamma(2-q)(a\Gamma(p) + \Gamma(p-q))}{(a+\Gamma(2-q))\Gamma(p-q)\Gamma(p)}.$$

Then we have $|\frac{\partial G(t,s)}{\partial t}| \leq r_3(1-s)^{p-q-2}$, for any $(t,s) \in [0,1] \times [0,1)$. On the other hand, $|\frac{\partial G(t,s)}{\partial t}| = 0 \leq r_3(1-s)^{p-q-2}$, for $s = 1$. Therefore,

$$|\frac{\partial G(t,s)}{\partial t}| \leq r_3(1-s)^{p-q-2}, \text{ for any } (t,s) \in [0,1] \times [0,1].$$

□

For convenience, we assume that the following hypotheses hold:
($\mathbf{H}_2$) $f \in C([0,1] \times [0,+\infty) \times (-\infty,+\infty), [0,+\infty))$ is an given function.
($\mathbf{H}_3$) $g_0, g_1 \in C([0,1],[0,+\infty))$ are given functions, such that the auxiliary function $\Phi(t,s)$ satisfies, $0 \leq m_0 := \min\{\Phi(t,s) : t,s \in [0,1]\} \leq \Phi(t,s) \leq \max\{\Phi(t,s) : t,s \in [0,1]\} := M_0 < 1$, and $\max\{|\Phi_t'(t,s)| : t,s \in [0,1]\} := M_1 < \Gamma(2-q) < 1$.

Let $X = \{u : u \in C([0,1]), {}^{C}D^{q}u \in C([0,1])\}$ be endowed with the maximum norm,

$$||u|| = \max\{\max_{0\leq t\leq 1} |u(t)|, \max_{0\leq t\leq 1} |{}^{C}D^{q}u(t)|\}.$$

Then X is a Banach space. Let $P = \{u \in X : u(t) \geq 0, 0 \leq t \leq 1\}$, it is easy to check that P is a cone on X.

Define a linear operator

$$A : X \longrightarrow X, Au(t) = \int_0^1 \Phi(t,s)u(s)\mathrm{d}s. \tag{4}$$

Lemma 2.3. *If (H_1)–(H_3) hold, then the operator A is a bounded linear operator, $A(P) \subset P$. Moreover $(I - A)$ is invertible and*

$$||(I-A)^{-1}|| \leq \max\{\frac{1}{1-M_0}, \frac{M_1 + (1-M_0)\Gamma(2-q)}{(1-M_0)\Gamma(2-q)}\}.$$

Proof. (i) It is clear that A is a linear operator.
$|Au(t)| = |\int_0^1 \Phi(t,s)u(s)\mathrm{d}s| \leq M_0||u||$ and $|^C D^q Au(t)| = |^C D^q \int_0^1 \Phi(t,s)u(s)\mathrm{d}s| \leq I_t^{1-q}\int_0^1 |\Phi_t'(t,s)|\mathrm{d}s\|u\| \leq \frac{M_1||u||}{\Gamma(2-q)}$.
Therefore

$$||A|| \leq \max\{M_0, \frac{M_1}{\Gamma(2-q)}\} < 1.$$

This shows that A is a bounded linear operator.

(ii) Let $u \in P$, then $u \in C([0,1])$, ${}^C D^q u \in C([0,1])$ and $u(t) \geq 0$. Because $\Phi(t,s)$ is continuous and nonnegative, it is easy to check that $Au \in C([0,1]), Au(t) \geq 0$. We can easily find that $\Phi_t'(t,s) = \frac{\Gamma(2-q)(g_1(s)-g_0(s))}{a+\Gamma(2-q)}$ is continuous.

Hence, we have
${}^C D^q Au(t) = I_t^{1-q}\int_0^1 \Phi_t'(t,s)u(s)\mathrm{d}s = \frac{t^{1-q}\int_0^1 (g_1(s)-g_0(s))u(s)\mathrm{d}s}{a+\Gamma(2-q)} \in C([0,1])$.
Therefore, $Au \in P$, which implies that $A(P) \subset P$.

(iii) We have proved in (i) that $||A|| \leq \max\{M_0, \frac{M_1}{\Gamma(2-q)}\} < 1$, which implies that $I - A$ is invertible.

To find the expression for $(I-A)^{-1}$, we use the theory of Fredholm integral equations. We have $u(t) = (I-A)^{-1}v(t)$ if and only if $u(t) = v(t) + Au(t)$ for $t \in I$. The definition of the operator A implies that

$$u(t) = v(t) + \int_0^1 \Phi(t,s)u(s)\mathrm{d}s. \tag{5}$$

The condition $||A|| < \max\{M_0, \frac{M_1}{\Gamma(2-q)}\} < 1$ implies that 1 is not an eigenvalue of the operator A. Hence (5) has a unique solution $u \in X$, for every $v \in X$. By successive substitutions in (5), we obtain

$$u(t) = (I-A)^{-1}v(t) = v(t) + \int_0^1 R(t,s)v(s)\mathrm{d}s, \tag{6}$$

where the resolvent kernel $R(t,s)$ is given by $R(t,s) = \Sigma_{j=1}^{\infty}\Phi_j(t,s)$,
here $\Phi_1(t,s) = \Phi(t,s)$, $\Phi_j(t,s) = \int_0^1 \Phi(t,\tau)\Phi_{j-1}(\tau,s)\mathrm{d}\tau$, $(j = 2, 3, \cdots)$. It is easy to show that

$$R_t'(t,s) = \Sigma_{j=1}^{\infty}\Phi_{j,t}'(t,s), \quad \Phi_{1,t}'(t,s) = \Phi_t'(t,s),$$

and

$$\Phi_{j,t}'(t,s) = \int_0^1 \Phi_t'(t,\tau)\Phi_{j-1}(\tau,s)\mathrm{d}\tau \ ,(j = 2, 3, \cdots).$$

Because $0 \le m_0 \le \Phi(t,s) \le M_0 < 1$ and $|\Phi'_t(t,s)| \le M_1 < \Gamma(2-q) < 1$, we have $m_0^j \le \Phi_j(t,s) \le M_0^j$ and $|\Phi'_{j,t}(t,s)| \le M_1 M_0^{j-1}$. Then

$$\frac{m_0}{1-m_0} \le R(t,s) \le \frac{M_0}{1-M_0} \text{ and } |R'_t(t,s)| \le \frac{M_1}{1-M_0}. \tag{7}$$

In view of (6) and (7), we obtain

$$\begin{aligned}
|(I-A)^{-1}v(t)| \le& |v(t)| + \int_0^1 |R(t,s)v(s)\mathrm{d}s| \\
\le& (1+\frac{M_0}{1-M_0})||v|| \\
=& \frac{1}{1-M_0}||v||. \\
|{}^C D^q (I-A)^{-1}v(t)| \le& |{}^C D^q v(t)| + |{}^C D^q \int_0^1 R(t,s)v(s)\mathrm{d}s| \\
\le& ||v|| + I_t^{1-q} \int_0^1 |R'_t(t,s)|\ |v(s)|\mathrm{d}s \\
\le& ||v|| + \frac{M_1||v||}{(1-M_0)\Gamma(2-q)} \\
=& \frac{[(1-M_0)\Gamma(2-q)+M_1]}{(1-M_0)\Gamma(2-q)}||v||.
\end{aligned}$$

Therefore $||(I-A)^{-1}|| \le \max\{\frac{1}{1-M_0}, \frac{M_1+(1-M_0)\Gamma(2-q)}{(1-M_0)\Gamma(2-q)}\}$. □

Now we introduce the fixed point theorem in a cone which due to Bai and Ge (See [3]), and it can be regarded as a generalization of the Leggett-Williams fixed point theorem.

Let E be a Banach space and $P \subset E$ be a cone. $\alpha, \beta : P \to [0, +\infty)$ are two nonnegative continuous convex functions satisfying

$$||u|| \le M \max\{\alpha(u), \beta(u)\}, \text{ for } u \in P, \tag{8}$$

where M is a positive constant, and

$$\Omega = \{u \in P \ : \ \alpha(u) < k, \beta(u) < L\} \ne \emptyset, \text{ for } k > 0, L > 0. \tag{9}$$

By (8) and (9), Ω is a bounded nonempty open subset in P.

Let $k > c > 0, L > 0$ be given, $\alpha, \beta : P \to [0, +\infty)$ be two nonnegative continuous convex functions satisfying (8) and (9), and γ be a nonnegative continuous concave function on the cone P. Define bounded convex sets

$$\begin{aligned}
P(\alpha, k; \beta, L) &= \{u \in P \ : \ \alpha(u) < k, \beta(u) < L\}, \\
\overline{P}(\alpha, k; \beta, L) &= \{u \in P \ : \ \alpha(u) \le k, \beta(u) \le L\}, \\
P(\alpha, k; \beta, L; \gamma, c) &= \{u \in P \ : \ \alpha(u) < k, \beta(u) < L, \gamma(u) > c\}, \\
\overline{P}(\alpha, k; \beta, L; \gamma, c) &= \{u \in P \ : \ \alpha(u) \le k, \beta(u) \le L, \gamma(u) \ge c\}.
\end{aligned}$$

Lemma 2.4 ([3]). *Let E be a Banach space, $P \subset E$ be a cone and $k_2 \geq d > b > k_1 > 0$, $L_2 \geq L_1 > 0$ be given. Assume that α, β are nonnegative continuous convex functions on P, such that (8) and (9) are satisfied, γ is an nonnegative continuous concave function on P, such that $\gamma(u) \leq \alpha(u)$, for all $u \in \overline{P}(\alpha, k_2; \beta, L_2)$ and sets $S : \overline{P}(\alpha, k_2; \beta, L_2) \to \overline{P}(\alpha, k_2; \beta, L_2)$ be completely continuous operator. Suppose*

(C1) $\{u \in \overline{P}(\alpha, d; \beta, L_2; \gamma, b) \ : \ \gamma(u) > b\} \neq \emptyset, \gamma(Su) > b,$ *for* $u \in \overline{P}(\alpha, d; \beta, L_2; \gamma, b)$;

(C2) $\alpha(Su) < k_1, \beta(Su) < L_1$, *for all* $u \in \overline{P}(\alpha, k_1; \beta, L_1)$;

(C3) $\gamma(Su) > b$, *for all* $u \in \overline{P}(\alpha, k_2; \beta, L_2; \gamma, b)$, *with* $\alpha(Su) > d$.

Then S has at least three fixed points $u_1, u_2, u_3 \in \overline{P}(\alpha, k_2; \beta, L_2)$. Further,

$$u_1 \in P(\alpha, k_1; \beta, L_1), \ u_2 \in \{\overline{P}(\alpha, k_2; \beta, L_2; \gamma, b) \ : \ \gamma(u) > b\},$$
$$u_3 \in \overline{P}(\alpha, k_2; \beta, L_2) \setminus \{\overline{P}(\alpha, k_2; \beta, L_2; \gamma, b) \cup \overline{P}(\alpha, k_1; \beta, L_1)\}.$$

3. Main results

Define a nonlinear operator $T : X \to X$ by

$$Tu(t) = \int_0^1 G(t, s) f(s, u(s), {}^C D^q u(s)) \mathrm{d}s. \tag{10}$$

In view of Lemma 2.1, (4) and (10), we obtain that u is solution of BVP(1) if and only if u is solution of the following equation:

$$u(t) = Tu(t) + Au(t), t \in I. \tag{11}$$

Clearly, u is a solution of (11) if and only if u is a solution of $u(t) = (I - A)^{-1} Tu(t)$, that is a fixed point of the operator $S := (I - A)^{-1} T$. By (6) and (10), we have

$$\begin{aligned} Su(t) = & \int_0^1 G(t, s) f(s, u(s), {}^C D^q u(s)) \mathrm{d}s \\ & + \int_0^1 R(t, s) \int_0^1 G(s, \tau) f(\tau, u(\tau), {}^C D^q u(\tau)) \mathrm{d}\tau \mathrm{d}s. \end{aligned}$$

Define functions

$$\alpha(u) = \max_{0 \leq t \leq 1} |u(t)|, \ \beta(u) = \max_{0 \leq t \leq 1} |{}^C D^q u(t)|, \ \gamma(u) = \min_{\xi \leq t \leq 1} |u(t)|.$$

Then $\alpha, \beta, \gamma : P \to [0, +\infty)$ are three continuous nonnegative functions such that $||u|| = \max\{\alpha(u), \beta(u)\}$, and (8), (9) hold; α, β are convex functions, γ is concave functions and $\gamma(u) \leq \alpha(u)$ holds, for all $u \in P$.

Theorem 3.1. *Suppose that* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *hold, and there exist constants* $k_2 \geq \frac{b r_1 (1 - m_0)}{r_2 (1 - M_0)(1 - m_0 \xi)} > b > k_1 > 0, \ L_2 \geq L_1 > 0$ *such that*

$$\frac{b(p - q)(1 - m_0)}{(1 - m_0 \xi) r_2} < \min\{\frac{k_2 (1 - M_0)(p - q)}{r_1}, \frac{\Gamma(2 - q)(1 - M_0)(p - q - 1) L_2}{(1 - M_0) r_3 + M_1 r_1}\},$$

and the following assumptions hold:
(A1) $f(t,u,v) < \min\{\frac{k_1(p-q)(1-M_0)}{r_1}, \frac{\Gamma(2-q)(1-M_0)(p-q-1)L_1}{(1-M_0)r_3+M_1r_1}\}$,
for $(t,u,v) \in [0,1]\times[0,k_1]\times[-L_1,L_1]$;
(A2) $f(t,u,v) > \frac{b(p-q)(1-m_0)}{(1-m_0\xi)r_2}$,
for $(t,u,v) \in [\xi,1]\times[b, \frac{br_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)}]\times[-L_2,L_2]$;
(A3) $f(t,u,v) \le \min\{\frac{k_2(1-M_0)(p-q)}{r_1}, \frac{\Gamma(2-q)(1-M_0)(p-q-1)L_2}{(1-M_0)r_3+M_1r_1}\}$,
for $(t,u,v) \in [0,1]\times[0,k_2]\times[-L_2,L_2]$.

Then BVP(1) has at least three positive solutions u_1, u_2*, and* u_3 *satisfying*

$$\max_{0\le t\le 1} u_1(t) \le k_1,\ \max_{0\le t\le 1} |^C D^q u_1(t)| \le L_1;$$

$$b < \min_{\xi\le t\le 1} u_2(t) \le \max_{0\le t\le 1} u_2(t) \le k_2,\ \max_{0\le t\le 1} |^C D^q u_2(t)| \le L_2;$$

$$\max_{0\le t\le 1} u_3(t) \le \frac{br_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)},\ \max_{0\le t\le 1} |^C D^q u_3(t)| \le L_2.$$

Proof. BVP(1) has a solution $u = u(t)$ if and only if u solves the operator equation

$$\begin{aligned} u(t) = Su(t) = &\int_0^1 G(t,s) f(s,u(s),{}^C D^q u(s))\mathrm{d}s \\ &+ \int_0^1 R(t,s)\int_0^1 G(s,\tau) f(\tau,u(\tau),{}^C D^q u(\tau))\mathrm{d}\tau\mathrm{d}s. \end{aligned}$$

It is easy to show that $S : P \to P$ is a completely continuous operator. Now, we show that the conditions of Lemma 2.4 are satisfied.

For $u \in \overline{P}(\alpha, k_2; \beta, L_2)$, it implies that $|u(t)| \le k_2$, $|^C D^q u(t)| \le L_2$ for $t \in [0,1]$. By (A3), Lemma 2.2 and (7), we have

$$\begin{aligned} \alpha(Su) = &\max_{0\le t\le 1} |\int_0^1 G(t,s) f(s,u(s),{}^C D^q u(s))\mathrm{d}s \\ &+ \int_0^1 R(t,s)\int_0^1 G(s,\tau) f(\tau,u(\tau),{}^C D^q u(\tau))\mathrm{d}\tau\mathrm{d}s| \\ \le & r_1\int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s \\ &+ \frac{M_0 r_1\int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s}{1-M_0} \\ = & \frac{r_1\int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s}{1-M_0} \\ \le & (\frac{r_1\int_0^1 (1-s)^{p-q-1}\mathrm{d}s}{1-M_0})\frac{k_2(1-M_0)(p-q)}{r_1} \\ = & k_2. \end{aligned}$$

By Lemma 2.2 and (7), we have

$$\begin{aligned}\beta(Su) =& \max_{0\le t\le 1} |{}^C D^q(\int_0^1 (G(t,s)f(s,u(s),{}^C D^q u(s))\mathrm{d}s\\
&+\int_0^1 R(t,s)\int_0^1 G(s,\tau)f(\tau,u(\tau),{}^C D^q u(\tau))\mathrm{d}\tau\mathrm{d}s)|\\
=&\max_{0\le t\le 1} |I_t^{1-q}(\int_0^1 G_t'(t,s)f(s,u(s),{}^C D^q u(s)\mathrm{d}s)\\
&+\int_0^1 R_t'(t,s)\int_0^1 G(s,\tau)f(\tau,u(\tau),{}^C D^q u(\tau))\mathrm{d}\tau\mathrm{d}s)|\\
\le&[\frac{r_3}{\Gamma(2-q)}+\frac{M_1 r_1}{(1-M_0)\Gamma(2-q)}]\int_0^1 (1-s)^{p-q-2} f(s,u(s),{}^C D^q u(s))\mathrm{d}s\\
\le&\frac{r_3(1-M_0)+M_1 r_1}{\Gamma(2-q)(1-M_0)}\cdot\frac{\Gamma(2-q)(1-M_0)(p-q-1)L_2}{r_3(1-M_0)+M_1 r_1}\int_0^1 (1-s)^{p-q-2}\mathrm{d}s\\
=&L_2.\end{aligned}$$

Then $S:\overline{P}(\alpha,k_2;\beta,L_2)\to\overline{P}(\alpha,k_2;\beta,L_2)$.

In the same way, by (A1), Lemma 2.2 and (7), we can obtain that $S:\overline{P}(\alpha,k_1;\beta,L_1)\to P(\alpha,k_1;\beta,L_1)$. Therefore condition (C2) of Lemma 2.4 is satisfied.

To check condition (C1) of Lemma 2.4, we choose $u_0=\frac{br_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)}$, $0\le t\le 1$. It is easy to see that

$$u_0=\frac{br_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)}\in\overline{P}(\alpha,\frac{br_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)};\beta,L_2;\gamma,b)|\gamma(u)>b\}\ne\emptyset.$$

For $u\in\overline{P}(\alpha,\frac{br_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)};\beta,L_2;\gamma,b)$, we have

$$b\le u(t)\le\frac{br_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)},\ \xi\le t\le 1,\ -L_2\le^C D^q u\le L_2.$$

By assumption (A2), $f(t,u(t),{}^C D^q u(t))>\frac{b(p-q)(1-m_0)}{(1-m_0\xi)r_2}$, we can obtain that

$$\begin{aligned}\gamma(Su) =& \min_{\xi\le t\le 1}|Su(t)|\\
=&\min_{\xi\le t\le 1}|\int_0^1 G(t,s)f(s,u(s),{}^C D^q u(s))\mathrm{d}s\\
&+\int_0^1 R(t,s)\int_0^1 G(s,\tau)f(\tau,u(\tau),{}^C D^q u(\tau))\mathrm{d}\tau\mathrm{d}s|\\
\ge& r_2\int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s\\
&+\frac{m_0}{1-m_0}\int_\xi^1\int_0^1 G(s,\tau)f(\tau,u(\tau),{}^C D^q u(\tau))\mathrm{d}\tau\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\geq r_2 \int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s \\
&\quad + \frac{m_0 r_2 (1-\xi)\int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s}{1-m_0} \\
&= \frac{(1-m_0\xi) r_2 \int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s}{1-m_0} \\
&> \frac{(1-m_0\xi) r_2}{1-m_0} \, \frac{(1-m_0)(p-q)b}{(1-m_0\xi) r_2} \int_0^1 (1-s)^{p-q-1}\mathrm{d}s \\
&= b.
\end{aligned}
$$

This implies that $\gamma(Su) > b$, for all $u \in \overline{P}(\alpha, \frac{br_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)}; \beta, L_2; \gamma, b)$.

Finally, we show that (C3) of Lemma 2.4 also holds.

For $u \in \overline{P}(\alpha, k_2; \beta, L_2; \gamma, b)$, with $\alpha(Su) > \frac{br_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)}$, we have

$$
\begin{aligned}
\alpha(Su) &= \max_{0\leq t\leq 1} \Big| \int_0^1 G(t,s) f(s,u(s),{}^C D^q u(s))\mathrm{d}s \\
&\quad + \int_0^1 R(t,s) \int_0^1 G(s,\tau) f(\tau,u(\tau),{}^C D^q u(\tau))\mathrm{d}\tau\mathrm{d}s \Big| \\
&\leq r_1 \int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s \\
&\quad + \frac{M_0 r_1 \int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s}{1-M_0} \\
&= \frac{r_1 \int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s}{1-M_0}.
\end{aligned}
$$

This implies that

$$
\int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s \geq \frac{(1-M_0)\alpha(Su)}{r_1}.
$$

Therefore,

$$
\begin{aligned}
\gamma(Su) &= \min_{\xi\leq t\leq 1} |Su(t)| \\
&= \min_{\xi\leq t\leq 1} \Big| \int_0^1 G(t,s) f(s,u(s),{}^C D^q u(s))\mathrm{d}s \\
&\quad + \int_0^1 R(t,s) \int_0^1 G(s,\tau) f(\tau,u(\tau),{}^C D^q u(\tau))\mathrm{d}\tau\mathrm{d}s \Big| \\
&\geq r_2 \int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s \\
&\quad + \frac{m_0}{1-m_0} \int_\xi^1 \int_0^1 G(s,\tau) f(\tau,u(\tau),{}^C D^q u(\tau))\mathrm{d}\tau\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\geq r_2 \int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s \\
&\quad + \frac{m_0 r_2 (1-\xi)\int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s}{1-m_0} \\
&= \frac{(1-m_0\xi) r_2 \int_0^1 (1-s)^{p-q-1} f(s,u(s),{}^C D^q u(s))\mathrm{d}s}{1-m_0} \\
&\geq \frac{(1-m_0\xi) r_2}{1-m_0} \, \frac{(1-M_0)\alpha(Su)}{r_1} \\
&> \frac{(1-m_0\xi) r_2}{(1-m_0)} \, \frac{(1-M_0)}{r_1} \, \frac{b r_1 (1-m_0)}{r_2(1-M_0)(1-m_0\xi)} \\
&= b.
\end{aligned}
$$

So the condition (C3) of Lemma 2.4 holds. In addition, as $\alpha(u_3) \leq \frac{r_1(1-m_0)\gamma(u_3)}{r_2(1-m_0\xi)(1-M_0)}$, we have $\max_{0\leq t\leq 1} u_3(t) \leq \frac{r_1(1-m_0)b}{r_2(1-m_0\xi)(1-M_0)}$.

From Lemma 2.4 we obtain that the operator S has at least three fixed points u_1, u_2, $u_3 \in \overline{P}(\alpha, k_2; \beta, L_2; \gamma, b)$, that is BVP(1) has at least three positive solutions u_1, u_2, and u_3 satisfying

$$\max_{0\leq t\leq 1} u_1(t) \leq k_1, \ \max_{0\leq t\leq 1} |{}^C D^q u_1(t)| \leq L_1;$$

$$b < \min_{\xi\leq t\leq 1} u_2(t) \leq \max_{0\leq t\leq 1} u_2(t) \leq k_2, \ \max_{0\leq t\leq 1} |{}^C D^q u_2(t)| \leq L_2;$$

$$\max_{0\leq t\leq 1} u_3(t) \leq \frac{b r_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)}, \ \max_{0\leq t\leq 1} |{}^C D^q u_3(t)| \leq L_2.$$

□

Remark 3.1. With Lemma 2.4 we have the result $\max_{0\leq t\leq 1} u_3(t) \leq k_2$, $\min_{\xi\leq t\leq 1} u_3(t) < b$. However, from BVP(1) the function α, γ holds additional relation

$$\gamma(u) = \min_{\xi\leq t\leq 1} u(t) \geq \frac{(1-m_0\xi) r_2 (1-M_0)\alpha(u)}{(1-m_0) r_1}, \text{ for } u \in P.$$

So we can obtain the better result $\max\limits_{0\leq t\leq 1} u_3(t) \leq \frac{b r_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)}$.

Theorem 3.2. *Suppose that there exist* $0 < k_1 < b_1 < \frac{b_1 r_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)} \leq k_2 < b_2 < \frac{b_2 r_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)} \leq \cdots \leq k_n$, $0 < L_1 \leq L_2 \leq \cdots \leq L_{n-1}$, $n \in N$, *such that for* $1 \leq i \leq n-1$,

$$\frac{b_i(p-q)(1-m_0)}{(1-m_0\xi) r_2} \leq \min\{\frac{k_{i+1}(1-M_0)(p-q)}{r_1}, \frac{\Gamma(2-q)(1-M_0)(p-q-1)L_{i+1}}{(1-M_0)r_3 + M_1 r_1}\},$$

and the following conditions are satisfied

(E1) $f(t,u,v) < \min\{\frac{k_i(p-q)(1-M_0)}{r_1}, \frac{\Gamma(2-q)(1-M_0)(p-q-1)L_i}{(1-M_0)r_3 + M_1 r_1}\}$, *for* $(t,u,v) \in [0,1] \times [0,k_i] \times [-L_i, L_i]$;

(E2) $f(t,u,v) > \frac{b_i(p-q)(1-m_0)}{(1-m_0\xi)r_2}$,
for $(t,u,v) \in [\xi,1] \times [b_i, \frac{b_i r_1(1-m_0)}{r_2(1-M_0)(1-m_0\xi)}] \times [-L_{i+1}, L_{i+1}], 1 \le i \le n-1$.

Then BVP (1) has at least $2n-1$ *positive solutions.*

Proof. When $n=1$, it follows from condition (E1) that $S : \overline{P}(\alpha, k_1; \beta, L_1) \to P(\alpha, k_1; \beta, L_1) \subseteq \overline{P}(\alpha, k_1; \beta, L_1)$, which means that at least one fixed point $u_1 \in P(\alpha, k_1; \beta, L_1)$ by the Schauder fixed point theorem. When $n=2$, it is clear that Theorem 3.2 holds. Then we can obtain at least three positive solutions u_2, u_3, u_4. Following this way, we can finish the proof by the induction method. □

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