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OBSERVATIONS ON A FURTHER IMPROVED $(\frac{G}{G})$ - EXPANSION METHOD AND THE EXTENDED TANH-METHOD FOR FINDING EXACT SOLUTIONS OF NONLINEAR PDES

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ABSTRACT. In the present article, we construct the exact traveling wave solutions of nonlinear PDEs in the mathematical physics via the (1+1)-dimensional Boussinesq equation by using the following two methods: (i) A further improved $(\frac{G}{G})$ - expansion method, where $G = G(\xi)$ satisfies the auxiliary ordinary differential equation $[G'(\xi)]^2 = aG^2(\xi)+bG^4(\xi)+cG^6(\xi)$, where $\xi = x - Vt$ while a, b, c and V are constants. (ii) The well known extended tanh- function method. We show that some of the exact solutions obtained by these two methods are equivalent. Note that the first method (i) has not been used by anyone before which gives more exact solutions than the second method (ii).

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1. Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors(see for example [1-49]) who are interested in nonlinear physical phenomena. Many powerful different methods have been presented by those authors. For integrable nonlinear differential equations, the inverse scattering transform method [2], the Hirota method [10], the truncated Painleve expansion method [43], the Backlund transform method [19,21] and the exp-function method [4,9,36,44,45] are used in looking for the exact solutions. Among non-integrable nonlinear differential equations there is a wide class of the equations that referred to as the partially integrable, becaues these equations become integrable for some values of their parameters. There are many different methods to

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look for the exact solutions of these equations. The most famous algorithms are the truncated Painleve expansion method [14], the Weierstrass elliptic function method [13], the tanh- function method [1,7,8,32,34,39,46] and the Jacobi elliptic function expansion method [6,16,18,30,37,38,40]. There are other methods which can be found in [12, 23-28].

Wang et al [30] have introduced a simple method which is called the $(\frac{G'}{G})$ - expansion method to look for traveling wave solutions of nonlinear PDEs, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = x - Vt$ while V, λ and μ are arbitrary constants. For further references see the articles [3,5,7,20,41,42,48,49]. Recently El-Wakil et al [7] and Parkes [20] have shown that the extended tanh- function method proposed by Fan [8] and the basic $(\frac{G'}{G})$ - expansion method proposed by Wang et al [30] are entirely equivalent in as much as they deliver exactly the same set of solutions to a given evolution equation. This observation has been pointed out recently by Kudryashov [15].

In this article, we introduce an alternative approach which is called a further improved $(\frac{G'}{G})$ - expansion method to find the exact traveling wave solutions of some nonlinear PDEs, where $G = G(\xi)$ satisfies the auxiliary ordinary differential equation $[G'(\xi)]^2 = aG^2(\xi) + bG^4(\xi) + cG^6(\xi)$, where a, b, c are constants. This approach has not been used by anyone before. It will play an important role in constructing many exact traveling wave solutions for the nonlinear PDEs via the (1+1)- dimensional Boussinesq equation .

The objective of this article is to show that the exact solutions of this equation obtained by using the further improved $(\frac{G'}{G})$ - expansion method and the well known extended tanh- function method are equivalent.

2. Description of a further improved $(\frac{G'}{G})$ - expansion method

Suppose we have the following nonlinear partial differential equation

$$F(u, u_t, u_x, u_y, u_{tt}, u_{xt}, ...) = 0, \tag{1}$$

where $u = u(x, t)$ is an unknown function, F is a polynomial in $u(x, t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. In the following we give the main steps of a further improved $(\frac{G'}{G})$ - expansion method:

Step 1 . The traveling wave variable

$$u(x, t) = u(\xi) , \qquad \xi = x - Vt, \tag{2}$$

where V is a constant, permits us reducing Eq. (1) to an ODE for $u = u(\xi)$ in the form

$$P(u, u', u'', u''', ...) = 0, \tag{3}$$

where $' = \frac{d}{d\xi}$.

Step 2. Suppose the solution of Eq.(3) can be expressed by a polynomial in ($\frac{G'}{G}$) as follows

$$u(\xi) = \sum_{i=0}^{n} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{4}$$

where $G = G(\xi)$ satisfies the following auxiliary equation

$$[G'(\xi)]^2 = aG^2(\xi) + bG^4(\xi) + cG^6(\xi), \tag{5}$$

where α_i ,a, b, c and V are arbitrary constants to be determined provided $\alpha_n \neq 0$. The positive integer "n" can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq (1) or (3) . More precisely, we define the degree of $u(\xi)$ as $D[u(\xi)] = n$

which gives rise to the degree of other expressions as follows

$$D[\frac{d^q u}{d\xi^q}] = n + q, \qquad D[u^p (\frac{d^q u}{d\xi^q})^s] = np + s(q + n). \tag{6}$$

Therefore, we can get the value of n in (4).

Step 3. Substituting (4) into (3) and using Eq (5), we obtain polynomials in $G^j(\xi)$, $G'(\xi)G^j(\xi)$ $(j = 0, \pm 1, \pm 2, ...)$. Equating each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for α_i ,a, b, c and V which can be solved by Maple or Mathematica.

Step 4 . The general solutions of the auxiliary equation (5) have been well known (see for example [35,47]) which can be writen in the form:

No	$G(\xi)$	No	$G(\xi)$
1	$\left[\frac{-a\ b\ \text{sech}^2(\sqrt{a}\xi)}{b^2-ac(1+\epsilon\tanh(\sqrt{a}\xi))^2}\right]^{1/2}$, $a > 0$	8	$\left[\frac{-a\ \sec^2(\sqrt{-a}\xi)}{b+2\epsilon\sqrt{-ac}\tan(\sqrt{-a}\xi)}\right]^{1/2}$, $a < 0, c > 0$
2	$\left[\frac{a\ b\ \text{csch}^2(\sqrt{a}\xi)}{b^2-ac(1+\epsilon\coth(\sqrt{a}\xi))^2}\right]^{1/2}$, $a > 0$	9	$\left[\frac{a\ \text{csch}^2(\sqrt{a}\xi)}{b+2\epsilon\sqrt{ac}\coth(\sqrt{a}\xi)}\right]^{1/2}$ $a > 0, c > 0$
3	$\left[\frac{2a}{\epsilon\sqrt{\Delta}\cosh(2\sqrt{a}\xi)-b}\right]^{1/2}$, $a > 0, \Delta > 0.$	10	$\left[\frac{-a\ \csc^2(\sqrt{-a}\xi)}{b+2\epsilon\sqrt{-ac}\cot(\sqrt{-a}\xi)}\right]^{1/2}$, $a < 0, c > 0$
4	$\left[\frac{2a}{\epsilon\sqrt{\Delta}\cos(2\sqrt{-a}\xi)-b}\right]^{1/2}$, $a < 0, \Delta > 0$	11	$\left[-\frac{a}{b}(1+\epsilon\tanh(\frac{1}{2}\sqrt{a}\xi)\right]^{1/2}$, $a > 0, \Delta = 0$
5	$\left[\frac{2a}{\epsilon\sqrt{-\Delta}\sinh(2\sqrt{a}\xi)-b}\right]^{1/2}$, $a > 0, \Delta < 0$	12	$\left[-\frac{a}{b}(1+\epsilon\coth(\frac{1}{2}\sqrt{a}\xi)\right]^{1/2}$, $a > 0, \Delta = 0$
6	$\left[\frac{2a}{\epsilon\sqrt{\Delta}\sin(2\sqrt{-a}\xi)-b}\right]^{1/2}$, $a < 0, \Delta > 0$	13	$\left[\frac{ae^{2\epsilon\sqrt{a}\xi}}{(e^{2\epsilon\sqrt{a}\xi}-4b)^2-64ac}\right]^{1/2}$, $a > 0,$
7	$\left[\frac{-a\ \text{sech}^2(\sqrt{a}\xi)}{b+2\epsilon\sqrt{ac}\tanh(\sqrt{a}\xi)}\right]^{1/2}$ $a > 0, c > 0$	14	$\left[\frac{\epsilon ae^{2\epsilon\sqrt{a}\xi}}{1-64ace^{4\epsilon\sqrt{a}\xi}}\right]^{1/2}$, $a > 0, b = 0$

where $\Delta = b^2 - 4ac$ and $\epsilon = \pm 1$.

Step 5. Substituting α_i, V and the general solution of Eq (5) into (4) we have many exact traveling wave solutions of the nonlinear partial differential equation (1).

3. Some applications

In this section, we apply the further improved $(\frac{G^{'}}{G})$ - expansion method to construct the exact traveling wave solutions for the (1+1)- dimensional Boussinesq equation which are very important nonlinear evolution equations in the mathematical physics and have been paid attention by many researchers.

3.1. Example 1. On solving the Boussinesq equation by a further improved $(\frac{G^{'}}{G})$ - expansion method. We start with the (1+1)- dimensional Boussinesq equation [11,22,29,33] in the form

$$u_{tt} - u_{xx} - (u^2)_{xx} + u_{xxxx} = 0. \tag{7}$$

This equation was proposed by Boussinesq for a model of nonlinear dispersive waves. It describes the propagation of long waves in shallow water. It also arises in other physical applications such as nonlinear lattice waves, iron sound waves in a plasma and in vibrations in nonlinear string. Moreover, it was applied to problems in the percolation of water in porous subsurface strata. In the recent years, a lot of research work on Boussinesq equation has invested. For example , its solitary wave solutions, shock wave solutions, periodic and other types of solutions are found in [33]. The relations between a nonlinear lattice Boussinesq equation and the KdV equation are studied in [22].

Let us now solve Eq. (7) by the proposed method. To this end, we see that the traveling wave variable (2) permits us converting Eq. (7) into the following ODE:

$$(V^2 - 1)u - u^2 + u'' = 0, \tag{8}$$

with zero constants of integration. Considering the homogeneous balance between the highest order derivative and the nonlinear term in (8), we deduce from (6) that $D(u'') = D\ (u^2)$. Therefore $n + 2 = 2n$ and hence $n = 2$. Thus, we get

$$u(\xi)\ = \alpha_2 \left(\frac{G^{'}}{G}\right)^2 + \alpha_1 \left(\frac{G^{'}}{G}\right) + \alpha_0. \tag{9}$$

From (5) and (9) we deduce that

$$u\ = \alpha_2 \left[a + bG^2 + cG^4\right] + \alpha_1 G^{-1} G' + \alpha_0, \tag{10}$$

$$u' = 2\alpha_2 G' \left[bG + 2cG^3\right] + \alpha_1 [bG^2 + 2cG^4], \tag{11}$$

$$u'' = 2\alpha_2 \left[2abG^2 + (8ac + 3b^2)G^4 + 14bcG^6 + 12c^2G^8\right] + 2\alpha_1 G'[bG + 4cG^3], \tag{12}$$

and so on.

Substituting (10) and (9) into (8) we have the following polynomial:

$$
\begin{aligned}
&G^8[24\alpha_2 c^2 - \alpha_2^2 c^2] + G^6[28\alpha_2 bc - 2\alpha_2^2 cb] + \\
&G^4[16\alpha_2 ac + 6\alpha_2 b^2 + \alpha_2 c(V^2-1) - \alpha_2^2 b^2 - 2\alpha_2^2 ac - 2\alpha_0\alpha_2 c - \alpha_1^2 c] + \\
&G^2[4\alpha_2 ab + \alpha_2 b(V^2-1) - 2\alpha_2^2 ba - 2\alpha_0\alpha_2 b - \alpha_1^2 b] + \\
&G^3 G'[8\alpha_1 c - 2\alpha_1\alpha_2 c] + GG'[2\alpha_1 b - 2\alpha_1\alpha_2 b] + \\
&G^{-1}G'[\alpha_1(V^2-1) - 2\alpha_0\alpha_1 - 2\alpha_2\alpha_1 a] + \\
&(V^2-1)(a\alpha_2 + \alpha_0) - \alpha_2^2 a^2 - \alpha_0^2 - 2\alpha_0\alpha_2 a - \alpha_1^2 a = 0.
\end{aligned} \tag{13}
$$

Equating each coefficient of the polynomial (13) to zero , we have a system of algebraic equations which can be solved by Maple or Mathematica to obtain the following two sets of solutions:

The set 1.

$$\alpha_2 = 24, \quad \alpha_1 = 0, \quad \alpha_0 = -8a\ , \quad V = \pm\sqrt{1+16a}, \quad b = 0. \tag{14}$$

The set 2.

$$\alpha_2 = 24, \quad \alpha_1 = 0, \quad \alpha_0 = -24a\ , \quad V = \pm\sqrt{1-16a}, \quad b = 0. \tag{15}$$

For the set 1, we have the following solution:

$$u(\xi) = 24\left(\frac{G'}{G}\right)^2 - 8a, \tag{16}$$

where

$$\xi = x \mp t\ \sqrt{1+16a}, \tag{17}$$

while for the set 2, we have the following solution:

$$u(\xi) = 24\left(\frac{G'}{G}\right)^2 - 24a, \tag{18}$$

where

$$\xi = x \mp t\ \sqrt{1-16a}, \tag{19}$$

According to the step 4 of section 2, we have the following families of exact solutions

Family 1. If $a > 0, \Delta > 0$,then the solution of Eq. (5) has the form

$$G(\xi) = \left[\frac{2a}{\epsilon\sqrt{\Delta}\cosh(2\sqrt{a}\xi) - b}\right]^{1/2} \tag{20}$$

Since $b = 0$, then $c < 0$. In this case we have the ratio

$$\frac{G'}{G} = -\sqrt{a}\ \tanh(2\sqrt{a}\xi). \tag{21}$$

Consequently, we have the following traveling wave solutions

For the set 1, we have

$$u(\xi) \ = 24a \ \tanh^2(2\sqrt{a}\xi) - 8a, \tag{22}$$

while for the set 2, we have

$$u(\xi) \ = -24a \ \mathrm{sech}^2(2\sqrt{a}\xi), \tag{23}$$

where

$$\xi = x \mp \sqrt{1 \pm 16a}\ t\ . \tag{24}$$

respectively.

Family 2. If $a < 0, \Delta > 0$,then the solution of Eq. (5) has the form

$$G(\xi) \ = \left[\frac{2a}{\epsilon\sqrt{\Delta}\cos(2\sqrt{-a}\xi) - b}\right]^{1/2} \text{or} \qquad G(\xi) \ = \left[\frac{2a}{\epsilon\sqrt{\Delta}\sin(2\sqrt{-a}\xi) - b}\right]^{1/2} \tag{25}$$

Since $b = 0$, then $c > 0$. In this case we have the ratio

$$\frac{G'}{G} \ = \sqrt{-a}\ \tan(2\sqrt{-a}\xi) \quad \text{or} \quad \frac{G'}{G} \ = -\ \sqrt{-a}\ \cot(2\sqrt{-a}\xi) \tag{26}$$

Consequently, we have the following traveling wave solutions

For the set 1, we have

$$u(\xi) \ = -24a\ \tan^2(2\sqrt{-a}\xi) - 8a \quad \text{or} \quad u(\xi) \ = -24a\ \cot^2(2\sqrt{-a}\xi) - 8a \tag{27}$$

while for the set 2, we have

$$u(\xi) \ = -24a\ \tan^2(2\sqrt{-a}\xi) - 24a \quad \text{or} \quad u(\xi) \ = -24a\ \csc^2(2\sqrt{-a}\xi) \tag{28}$$

where

$$\xi = x \mp \sqrt{1 \pm 16a}\ t\ . \tag{29}$$

respectively.

Family 3. If $a > 0, \Delta < 0$,then the solution of Eq. (5) has the form

$$G(\xi) \ = \left[\frac{2a}{\epsilon\sqrt{-\Delta}\sinh(2\sqrt{a}\xi) - b}\right]^{1/2} \tag{30}$$

Since $b = 0$, then $c > 0$. In this case we have the ratio

$$\frac{G'}{G} \ = -\sqrt{a}\ \coth(2\sqrt{a}\xi). \tag{31}$$

Consequently, we have the following traveling wave solutions

For the set 1, we have

$$u(\xi) \ = 24a\ \coth^2(2\sqrt{a}\xi) - 8a, \tag{32}$$

while for the set 2, we have

$$u(\xi) \ = 24a\ \ \mathrm{csch}^2(2\sqrt{a}\xi), \tag{33}$$

where

$$\xi = x \mp \sqrt{1 \pm 16a}\ \ t\ . \tag{34}$$

respectively

.

Family 4. If $a > 0, c > 0$,then the solution of Eq. (5) has the form

$$G(\xi) = \left[\frac{-a\ \text{sech}^2(\sqrt{a}\xi)}{b + 2\epsilon\sqrt{ac}\tanh(\sqrt{a}\xi)}\right]^{1/2} \quad \text{or} \quad G(\xi) = \left[\frac{a\ \text{csch}^2(\sqrt{a}\xi)}{b + 2\epsilon\sqrt{ac}\coth(\sqrt{a}\xi)}\right]^{1/2} \tag{35}$$

Since $b = 0$, then we have the ratio

$$\frac{G'}{G} = -\frac{1}{2}\sqrt{a}\ [\tanh(\sqrt{a}\xi) + \coth(\sqrt{a}\xi)] \quad \text{or} \quad \frac{G'}{G} = -\frac{a\ \epsilon}{2\sqrt{c}}\ [\text{csch}^2(\sqrt{a}\xi)] \tag{36}$$

Consequently, we have the following traveling wave solutions

For the set 1, we have

$$u(\xi) = 6a\ \ [\tanh(\sqrt{a}\xi) + \coth(\sqrt{a}\xi)]^2 - 8a, \quad \text{or} \quad u(\xi) = \frac{6a^2}{c}\text{csch}^4(\sqrt{a}\xi) - 8a \tag{37}$$

while for the set 2, we have

$$u(\xi) = 6a\ \ [\tanh(\sqrt{a}\xi) + \coth(\sqrt{a}\xi)]^2 - 24a, \quad \text{or}\ u(\xi) = \frac{6a^2}{c}\text{csch}^4(\sqrt{a}\xi) - 24a, \tag{38}$$

where

$$\xi = x \mp \sqrt{1 \pm 16a}\ t\ . \tag{39}$$

respectively.

Family 5. If $a < 0, c > 0$,then the solution of Eq. (5) has the form

$$G(\xi) = \left[\frac{-a\ \sec^2(\sqrt{-a}\xi)}{b + 2\epsilon\sqrt{-ac}\tan(\sqrt{-a}\xi)}\right]^{1/2} \quad \text{or} \quad G(\xi) = \left[\frac{-a\ \csc^2(\sqrt{-a}\xi)}{b + 2\epsilon\sqrt{-ac}\cot(\sqrt{-a}\xi)}\right]^{1/2} \tag{40}$$

Since $b = 0$, then we have the ratio

$$\frac{G'}{G} = -\frac{1}{2}\sqrt{-a}\ [\tan(\sqrt{-a}\xi) - \cot(\sqrt{-a}\xi)]. \tag{41}$$

Consequently, we have the following traveling wave solutions

For the set 1, we have

$$u(\xi) = -6a\ \ [\tan(\sqrt{-a}\xi) - \cot(\sqrt{-a}\xi)]^2 - 8a, \tag{42}$$

while for the set 2, we have

$$u(\xi) = -6a\ \ [\tan(\sqrt{-a}\xi) - \cot(\sqrt{-a}\xi)]^2 - 24a, \tag{43}$$

where

$$\xi = x \mp \sqrt{1 \pm 16a}\ t\ . \tag{44}$$

respectively.

Family 6. If $a > 0, b = 0$ then the solution of Eq. (5) has the form

$$G(\xi) \ = \left[\frac{\epsilon a e^{2\epsilon\sqrt{a}\xi}}{1-64ace^{4\epsilon\sqrt{a}\xi}}\right]^{1/2} \tag{45}$$

Consequently we have the ratio

$$\frac{G'}{G} \ = \frac{\epsilon}{8\sqrt{c}}\coth(\frac{\epsilon\xi}{4\sqrt{c}}) \tag{46}$$

where $64ac = 1$. Consequently, we have the following traveling wave solutions

For the set 1, we have

$$u(\xi) \ = 24a \quad \coth^2(2\sqrt{a}\epsilon\xi) - 8a \tag{47}$$

while for the set 2, we have

$$u(\xi) \ = 24a \quad \coth^2(2\sqrt{a}\epsilon\xi) - 24a, \tag{48}$$

where

$$\xi = x \mp \sqrt{1 \pm 16at} \ . \tag{49}$$

respectively. Similarly, we can write down the other families of exact solutions of Eq. (7) which are omitted for convenience.

3.2. Example 2. On solving the Boussinesq equation by the extended tanh-function method. With reference to the well known extended tanh-function method [1,7,8,32,34,39,46], the solution of the Boussinesq equation (7) can be writen in the form:

$$u(\xi) = \alpha_2\phi^2(\xi) + \alpha_1\phi(\xi) + \alpha_0, \tag{50}$$

where $\phi(\xi)$ satisfies the Riccati equation

$$\phi'(\xi) = R + \phi^2(\xi) \tag{51}$$

The Riccati equation (51) have the following solutions:

(i) If $R < 0$, then

$$\phi(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi), \quad \text{or} \quad \phi(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\xi) \tag{52}$$

(ii) If $R > 0$, then

$$\phi(\xi) = \sqrt{R}\tan(\sqrt{R}\xi), \ \text{or} \quad \phi(\xi) = -\sqrt{R}\cot(\sqrt{R}\xi) \tag{53}$$

(iii) If $R = 0$, then

$$\phi(\xi) = \frac{-1}{\xi}. \tag{54}$$

Substituting (50) along with (51) into (8) we get the following polynomail:

$$(-\alpha_2^2 + 6\alpha_2)\phi^4 + (-2\alpha_2\alpha_1 + 2\alpha_1)\phi^3 + (V^2\alpha_2 - \alpha_1^2 - \alpha_2 - 2\alpha_2\alpha_0 + 8\alpha_2 R)\phi^2$$
$$+(-\alpha_1 + V^2\alpha_1 - 2\alpha_1\alpha_0 + 2\alpha_1 R)\phi + 2\alpha_2 R^2 - \alpha_0 + V^2\alpha_0 - \alpha_0^2 = 0 \tag{55}$$

Equating the coefficients of this polynomial to zero and solving the algebraic equations by Maple or Mathematica, we have the following two sets of solutions:

The set 3

$$\alpha_2 = 6, \quad \alpha_1 = 0, \quad \alpha_0 = 2R, \quad V = \pm\sqrt{1-4R}, \tag{56}$$

The set 4

$$\alpha_2 = 6, \quad \alpha_1 = 0, \quad \alpha_0 = 6R, \quad V = \pm\sqrt{1+4R} \tag{57}$$

Thus, the exact solutions of Eq.(7) have the following forms:

For the set 3 we deduce for $R < 0$ that

$$u(\xi) = -6R\ \tanh^2(\sqrt{-R}\xi) + 2R, \quad \text{or} \quad u(\xi) = -6R\ \coth^2(\sqrt{-R}\xi) + 2R., \tag{58}$$

while for the set 4 we deduce for $R < 0$ that

$$u(\xi) = -6R\ \tanh^2(\sqrt{-R}\xi) + 6R, \quad \text{or} \quad u(\xi) = -6R\ \coth^2(\sqrt{-R}\xi) + 6R.. \tag{59}$$

For the set 3 we deduce for $R > 0$ that

$$u(\xi) = 6R\ \tan^2(\sqrt{R}\xi) + 2R, \quad \text{or} \quad u(\xi) = 6R\ \cot^2(\sqrt{R}\xi) + 2R., \tag{60}$$

while for the set 4 we deduce for $R > 0$ that

$$u(\xi) = 6R\ \tan^2(\sqrt{R}\xi) + 6R, \quad \text{or} \quad u(\xi) = 6R\ \cot^2(\sqrt{R}\xi) + 6R., \tag{61}$$

where

$$\xi = x \mp \sqrt{1 \mp 4R}t$$

respectively.

From the previous results, we have the following remarks:
Remark 1. If we put $R = -4a$ where $a > 0$ then the results (58),(59) are equivalent to the results (22), (32), (23) and (33) respectively .
Remark 2. If we put $R = -4a$ where $a < 0$ then the results (60),(61) are equivalent to the results (27) and (28) respectively .

From these remarks we have the following observation :
" The exact solutions of the Boussinesq equation obtained using the extended tanh- function method are equivalent to its exact solutions obtained using the further improved $(\frac{G'}{G})$- expansion method."

4. Conclusions

In summary, we have found the exact solutions of the (1+1)- dimensional Bussinesq equation (3.1) using two methods via the further improved $(\frac{G}{G})$ - expansion method and the extended tanh-function method. We have arrived at the observation that these exact solutions are equivalent.

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