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RELATION BETWEEN METRIC AND FUZZY METRIC SPACES AND SOME FIXED POINT THEOREMS

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Abstract. In this work we have considered several common fixed point results in metric spaces for weak compatible mappings. By applications of these results we establish some fixed point theorems in fuzzy metric spaces.

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1. Introduction

In this paper we establish some fixed point results in a fuzzy metric space by applications of certain fixed point theorems in metric spaces. Also we prove some fixed point results in metric spaces. Fuzzy metric space was first introduced by Kramosil and Michalek [8]. Subsequently, George and Veeramani had given a modified definition of fuzzy metric spaces[1]. Fixed point results in such spaces have been established in a large number of works. Some of these works are noted in [2], [10], [11], [14],[15] and [16].

Definition 1.1 ([1]). A binary operation $* : [0,1]\times[0,1] \longrightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions:
(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a*b = ab$ and $a*b = \min(a, b)$.

Definition 1.2 ([1]). A 3-tuple $(X, M, *)$ is called a fuzzy metric space (in the sense of George and Veeramani) if X is an arbitrary (non-empty) set, $*$ is a

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continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Here we have considered another definition of fuzzy metric space (non-Archimedean). We describe the space along with some associated concepts in the following.

Definition 1.3 ([12])**.** A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous $t-$norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, z, t) * M(z, y, s) \leq M(x, y, t \vee s)$, where $(t \vee s) = max\{s, t\}$,
(5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

All fuzzy metric in this paper are assumed to be non-Archimedean. The following properties of M noted in the theorem below are easy consequences of the definition.

Theorem 1.1. *(i) $M(x, y, t)$ is nondecreasing with respect to t for each $x, y \in X$.*

(ii) $M(x, y, t) \geq M(x, z, t) * M(z, y, t)$ for all $x, y, z \in X$ and $t > 0$.

Example 1.2. (1) Let $a * b = ab$ for all $a, b \in [0, 1]$ and M be the fuzzy set on $X^2 \times$:$]0, +\infty[$ defined by

$$M(x, y, t) = \exp^{-}\frac{d(x, y)}{t},$$

where d is an ordinary metric on set X. Then $(X, M, *)$ is a fuzzy metric space.

(2) Let $a * b = ab$ for all $a, b \in [0, 1]$ and M be the fuzzy set on $\mathbb{R}^+ \times \mathbb{R}^+ \times (0, +\infty)$ defined by

$$M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}} ,$$

for all $x, y \in \mathbb{R}^+$. Then $(X, M, *)$ is a fuzzy metric space.

Example 1.3. If M be the fuzzy set on $X^2 \times (0, +\infty)$ defined by

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

where d is any metric on X, then it is easy to see that M satisfies all the conditions of definition 1.3, but not all of the conditions of definition 1.2 are satisfied for each $x, y, z \in X$ and $t > 0$.

Let $(X, M, *)$ be a fuzzy metric space . For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Let $(X, M, *)$ be a fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exits $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. This definition of Cauchy sequence is identical with that given by George and Veeramani[1]. The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent.

Let Φ denote a family of mappings such that for each $\phi \in \Phi$,

(1) $\phi : (0, 1] \longrightarrow [0, \infty)$,
(2) ϕ is continuous and decreasing,
(3) $\phi(x) = 0 \Longleftrightarrow x = 1$ and $\phi(x.y) \leq \phi(x) + \phi(y)$, for every $x, y \in (0, 1]$.

2. Main results

Lemma 2.1. *Let $(X, M, *)$ be a fuzzy metric space with $a * b \geq ab$ for all $a, b \in [0, 1]$ and $M(x, y, .)$ having discontinuity at 0, for all $x, y \in X$. If we define $d : X^2 \longrightarrow [0, \infty)$ by $d(x, y) = sup_\alpha \int_\alpha^1 \phi(M(x, y, t))dt$, then d is a metric on X for fixed $0 < \alpha < 1$.*

Proof. It is clear from the definition that $d(x, y)$ is well defined for each $x, y \in X$.
(i) $d(x, y) \geq 0$ for all $x, y \in X$ is trivial.

$$\begin{aligned} (ii)\ d(x, y) &= 0 \Longleftrightarrow \phi(M(x, y, t)) = 0 \text{ for all } t > 0 \\ &\Longleftrightarrow M(x, y, t) = 1 \text{ for all } t > 0 \Longleftrightarrow x = y. \end{aligned}$$

$$\begin{aligned} (iii)\ d(x, y) &= sup_\alpha \int_\alpha^1 \phi(M(x, y, t))dt \\ &= sup_\alpha \int_\alpha^1 \phi(M(y, x, t))dt = d(y, x). \end{aligned}$$

$$\begin{aligned} (iv) Since\ M(x, y, t) &\geq M(x, z, t) * M(z, y, t) \\ &\geq M(x, z, t).M(z, y, t). \end{aligned}$$

and also since ϕ is decreasing, it follows that,

$$\begin{aligned} d(x,y) &= sup_\alpha \int_\alpha^1 \phi(M(x,y,t))dt \\ &\leq sup_\alpha \int_\alpha^1 \phi(M(x,z,t).M(z,y,t))dt \\ &\leq sup_\alpha \int_\alpha^1 \phi(M(x,z,t))dt + sup_\alpha \int_\alpha^1 \phi(M(z,y,t))dt \\ &= d(x,z) + d(z,y) \end{aligned}$$

This proves that d is a metric on X. □

Theorem 2.2. *Let $(X, M, *)$ be a fuzzy metric space with $a * b \geq ab$ for all $a, b \in [0,1]$ and $M(x,y,.)$ having discontinuity at 0, for all $x, y \in X$. If we define $d : X^2 \longrightarrow [0,\infty)$ by $d(x,y) = sup_\alpha \int_\alpha^1 \log_a^{M(x,y,t)} dt$, then d is a metric on X for $0 < a < 1$.*

Proof. The proof follows from the above lemma 1.7 by choosing $\phi(x) = \log_a^x$. □

Let Ψ denote a family of mappings such that for each $\psi \in \Psi$,
$\psi : [0,\infty) \longrightarrow [0,\infty)$,

(1) ψ is continuous ,
(2) $\psi(t)$ is increasing and
(3) $\psi(0) = 0$ and $\psi(t) < t$ for every $t > 0$.

Lemma 2.3. *Let $\psi \in \Psi$, then corresponding to this ψ, there exists a $\gamma : (0,1] \longrightarrow (0,1]$ such that γ is a continuous, increasing function such that $\gamma(t) > t$ for $0 < t < 1$ and $\gamma(1) = 1$.*

Proof. Let $\lambda : [0,\infty) \longrightarrow (0,1]$ be such that $\lambda(t) = a^{\psi(t)}$ for every $t \in [0,\infty)$ and $0 < a < 1$. It is easy to see that λ is a continuous, decreasing function and $\lambda(0) = 1$. Now, we define $\eta : (0,1] \longrightarrow [0,\infty)$ such that $\eta(t) = \log_a^t$ for every $t \in (0,1]$ and $0 < a < 1$. We define $\gamma : (0,1] \longrightarrow (0,1]$ such that

$$\gamma(t) = \lambda(\eta(t)) = \lambda(\log_a^t) = a^{\psi(\log_a^t)}$$

for every $t \in (0,1]$ and $0 < a < 1$. Then γ is a continuous, increasing function and $\gamma(t) > t$ for $0 < t < 1$ and $\gamma(1) = 1$. □

Example 2.4. Let $\psi : [0,\infty) \longrightarrow [0,\infty)$ be defined as $\psi(x) = kx$ where $0 < k < 1$. Now corresponding to this function ψ, we define $\gamma : (0,1] \longrightarrow (0,1]$ by $\gamma(t) = t^k$ where $0 < t \leq 1$.

Lemma 2.5. *Let $(X, M, *)$ be a fuzzy metric space with $a * b \geq ab$ for all $a, b \in [0,1]$ and $M(x,y,.)$ having discontinuity at 0, for all $x, y \in X$. We define $d : X^2 \longrightarrow [0,\infty)$ by $d(x,y) = sup_\alpha \int_\alpha^1 \cot(\frac{\pi}{2}M(x,y,t))dt$, then d is a metric on X.*

Proof.

(i) $d(x, y) \geq 0$ is trivial.
(ii)

$$\begin{aligned} d(x,y) &= 0 \Longleftrightarrow \cot(\frac{\pi}{2}M(x,y,t)) = 0 \text{ for all } t > 0 \\ &\Longleftrightarrow M(y,x,t) = 1 \text{ for all } t > 0 \Longleftrightarrow x = y. \end{aligned}$$

(iii)

$$\begin{aligned} d(x,y) &= sup_\alpha \int_\alpha^1 \cot(\frac{\pi}{2}M(x,y,t))dt \\ &= sup_\alpha \int_\alpha^1 \cot(\frac{\pi}{2}M(x,y,t))dt = d(y,x). \end{aligned}$$

(iv)

$$\begin{aligned} \text{Since } M(x,y,t) &\geq M(x,z,t) * M(z,y,t) \\ &\geq \min\{M(x,z,t), M(z,y,t)\}. \end{aligned}$$

and also since $0 < \frac{\pi}{2}M(x,y,t) \leq \frac{\pi}{2}$ it follows that,

$$\begin{aligned} d(x,y) &= sup_\alpha \int_\alpha^1 \cot(\frac{\pi}{2}M(x,y,t))dt \\ &\leq sup_\alpha \int_\alpha^1 \cot[\frac{\pi}{2}(M(x,z,t) * M(z,y,t))]dt \\ &= sup_\alpha \int_\alpha^1 \cot(\frac{\pi}{2}\min\{M(x,z,t), M(z,y,t)\})dt \\ &\leq sup_\alpha \int_\alpha^1 \cot(\frac{\pi}{2}M(x,z,t))dt \\ &\quad + sup_\alpha \int_\alpha^1 \cot(\frac{\pi}{2}M(z,y,t))dt \\ &= d(x,z) + d(z,y), \end{aligned}$$

that is d is a metric on X. □

Remark 2.1. Let $a, b \in (0, 1]$, then it is a standard result that

$$\text{Arccot}(\min\{a,b\}) \leq \text{Arccot}(a) + \text{Arccot}(b) - \frac{\pi}{4}$$

Lemma 2.6. *Let $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $(X, M, *)$ be a fuzzy metric space. We define $d : X^2 \longrightarrow [0, \infty)$ by* $d(x, y) = sup_\alpha \int_\alpha^1 (\frac{4}{\pi} Arccot(M(x, y, t)) - 1)dt$, *then d is a metric on X. Also* $0 \leq d(x, y) < 1$.

Proof. (i) $0 \leq d(x, y) < 1$ is trivial.

(ii)

$$\begin{aligned} d(x,y)=0 &\iff \frac{4}{\pi}\text{Arccot}(M(x,y,t))-1=0 \text{ for all } t>0 \\ &\iff \text{Arccot}(M(x,y,t))=\frac{\pi}{4} \text{ for all } t>0. \\ &\iff M(x,y,t)=1 \text{ for all } t>0 \iff x=y. \end{aligned}$$

(iii) $d(x,y)=d(y,x)$ is trivial.

(iv) Since

$$\begin{aligned} M(x,y,t) &\geq M(x,z,t)*M(z,y,t) \\ &= \min\{M(x,z,t),M(z,y,t)\}, \end{aligned}$$

it follows that,

$$\begin{aligned} \text{Arccot}(M(x,y,t)) &\leq \text{Arccot}[M(x,y,t)*M(z,y,t)] \\ &= \text{Arccot}(\min\{M(x,z,t),M(z,y,t)\}) \\ &\leq \text{Arccot}(M(x,z,t)) \\ &\quad +\text{Arccot}(M(z,y,t))-\frac{\pi}{4} \end{aligned}$$

Hence,

$$\begin{aligned} d(x,y) &= \sup_{\alpha}\int_{\alpha}^{1}(\frac{4}{\pi}\text{Arccot}(M(x,y,t))-1)dt \\ &\leq \sup_{\alpha}\int_{\alpha}^{1}(\frac{4}{\pi}\text{Arccot}(M(x,z,t))-1)dt \\ &\quad +\sup_{\alpha}\int_{\alpha}^{1}(\frac{4}{\pi}\text{Arccot}(M(z,y,t))-1)dt \\ &= d(x,z)+d(z,y), \end{aligned}$$

that is d is a metric on X.

□

Remark 2.2. Let $(X,M,*)$ be a fuzzy metric space with $a*b\geq ab$ for all $a,b\in[0,1]$ and $M(x,y,.)$ having discontinuity at 0, for all $x,y\in X$. If sequence $\{x_n\}$ in X converges to x, that is, for every $0<\epsilon<1$ there exist $n_0\in\mathbb{N}$ such that $M(x_n,x,t)>1-\epsilon$ for $\forall n\geq n_0$ and each $t>0$, then $d(x_n,x)\longrightarrow 0$ where $d(x,y)=\sup_{\alpha}\int_{\alpha}^{1}\log_a^{M(x,y,t)}dt$. Also it is a Cauchy sequence if for each $0<\varepsilon<1$ and $t>0$, there exits $n_0\in\mathbb{N}$ such that $M(x_n,x_m,t)>1-\varepsilon$ for each $n,m\geq n_0$. It follows that $d(x_n,x_m)=\sup_{\alpha}\int_{\alpha}^{1}\log_a^{M(x_n,x_m,t)}dt<\sup_{\alpha}\int_{\alpha}^{1}\log_a^{1-\varepsilon}dt<\eta$, for every $\eta=(1-\alpha)\log_a^{1-\varepsilon}$. Thus $\{x_n\}$ in (X,d) is a Cauchy sequence.

In 1976, Jungck[4] introduced the notion of commuting mappings to find common fixed point results in metric spaces. Later on, in[5] Jungck proposed the notion of compatible mappings which is a generalization of the concept of commuting mapping. Some common fixed point theorems for compatible mappings and their generalizations are addressed in [6], [7] [9] and [17]. In this paper we consider weak compatible mappings.

Definition 2.1 ([13])**.** Let A and S be mappings from a metric space X into itself. Then the mappings are said to be *weak compatible* if they commute at a coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

The Main Results

Theorem 2.7 ([3])**.** *Let A, B, S, T, L and M be self maps on a metric space (X, d) satisfying:*

(i) $L(X) \subseteq ST(X), M(X) \subseteq AB(X)$,

(ii) *there exists $k \in (0, 1)$ such that for each $x, y \in X$,*
$d(Lx, My) \leq k.N(x, y)$, *where*

$$N(x,y) = \max\{d(ABx, Ly), d(STy, My), d(ABx, STy), \frac{1}{2}(d(STy, Ly) + d(ABx, My))\};$$

(iii)*one of $L(X), M(X), AB(X)$ or $ST(X)$ is a complete subset of X, then*

(a) M and ST have a coincidence point,

(b) L and AB have a coincidence point.

Further if

(iv) $AB = BA, ST = TS, LB = BL, MT = TM$,

(v) *the pairs (L, AB) and (M, ST) are weakly compatible.*

Then A, B, S, T, L and M have a unique common fixed point in X.

Remark 2.3. Theorem 2.1 improves, extends and generalizes the results of Mishra [9] and Jungck [6].

We next apply theorem 2.1 to establish the following theorem in fuzzy metric spaces.

Theorem 2.8. *Let $(X, M, *)$ be a fuzzy metric space with $a*b \geq ab$ for all $a, b \in [0, 1]$ and $M(x, y, .)$ having discontinuity at 0, for all $x, y \in X$. Let A, B, S, T, L and H be self maps on X satisfying:*

(i) $L(X) \subseteq ST(X), H(X) \subseteq AB(X)$;

(ii) *there exists $k \in (0, 1)$ such that for each $x, y \in X$,*
$M(Lx, Hy, t) \geq$

$$\min \begin{pmatrix} M(ABx, Ly, t), M(STy, Hy, t), \\ M(ABx, STy, t), \sqrt{M(STy, Ly, t).M(ABx, Hy, t))} \end{pmatrix}^k$$

(iii)*one of $L(X), H(X), AB(X)$ or $ST(X)$ is a complete subset of X, then*

(a) H and ST have a coincidence point,

(b) L and AB have a coincidence point.

Further if

(iv) $AB = BA, ST = TS, LB = BL, HT = TH$,

(v) *the pairs* (L, AB) *and* (H, ST) *are weakly compatible.*

Then A, B, S, T, L *and* H *have a unique common fixed point in* X.

Proof. We define $d(x, y) = sup_\alpha \int_\alpha^1 \log_a^{M(x,y,t)} dt$ for every $x, y \in X$ where $0 < a < 1$. Then by lemma 1.8 and Remark 1.14, (X, d) is a complete metric space. From the above inequality, we get,

$sup_\alpha \int_\alpha^1 \log_a^{M(Lx,Hy,t)} dt \leq$

$$k \max \begin{pmatrix} sup_\alpha \int_\alpha^1 \log_a^{M(ABx,Ly,t)} dt, sup_\alpha \int_\alpha^1 \log_a^{M(STy,Hy,t)} dt, \\ sup_\alpha \int_\alpha^1 \log_a^{M(ABx,STy,t)} dt, \frac{1}{2}(sup_\alpha \int_\alpha^1 \log_a^{M(STy,Ly,t)} dt \\ +sup_\alpha \int_\alpha^1 \log_a^{M(ABx,Hy,t)} dt) \end{pmatrix},$$

which is,

$$d(Lx, Hy) \leq k \max \begin{pmatrix} d(ABx, Ly), d(STy, Hy), \\ d(ABx, STy), \frac{1}{2}(d(STy, Ly) + d(ABx, Hy)) \end{pmatrix}.$$

Hence all the conditions of Theorem 2.1 hold. Hence the conclusion of Theorem 2.3 follows by an application of Theorem 2.1 . □

Next we establish the following result in metric spaces.

Theorem 2.9. *Let* A, B, S *and* T *be self maps on a metric space* (X, d) *satisfying:*

(i) $A(X) \subseteq T(X), B(X) \subseteq S(X)$ *and* $T(X)$ *or* $S(X)$ *is a closed subset of* X*;*

(ii) *there exist positive real numbers* a, b, c, e *such that* $a + b + c + 2e \leq 1$ *and for each* $x, y \in X$,

$$\begin{aligned} d(Ax, By) &\leq ad(Sx, Ty) + bd(Sx, Ax) + cd(Ty, By) + e(d(Sx, By) \\ &\quad +d(Ty, Ax)); \end{aligned}$$

(iii) *the pairs* (A, S) *and* (B, T) *are weakly compatible.*

Then A, B, S *and* T *have a unique common fixed point in* X.

Proof. Let x_0 be an arbitrary point in X. By (i), we can choose a point x_1 in X such that $y_0 = Ax_0 = Tx_1$ and $y_1 = Bx_1 = Sx_2$. In general, there exists a sequence $\{y_n\}$ such that, $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 1, 2, \cdots$. We claim that the sequence $\{y_n\}$ is a Cauchy sequence.

By (ii), we have,

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Ax_{2n}, Bx_{2n+1}) \\ &\leq ad(Sx_{2n}, Tx_{2n+1}) + bd(Sx_{2n}, Ax_{2n}) \\ &\quad +cd(Tx_{2n+1}, Bx_{2n+1}) + e(d(Sx_{2n}, Bx_{2n+1}) \\ &\quad +d(Tx_{2n+1}, Ax_{2n})) \\ &= ad(y_{2n-1}, y_{2n}) + bd(y_{2n-1}, y_{2n}) + cd(y_{2n}, y_{2n+1}) \\ &\quad +e(d(y_{2n-1}, y_{2n+1}) \end{aligned}$$

$$+d(y_{2n}, y_{2n})).$$

If we put $d_n = d(y_n, y_{n+1})$, then by above inequality we have,

$$d_{2n} \leq ad_{2n-1} + bd_{2n-1} + cd_{2n} + e(d(y_{2n-1}, y_{2n+1}) + 0).$$

Hence,

$$d_{2n} \leq ad_{2n-1} + bd_{2n-1} + cd_{2n} + ed_{2n-1} + ed_{2n}.$$

Hence we have,

$$\begin{aligned} d_{2n} &\leq \frac{a+b+e}{1-c-e} d_{2n-1} \\ &= td_{2n-1}, \end{aligned}$$

where $0 < t = \frac{a+b+e}{1-c-e} < 1$.

Similarly, it follows that

$$\begin{aligned} d_{2n+1} &\leq \frac{a+c+e}{1-e-b} d_{2n} \\ &= t' d_{2n}, \end{aligned}$$

where $0 < t' = \frac{a+c+e}{1-e-b} < 1$. If we set $k = \max\{t, t'\} < 1$, then for every $n \in \mathbf{N}$ by above inequalities we get $d_n \leq kd_{n-1}$.

Hence,

$$d_n \leq kd_{n-1} \leq k^2 d_{n-2} \leq \cdots \leq k^n d_0.$$

That is,

$$d(y_n, y_{n+1}) \leq k^n d(y_0, y_1)$$

If $m \geq n$, then

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\ &\leq k^n d(y_0, y_1) + k^{n+1} d(y_0, y_1) \cdots + k^{m-1} d(y_0, y_1) \\ &\leq \frac{k^n}{1-k} d(y_0, y_1) \to 0 \end{aligned}$$

as $n \to \infty$. It follows that, the sequence $\{y_n\}$ is Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to $y \in X$. Then

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = y.$$

Let $T(X)$ be a closed subset of X, then there exists $v \in X$ such that $Tv = y$.

We now prove that $Bv = y$. By (ii), we get

$$\begin{aligned} \lim_{n\to\infty} d(Ax_{2n}, Bv) &\leq \lim_{n\to\infty} [ad(Sx_{2n}, Tv) + bd(Ax_{2n}, Sx_{2n}) \\ &\quad + cd(Bv, Tv) + e(d(Bv, Sx_{2n}) + d(Ax_{2n}, Tv))] \end{aligned}$$

and so

$$\begin{aligned} d(y, Bv) &\leq ad(y, Tv) + bd(y, y) + cd(Bv, y) + e(d(Bv, y) + d(y, Tv)) \\ &< d(y, Bv). \end{aligned}$$

It follows that $Bv = y = Tv$. Since B and T are two weakly compatible mappings, we have $BTv = TBv$ and so $By = Ty$.

Next, we prove that $By = y$. By (ii), we get

$$\begin{aligned}\lim_{n\to\infty} d(Ax_{2n}, By) &\leq \lim_{n\to\infty} [ad(Sx_{2n}, Ty) + bd(Ax_{2n}, Sx_{2n}) \\ &\quad +cd(By, Ty) + e(d(By, Sx_{2n}) + d(Ax_{2n}, Ty))].\end{aligned}$$

Hence,

$$\begin{aligned}d(y, By) &\leq ad(y, Ty) + bd(y, y) + cd(By, Ty) + e(d(By, y) + d(y, Ty)) \\ &< d(y, By)\end{aligned}$$

and so $By = y$.

Since $B(X) \subseteq S(X)$, there exists $w \in X$ such that $Sw = y$. We prove that $Aw = y$. By (ii) we have

$$\begin{aligned}d(Aw, By) &\leq ad(Sw, Ty) + bd(Aw, Sw) + cd(By, Ty) + e(d(By, Sw) \\ &\quad +d(Aw, Ty))\end{aligned}$$

and it follows that

$$\begin{aligned}d(Aw, y) &\leq ad(Sw, y) + bd(Aw, Sw) + cd(y, Ty) + e(d(y, Sw) \\ &\quad +d(Aw, Ty)) \\ &< d(Aw, y).\end{aligned}$$

This implies that $Aw = y$ and hence $Aw = Sw = y$. Since A and S are weakly compatible, then $ASw = SAw$ and so $Ay = Sy$.

Now, we prove that $Ay = y$. From (ii), we have

$$\begin{aligned}d(Ay, By) &\leq ad(Sy, Ty) + bd(Ay, Sy) + cd(By, Ty) + e(d(By, Sy) \\ &\quad +d(Ay, Ty))\end{aligned}$$

it follows that

$$\begin{aligned}d(Ay, y) &\leq ad(Sy, y) + bd(Ay, Sy) + cd(By, y) + e(d(y, Sy)) \\ &\quad +d(Ay, y)) \\ &< d(Ay, y)\end{aligned}$$

and hence $Ay = y$ and therefore $Ay = Sy = By = Ty = y$. That is y is a common fixed point for A, B, T, S.

The proof is similar when $S(X)$ is assumed to be a closed subset of X.

Now to prove the uniqueness. Assume that x is another common fixed point of A, B, S and T. Then

$$\begin{aligned}d(x, y) &= d(Ax, By) \\ &\leq ad(Sx, Ty) + bd(Ax, Sx) + cd(By, Ty) + e(d(Sx, By) \\ &\quad +d(Ax, Ty))\end{aligned}$$

and so

$$\begin{aligned}d(x, y) &\leq ad(x, y) + bd(x, x) + cd(y, y) + e(d(x, y) \\ &\quad +d(x, y))\end{aligned}$$

$$< \quad d(x,y).$$

Thus it follows that $x = y$. □

We now apply the above theorem to prove the following fixed point result in fuzzy metric spaces.

Theorem 2.10. *Let $(X, M, *)$ be a fuzzy metric space with $a * b \geq ab$ for all $a, b \in [0,1]$ and $M(x, y, .)$ having discontinuity at 0, for all $x, y \in X$. Let A, B, S and T be self maps on X satisfying:*

(i) *$A(X) \subseteq T(X), B(X) \subseteq S(X)$ and $T(X)$ or $S(X)$ is a closed subset of X;*

(ii) *there exists positive real numbers a, b, c, e such that $a+b+c+2e \leq 1$ and for each $x, y \in X$,*

$$M(Ax, By, t) \geq \begin{array}{l} [M^a(Sx, Ty, t) * M^b(Sx, Ax, t)] \\ *[M^c(Ty, By, t) * [M^e(Sx, By, t) * M^e(Ty, Ax, t)]] \end{array} \quad ;$$

(iii) *the pairs (A, S) and (B, T) are weakly compatible.*

Then A, B, S and T have a unique common fixed point in X.

Proof. We define $d(x, y) = sup_\alpha \int_\alpha^1 \log_k^{M(x,y,t)} dt$ for every $x, y \in X$ and $0 < k < 1$. From inequality (ii) above, we get,

$$\begin{aligned}
& \sup_\alpha \int_\alpha^1 \log_k^{M(Ax,By,t)} dt \\
\leq \; & \sup_\alpha \int_\alpha^1 \log_k^{\left(\begin{array}{l} [M^a(Sx,Ty,t) * M^b(Sx,Ax,t)] \\ *[M^c(Ty,By,t)* \\ [M^e(Sx,By,t) * M^e(Ty,Ax,t)]] \end{array}\right)} dt \\
\leq \; & \sup_\alpha \int_\alpha^1 \log_k^{\left(\begin{array}{l} M^a(Sx,Ty,t)M^b(Sx,Ax,t) \\ \quad M^c(Ty,By,t) \\ \quad [M^e(Sx,By,t)M^e(Ty,Ax,t)] \end{array}\right)} dt \\
= \; & \begin{array}{l} a \sup_\alpha \int_\alpha^1 \log_k^{M(Sx,Ty,t)} dt \\ +b \sup_\alpha \int_\alpha^1 \log_k^{M(Sx,Ax,t)} dt \\ +c \sup_\alpha \int_\alpha^1 \log_k^{M(Ty,By,t)} dt \\ +e(\sup_\alpha \int_\alpha^1 \log_k^{M(Sx,By,t)} dt \\ +\sup_\alpha \int_\alpha^1 \log_k^{M(Ty,Ax,t)} dt) \end{array}
\end{aligned}$$

It follows that,

$$d(Ax, By) \leq \quad ad(Sx, Ty) + bd(Sx, Ax) + cd(Ty, By) + e(d(Sx, By) + d(Ty, Ax)) \quad .$$

Hence all of conditions Theorem 2.4 hold. Thus A, B, S and T have a unique common fixed point in X. □

For any set C, ∂C denotes the boundary of C. The following theorem was proved by Rhoades.

Theorem 2.11 ([13])**.** *Let (X, d) be a metric space, C a nonempty closed subset of X. Let $g : C \longrightarrow X$, $f : C \longrightarrow C$ satisfy the following conditions:*

(i) *there exists a constant $\lambda \in (0, 1)$ such that for each $x, y \in C$,* $d(x, y) \leq \lambda.N(x, y)$, *where*

$$N(x, y) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(gy, fx)\},$$

(ii) $g(C) \cap C \subset f(C)$,

(iii) $g(\partial C) \subset C$,

(iv) $f(\partial C) \subset C$,

(v) *f and g are weakly compatible.*

Then f and g have a unique common fixed point in C.

Theorem 2.12. *Let $(X, M, *)$ be a fuzzy metric space with $a * b \geq ab$ for all $a, b \in [0, 1]$ and $M(x, y, .)$ having discontinuity at 0, for all $x, y \in X$. Let C a nonempty closed subset of X. Let $g : C \longrightarrow X$, $f : C \longrightarrow C$ satisfy the following conditions:*

(i) *there exists a constant $\lambda \in (0, 1)$ such that , for each $x, y \in C$,* $M(x, y, t) \geq L(x, y)^{\lambda}$, *where*

$$L(x, y) = \min\begin{pmatrix} M(fx, fy, t), M(fx, gx, t), \\ M(fy, gy, t), \\ M(fx, gy, t), M(gy, fx, t) \end{pmatrix},$$

(ii) $g(C) \cap C \subset f(C)$,

(iii) $g(\partial C) \subset C$,

(iv) $f(\partial C) \subset C$,

(v) *f and g are weakly compatible.*

Then f and g have a unique common fixed point in C.

Proof. We define $d(x, y) = \sup_\alpha \int_\alpha^1 \log_a^{M(x,y,t)} dt$ for every $x, y \in X$ where $0 < a < 1$. From the inequality (i) above, we get,

$$\begin{aligned} \sup_\alpha \int_\alpha^1 \log_a^{M(x,y,t)} dt \quad \leq \quad & \lambda \max\{\sup_\alpha \int_\alpha^1 \log_a^{M(fx,fy,t)} dt, \\ & \sup_\alpha \int_\alpha^1 \log_a^{M(fx,gx,t)} dt, \\ & \sup_\alpha \int_\alpha^1 \log_a^{M(fy,gy,t)} dt, \\ & \sup_\alpha \int_\alpha^1 \log_a^{M(fx,gy,t)} dt, \end{aligned}$$

$$\sup_{\alpha} \int_{\alpha}^{1} \log_a^{M(gy,fx,t)} dt\}$$

Hence we have

$$d(x,y) \leq \lambda \max\{d(fx,fy), d(fx,gx), d(fy,gy), d(fx,gy), d(gy,fx)\}$$

Hence all of conditions Theorem 2.6 hold. Thus by Theorem 2.6 maps f and g have a unique common fixed point in C. This completes the proof of the theorem. □

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