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STABILITY OF A TWO-STRAIN EPIDEMIC MODEL WITH AN AGE STRUCTURE AND MUTATION†

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Abstract. A two-strain epidemic model with an age structure mutation and varying population is studied. By means of the spectrum theory of bounded linear operator in functional analysis, the reproductive numbers according to the strains, which associates with the growth rate λ^* of total population size are obtained. The asymptotic stability of the steady states are obtained under some sufficient conditions.

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1. Introduction

Increase in disease virulence accelerates the transmission of infectious between hosts. Simultaneously it reduces the longevity of infection within hosts. Change in virulence can alter host-pathogen dynamics, significantly. Mathematical modelling can aid that process by providing insight into the mechanisms that sustain microorganisms genetic diversity. Competitive exclusion and coexistence of strains in gonorrhea and other sexually transmitted diseases are discussed in [1, 2, 11]. Some of those are super-infection[3, 4, 5], coinfection [6], cross-immunity [7], density-dependent host mortality [8]. Because of their paramount importance in biology and public health, multi-strain models attract significant attention. Results of these research efforts have been summarized in three topical reviews [12, 14, 15]. Pathogen mutations that circumvent the protective effects of a patient's immune response are common in infections diseases such as measles, hepatitis B, HIV, West Nile virus, and influenza. Mutation [9] is an important nature of the multi-strains. Mutation has impact on the complex

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nature of the strain epidemic model. In our paper, we consider the mutation of two strains in an epidemic model.

Host age structure is often an important factor which affects the dynamics of disease transmission. The age-structured epidemic models also have enriched the knowledge of epidemic models. In addition, the age-structured epidemic models are generally be described by the first-order partial differential equations with nonlocal boundary conditions, and it is very difficult to theoretically analyze the dynamical behaviors of the models. Therefore, it is necessary to study age-structured epidemic model with multiple strains. To our knowledge, there are few papers to consider age-structured epidemic models with multiple strains. Li et al [17] discuss two trains model with superinfection. They obtained the reproductive numbers according to the strains and proved the local asymptotical stability of the boundary equilibria. They also give the existence of the endemic equilibrium using the fixed theory. However, they didn't give the explicit formula of the endemic equilibria. Most existing studies fucus on the total population is a constant. In reality, this assumption is not realistic and has some limits. In this paper, we mainly study an age-structured epidemic model with mutation and varying population, and focus on the local behaviors of the model.

This paper is organized as follows: Sec.2 introduces age-structure two strains epidemic model with mutation. In Sec.3, we obtain existence of the positive solution. In Sec.4, existence of the steady states are studied, In Sec.5, we analysis the spectral of the linear operator at steady states. Sec.5 gives locally asymptotical stability of the steady states. In Sec.6, we conclude the main contents in this paper and give some discussions.

2. The model formulation

In this section, we introduce an age-structured two-strain model with mutation. The total population $P(t,a)$ is divided into four classes: susceptible $S(t,a)$, infected with strain one $I(t,a)$, infected with strain two $J(t,a)$ and recovered $R(t,a)$. Susceptible individuals can become the infected by strain one and move to the class $I(t,a)$, or infected by strain two and move to the class $J(t,a)$. We assume that those infected with strain one can mutate into strain two. This process is referred as mutation. We take the transmission rate $\lambda_1(t,a)$ in the separable intercohort constitutive form for the force of infection generated by $I(t,a)$

$$\lambda_1(t,a) = k(a)\frac{\int_0^{a_+} h_1(a)I(t,a)da}{\int_0^{a_+} P(t,a)da}$$

where $h_1(a)$ is the age-specific infectiousness for strain one, and $k(a)$ is the age-specific susceptibility of susceptible individuals. We note that in essence what we have assumed is that the age-specific contact rate of individuals age a with individuals age b is separable: $c(a,b) = c_1(a)c_2(b)$. The assumption for separability of the contact rates is important part of our results. Without it different techniques may be necessary to deal with the problem. The term

$c_1(a)$ can be absorbed in the coefficient $k(a)$, while the term $c_2(b)$ is absorbed into the function $h_1(b)$. Similarly, we take $\lambda_2(a,t)$ in the separable intercohort constitutive form for the force of infection generated by $J(a,t)$

$$\lambda_2(t,a) = k(a)\frac{\int_0^{a_+} h_2(a)J(t,a)da}{\int_0^{a_+} P(t,a)da}$$

where $h_2(a)$ is the age-specific infectiousness for strain two. In this article we assume that the susceptibilities for the two strains are the same and given by $k(a)$ but our results can be easily and trivially extended to the case when they are different, that is $k_1(a) \neq k_2(a)$. The functions $h_i(a), k(a)$ have compact support and satisfy

$$\begin{aligned}&k(a), h_i(a) \in L^\infty(0,+\infty), k(a), h_i(a) \geq 0, \text{ on } a \in (0,a^+), i = 1,2\\ &\text{and } k(a), h_i(a) = 0, \text{ for } a > a^+.\end{aligned}$$

In addition, the other parameters satisfy

$$\begin{aligned}&\gamma_i(a), \alpha(a), \beta(a) \in L^\infty(0,+\infty), i = 1,2,\\ &\mu(a) \in L^1_{loc}(0,a^+), \text{ and } \int_0^{a^+} \mu(a)da = \infty,\\ &\mu(a)\pi(a) \text{ is a.e. boundary at } (0,a^+).\end{aligned}$$

where $\pi(a) = e^{-\int_0^a \mu(\tau)d\tau}$. All the infected individuals can cured and remove into the removed classes $R(t,a)$ at rates $\gamma_1(a)$ and $\gamma_2(a)$, respectively. Then the joint dynamics of the age-structured epidemiological model for the transmission of TB can be written as

$$\begin{cases} \frac{\partial S(t,a)}{\partial t} + \frac{\partial S(t,a)}{\partial a} = -\lambda_1(t,a)S(t,a) - \lambda_2(t,a)S(t,a) - \mu(a)S(t,a), \\ \frac{\partial I(t,a)}{\partial t} + \frac{\partial I(t,a)}{\partial a} = \lambda_1(t,a)S(t,a) - \mu(a)I(t,a) - \gamma_1(a)I(t,a) - \alpha(a), \\ \frac{\partial J(t,a)}{\partial t} + \frac{\partial J(t,a)}{\partial a} = \lambda_2(t,a)S(t,a) + \alpha(a)I(t,a) - \mu(a)J(t,a) - \gamma_2(a)I(t,a), \\ \frac{\partial R(t,a)}{\partial t} + \frac{\partial R(t,a)}{\partial a} = \gamma_1(a)I(t,a) + \gamma_2(a)J(t,a) - \mu(a)R(t,a), \\ S(t,0) = \int_0^{a^+} \beta(a)P(t,a)da, I(t,0) = J(t,0) = R(t,0) = 0, \\ S(0,a) = S_0(a) \geq 0, I(0,a) = I_0(a) \geq 0, J(0,a) = I_0(a) \geq 0, R(0,a) = R_0(a) \geq 0. \end{cases} \quad (2.1)$$

where $\beta(a)$ is the age-specific per capita birth rate. Summing the equations of (2.1) we obtain the following problem for the total population density $P(t,a) = S(t,a) + I(t,a) + J(t,a) + R(t,a)$.

$$\begin{cases} \frac{\partial P(t,a)}{\partial t} + \frac{\partial P(t,a)}{\partial a} = -\mu(a)P(t,a), \\ P(t,0) = \int_0^{a^+} \beta(a)P(t,a)da, \ P(0,a) = P_0(a) \geq 0. \end{cases} \quad (2.2)$$

3. Existence of positive solutions

Consider the Banach space

$$\mathbf{X} = L^1(0, a^+) \times L^1(0, a^+) \times L^1(0, a^+) \times L^1(0, a^+)$$

endowed with the norm

$$\|f\| = \sum_{i=1}^{4} \|f_i\| \text{ for } f(a) = (f_1(a), f_2(a), f_3(a), f_4(a)) \in \mathbf{X},$$

where $\|.\|$ is the norm of $L^1(0, a^+)$. The state space of system (2.1) is

$$\Gamma = \{(S,\ I,\ J,\ P) \mid 0 \le S + I + J \le P,\ S \ge 0,\ I \ge 0,\ J \ge 0\}.$$

where $\mathbf{X}_+ = L^1_+(0, a^+) \times L^1_+(0, a^+) \times L^1_+(0, a^+) \times L^1_+(0, a^+)$, and $L^1_+(0, a^+)$ denotes the positive cone of $L^1(0, a^+)$. We define the following notations

$$H(t) = \int_0^{a_+} h_1(a) I(t, a) da + \int_0^{a_+} h_2(a) J(t, a) da$$

and

$$H_1(t) = \int_0^{a_+} h_1(a) I(t, a) da, H_2(t) = \int_0^{a_+} h_2(a) J(t, a) da.$$

Let $\mathcal{A} : D(\mathcal{A}) \to \mathcal{X}$ be a linear operator defined by

$$\mathcal{A} = \begin{pmatrix} -\frac{\partial}{\partial a} - \mu a & 0 & 0 & 0 \\ 0 & -\frac{\partial}{\partial a} - \mu(a) - \alpha(a) - \gamma_1(a) & 0 & 0 \\ 0 & \alpha(a) & -\frac{\partial}{\partial a} - \mu(a) - \gamma_2(a) & 0 \\ 0 & 0 & 0 & -\frac{\partial}{\partial a} - \mu(a) \end{pmatrix}$$

and the domain $\mathcal{D}(\mathcal{A})$ is given as

$$\mathcal{D}(\mathcal{A}) = \left\{ f \;\middle|\; f_i(.) \text{ is absolutely continuous}, \right.$$
$$\left. f_1(0) = f_4(0) = \int_0^{a_+} \beta(a) f_4(a) da, f_2(0) = f_3(0) \right\}.$$

And as same time we define nonlinear operator $\mathcal{F} : \mathbf{X} \to \mathbf{X}$ as

$$\mathcal{F}(f)(a) = \begin{pmatrix} -k(a) f_1(a) \frac{\mathcal{H}_1(f)}{\mathcal{N}(f)} - k(a) f_1(a) \frac{\mathcal{H}_2(f)}{\mathcal{N}(f)} \\ f_1(a) \frac{\mathcal{H}_1(f)}{\mathcal{N}(f)} \\ f_1(a) \frac{\mathcal{H}_2(f)}{\mathcal{N}(f)} \\ 0 \end{pmatrix},$$

where

$$\mathcal{H}_1 : X \to R, \mathcal{H}_1(f) = \int_0^{a_+} h_1(a) f_2(a) da,$$

$$\mathcal{H}_2 : X \to R, \mathcal{H}_2(f) = \int_0^{a_+} h_2(a) f_3(a) da,$$

$$\mathcal{N} : X \to R, \mathcal{N}(f) = \int_0^{a_+} f_4(a)da.$$

$\mathcal{F}(0) = (0, 0, 0, 0)^T$. We define the functional φ :

$$< \varphi, f >:= \mathcal{N}(f) = \int_0^{a_+} f_4(a)da.$$

Since $\mu(a)\pi(a)$ is almost boundary on $(0, a^+)$, $\forall f \in \mathbf{X}$, $< \varphi, \mathcal{A}f >$ is a boundary operator on $\mathbf{X}$ i.e $\varphi \in D(\mathcal{A}^*)$, where $\mathcal{A}^*$ is adjoint operator of $\mathcal{A}$. It is easy to know that $\mathcal{H}_i, i = 1, 2$ are bounded linear operators on $L^1(0, a^+)$.

Let $u(t) = (S(t, \cdot), I(t, \cdot), J(t, \cdot), P(t, \cdot))$. Thus, we can rewrite the system as an abstract Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = \mathcal{A}u(t) + \mathcal{F}(u(t)), \\ u(0) = u_0 \in \mathbf{X} \end{cases} \tag{3.1}$$

where $u_0 = (S_0(a), I_0(a), J_0(a), P_0(a))^T$.

For $\mathcal{A}$ and $\mathcal{F}$, we have the following results.

Lemma 3.1. *The operator $\mathcal{A}$ generates a C_0 semigroup $e^{t\mathcal{A}}$ and the space Γ is positively invariant with respect to the semiflow $e^{t\mathcal{A}}$.*

Lemma 3.2. *The operator $\mathcal{F}$ is continuous and Fréchet differentiable on $\mathbf{X}$.*

By Lemma 3.1 and 3.2 and following Inaba[13] and Webb[16], we have the following theorem.

Theorem 3.3. *For each $u_0 \in \mathbf{X}_+$, there are a maximal interval of existence $[0, t_0)$ and a unique continuous mild solution $u(t, u_0) \in \mathbf{X}_+, t \in [0, t_0)$ for (3.1) such that*

$$u(t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{\mathcal{A}(t-\tau)}\mathcal{F}(u(\tau))d\tau.$$

The operator $B : D(B) \subset \mathbf{X} \to \mathbf{X}$ is defined by

$$B = \mathcal{A} + \mathcal{F}'(E_*), \ D(B) = D(\mathcal{A}).$$

It is easy to see that B is a boundary perturbation on $\mathcal{A}$. We assume $E(t)$ is a solution of (3.1), and define

$$w(t) = E(t) - E_* \in ker\phi,$$

where

$$ker\phi = \{z \in X| < \phi, z >= 0\}.$$

In the following, we define another linear operator S on $\mathbf{X}$

$$Sz = Bz- < \phi, \quad Bz > E_*, \quad D(S) = D(B).$$

Note that e^{St} is a semigroup generated by S and $\mathbf{X}_\phi = ker\phi$, and S_ϕ is the linear operator restricted on $\mathbf{X}_\phi$. According to [10], we have the following lemmas.

Lemma 3.4. *If* $\|e^{S_\phi t}\| \le Me^{\omega t}, \omega < \lambda^*$, *where* λ^* *is a characteristic value of nonlinear homogeneous operator* $\mathcal{A}+\mathcal{F}$, *thus*

$$\lim_{t\to\infty} w(t) = 0.$$

Lemma 3.5. *If the semigroup generated by* S_ϕ *is finally compact and*

$$\sup_{\lambda\in\sigma(B)} Re\lambda \neq \lambda^*,$$

thus

$$\lim_{t\to\infty} w(t) = 0.$$

4. Existence of steady state

In this section, we discuss existence of the persistent solutions of (2.1). Denote

$$S(t,a) = e^{\lambda t}S^*(a), I(t,a) = e^{\lambda t}I^*(a), J(t,a) = e^{\lambda t}J^*(a), P(t,a) = e^{\lambda t}P^*(a),$$

which satisfies the following equations

$$\begin{cases} \lambda S^*(a) + \dfrac{dS^*(a)}{da} = -\mu(a)S^*(a) - k(a)\dfrac{H_1^*}{N^*}S^*(a) - k(a)\dfrac{H_2^*}{N^*}S^*(a), \\ \lambda I^*(a) + \dfrac{dI^*(a)}{da} = -(\mu(a)+\alpha(a)+\gamma_1(a))I^*(a) + k(a)\dfrac{H_1^*}{N^*}S^*(a), \\ \lambda J^*(a) + \dfrac{dJ^*(a)}{da} = -(\mu(a)+\gamma_2(a))J^*(a) + \alpha(a)I^*(a) + k(a)\dfrac{H_2^*}{N^*}S^*(a), \\ \lambda P^*(a) + \dfrac{dP^*(a)}{da} = -\mu(a)P^*(a), \\ S^*(0) = P^*(0) = \displaystyle\int_0^{a_+}\beta(a)P^*(a)da, I^*(0) = J^*(0) = 0. \end{cases} \tag{4.1}$$

where $H_1^* = \int_0^{a_+} h_1(a)I^*(a)da, H_2^* = \int_0^{a_+} h_2(a)J^*(a)da, N^* = \int_0^{a_+} P^*(a)da.$ In fact, the existence of persistent solutions of (4.1) is equivalent to the existence of persistent solutions of (2.1). Solving the fourth equation of (4.1), we get

$$P^*(a) = b_0 e^{-\lambda^* a}\pi(a),$$

where $\pi(a) = e^{-\int_0^a \mu(a)da}$. λ^* is unique real root of the following characteristic equation

$$\int_0^{a_+}\beta(a)e^{-\lambda a}\pi(a)da = 1.$$

and then

$$N^* = \int_0^{a_+} P^*(a)da = 1, b_0\int_0^{a_+}\pi(a)e^{-\lambda^* a}da = 1.$$

Solving the other three equations of (4.1),

(1) we obtain disease-free steady state $E_0(S^0, I^0, J^0, P^0)$, where

$$S^0(a) = P^0(a) = P^*(a), I^0 = J^0 = 0, H_1^* = H_2^* = 0;$$

(2) we get strain two dominated steady state $E_1(S_1^*, I_1^*, J_1^*, P_1^*)$, where

$$\begin{aligned} I_1^* &= 0, \\ S_1^* &= b_0\pi(a)e^{-\lambda^* a}e^{-H_{21}^*\int_0^a k(\tau)d\tau}, \\ J_1^* &= b_0 H_{21}^*\pi(a)e^{-\lambda^* a}\int_0^a k(\sigma)e^{-\int_\sigma^a \gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^\sigma k(\tau)d\tau}d\sigma. \end{aligned} \tag{4.2}$$

Substituting J_1^* into H_{21}^*, it leads to

$$H_{21}^* = H_{21}^* b_0\int_0^{a_+} h_2(a)\pi(a)e^{-\lambda^* a}\int_0^a k(\sigma)e^{-\int_\sigma^a \gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^\sigma k(\tau)d\tau}d\sigma da. \tag{4.3}$$

Both dividing H_{21}^* of (4.3), it is easy to get the following equation

$$\begin{aligned} R_2(H_{21}^*) = b_0\int_0^{a_+} h_2(a)\pi(a)e^{-\lambda^* a}\int_0^a k(\sigma)e^{-\int_\sigma^a \gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^\sigma k(\tau)d\tau}d\sigma da \\ = 1. \end{aligned} \tag{4.4}$$

It is easy to know that $R_2(H_{21}^*) \to 0$, as $H_{21}^* \to +\infty$; and $R_2(H_{21}^*) \to +\infty$, as $H_{21}^* \to -\infty$. Therefore, if $R_2(0) > 1$, (4.4) has a unique positive solution H_{21}^*;

(3) we have endemic steady state $E_*(S_*, I_*, J_*, P_*)$, where

$$\begin{aligned} S_* =& b_0\pi(a)e^{-\lambda^* a}e^{-H^*\int_0^a k(\tau)}d\tau, \\ I_* =& b_0 H_1^*\pi(a)e^{-\lambda^* a}\int_0^a k(\sigma)e^{-\int_\sigma^a(\alpha(\tau)+\gamma_1(\tau))d\tau}e^{-H^*\int_0^\sigma k(\sigma)d\sigma}d\sigma, \\ J_* =& b_0 H_2^*\pi(a)e^{-\lambda^* a}\int_0^a k(\sigma)e^{-\int_\sigma^a \gamma_2(\tau)d\tau}e^{-H^*\int_0^\sigma k(\sigma)d\sigma}d\sigma + b_0 H_1^*\pi(a)e^{-\lambda^* a} \\ & \int_0^a k(\sigma)e^{-H^*\int_0^\sigma k(\tau)d\tau}\int_\sigma^a \alpha(\xi)e^{-\int_\xi^a \gamma_2(\tau)d\tau}e^{-\int_\sigma^\xi(\alpha(\tau)+\gamma_1(\tau))d\tau}d\xi d\sigma. \end{aligned}$$

Substituting I^* into H_1^*, it leads to

$$H_1^* = b_0 H_1^*\int_0^{a_+} h_1(a)\pi(a)e^{-\lambda^* a}\int_0^a k(\sigma)e^{-\int_\sigma^a(\alpha(\tau)+\gamma_1(\tau))d\tau}e^{-H^*\int_0^\sigma k(\tau)d\tau}d\sigma da, \tag{4.5}$$

and substituting J^* into H_2^*, it leads to

$$\begin{aligned} H_2^* =& b_0 H_2^*\int_0^{a_+} h_2(a)\pi(a)e^{-\lambda^* a}\int_0^a k(\sigma)e^{-\int_\sigma^a \gamma_2(\tau)d\tau}e^{-H^*\int_0^\sigma k(\tau)d\tau}d\sigma da \\ & + b_0 H_1^*\int_0^{a_+} h_2(a)\pi(a)e^{-\lambda^* a}\int_0^a k(\sigma)e^{-H^*\int_0^\sigma k(\tau)d\tau}\int_\sigma^a \alpha(\xi) \\ & e^{-\int_\xi^\sigma \gamma_2(\tau)d\tau}e^{-\int_\sigma^\xi(\alpha(\tau)+\gamma_1(\tau))d\tau}d\xi d\sigma da. \end{aligned} \tag{4.6}$$

Let

$$R_1(H^*) = b_0\int_0^{a_+} h_1(a)\pi(a)e^{-\lambda^* a}\int_0^a k(\sigma)e^{-\int_\sigma^a(\alpha(\tau)+\gamma_1(\tau))d\tau}e^{-H^*\int_0^\sigma k(\tau)d\tau}d\sigma da,$$

$$R_2(H^*) = b_0\int_0^{a_+} h_2(a)\pi(a)e^{-\lambda^* a}\int_0^a k(\sigma)e^{-\int_\sigma^a \gamma_2(\tau)d\tau}e^{-H^*\int_0^\sigma k(\tau)d\tau}d\sigma da.$$

Thus (4.5) and (4.6) are equivalent to the following equation

$$\begin{vmatrix} R_1(H^*)-1 & 0 \\ M(H^*) & R_2(H^*)-1 \end{vmatrix} = 0 \tag{4.7}$$

where

$$M(H^*) = -\,b_0 \int_0^{a_+} h_2(a)\pi(a)e^{-\lambda^* a} \int_0^a k(\sigma)e^{-H^* \int_0^\sigma k(\tau)d\tau} \int_\sigma^a \alpha(\xi)e^{-\int_\xi^\sigma \gamma_2(\tau)d\tau} e^{-\int_\sigma^\xi (\alpha(\tau)+\gamma_1(\tau))d\tau} d\xi d\sigma da.$$

From (4.7), we get

$$R_i(H^*) = 1, i = 1, 2.$$

Note that $\lim_{H^* \to +\infty} R_i(H^*) = 0$, $\lim_{H^* \to -\infty} R_i(H^*) = 0, i = 1, 2$. Therefore, if $R_{i0} = R_i(0) > 1, i = 1, 2$, it has a positive solution H^*. Concluding above the results, we get the following theorem.

Theorem 4.1. *If $R_{i0} = R_i(0) < 1, i = 1, 2$ (3.1) has disease-free state steady $E_0 = (S^0, I^0, J^0, P^0)$ compared unique λ^*.*
If $R_{20} > 1$, and $R_{10} < 1$, (3.1) has two steady states, one is disease-free steady state E_0 and the other is strain two dominated steady state $E_1^* = (S_1^*, 0, J_1^*, P_1^*)$.
If $R_{10} > 1$ and $R_{20} > 1$, (3.1) has two steady states, one is disease-free equilibrium E_0 and the other is endemic steady state $E_* = (S^*, I^*, J^*, P^*)$.

5. Analysis of linear operators at the steady states

In this section, we mainly analysis linear operators at the steady states. We can change the problem into analysis the spectral of linearize operator at the steady states in (2.1). $\mathcal{B}$ denotes the linear operator $\mathcal{A} + \mathcal{F}'(E_0)$ at E_0; $\mathcal{B}_1$ denotes the linear operator $\mathcal{A} + \mathcal{F}'(E_1^*)$ at E_1; $\mathcal{B}_2$ denotes the linear operator $\mathcal{A} + \mathcal{F}'(E^*)$ at E^*.

Firstly, we analysis the spectral of $\mathcal{B}$. We have to find the nontrivial solution of

$$(\mathcal{B} - \lambda)v = 0$$

where $v = (s, i, j, p)^T$, i.e.

$$\begin{cases} \dfrac{ds}{da} = -\lambda s - \mu(a)s - \hat{x}k(a)S^0(a) - \hat{y}k(a)S^0(a), \\ \dfrac{di}{da} = -\lambda i - \mu(a) - \alpha(a)i - \gamma_1(a)i + \hat{x}k(a)S^0(a), \\ \dfrac{dj}{da} = -\lambda j - \mu(a)j - \gamma_2(a)j + \alpha(a)i + \hat{y}k(a)S^0(a), \\ \dfrac{dp}{da} = -\lambda p - \mu(a)p \end{cases} \tag{5.1}$$

with the boundary conditions

$$\begin{aligned} &s(0)=p(0)=\int_0^{a^+}\beta(a)p(a)da,\ \ i(0)=j(0)=0,\ \ \hat{x}=\int_0^{a^+}h_1(a)i(a)da,\\ &\hat{y}=\int_0^{a^+}h_2(a)j(a)da,\ \ \hat{n}=\int_0^{a^+}p(a)da,\ \ \hat{z}=\hat{x}+\hat{y}. \end{aligned} \tag{5.2}$$

Denote λ^* is the real solution of the characteristic equation (5.3),

$$\int_0^{a_+}\beta(a)e^{-\lambda a}\pi(a)da=1. \tag{5.3}$$

$\alpha_j, j=1,2,\cdots$ are the complex roots of the characteristic equation (5.3).

Now we discuss two following cases:
(1) If $\lambda=\lambda^*$ or $\lambda=\alpha_j$ for some j, then the fourth equation of (5.1) with (5.2) has nontrivial solution p, so system (5.1) has nontrivial solution $(p,0,0,p)$ at least, and λ^*, α_j are the characteristic value of $\mathcal{B}_0$.
(2) If $\lambda\neq\lambda^*$ and $\lambda\neq\alpha_j, j=1,2,\cdots$, then the fourth equation of (5.1) with (5.2) has only trivial solution, i.e $p(a)=0$, that is $s(0)=p(0)=0, \hat{n}=0$. Solving the first three equations of (5.1), we get

$$\begin{aligned} s(a)=&-b_0\pi(a)\hat{z}\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}d\sigma,\\ i(a)=&b_0\pi(a)\hat{x}\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}e^{-\int_\sigma^a(\alpha(\tau)+\gamma_1(\tau))d\tau}d\sigma,\\ j(a)=&b_0\pi(a)\hat{y}\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}e^{-\int_\sigma^a\gamma_2(\tau)d\tau}d\sigma+b_0\pi(a)\hat{x}\int_0^a\\ &\alpha(\sigma)e^{-\int_\sigma^a\gamma_2(\tau)d\tau}\int_0^\sigma k(\xi)e^{\lambda(\xi-a)}e^{-\lambda^*\xi}e^{-\int_\xi^\sigma(\alpha(\tau)+\gamma_1(\tau))d\tau}d\xi d\sigma. \end{aligned} \tag{5.4}$$

Changing the integral order of $j(a)$, we obtain

$$\begin{aligned} j(a)=&b_0\pi(a)\hat{y}\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}e^{-\int_\sigma^a\gamma_2(\tau)d\tau}d\sigma+b_0\pi(a)\hat{x}\int_0^a\\ &k(\sigma)e^{\lambda(\sigma-a)}e^{-\lambda^*\sigma}\int_\sigma^a\alpha(\eta)e^{-\int_\eta^a\gamma_2(\tau)d\tau}e^{-\int_\sigma^\eta(\alpha(\tau)+\gamma_1(\tau)d\tau}d\eta d\sigma. \end{aligned} \tag{5.5}$$

Substituting (5.5) into (5.2), and dividing $\hat{x}$, we obtain

$$\left(b_0\int_0^{a^+}h_1(a)\pi(a)\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}e^{-\int_\sigma^a(\alpha(\tau)+\gamma_1(\tau))d\tau}d\sigma da-1\right)\hat{x}=0$$

$$\begin{aligned} &\hat{x}b_0\int_0^{a^+}h_2(a)\pi(a)\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}e^{-\int_\sigma^a\gamma_2(\tau)d\tau}d\sigma da+\\ &\quad\left(b_0\int_0^{a^+}h_2(a)\pi(a)\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}e^{-\int_\sigma^a\gamma_2(\tau)d\tau}d\sigma da-1\right)\hat{y}=0, \end{aligned}$$

that is

$$\begin{aligned} F_1(\lambda) &= b_0 \int_0^{a^+} h_1(a)\pi(a) \int_0^a k(\sigma)e^{-\lambda(a-\sigma)} e^{-\lambda^*\sigma} e^{-\int_\sigma^a (\alpha(\tau)+\gamma_1(\tau))d\tau} d\sigma da = 1, \\ F_2(\lambda) &= b_0 \int_0^{a^+} h_2(a)\pi(a) \int_0^a k(\sigma)e^{-\lambda(a-\sigma)} e^{-\lambda^*\sigma} e^{-\int_\sigma^a \gamma_2(\tau)d\tau} d\sigma da = 1. \end{aligned} \tag{5.6}$$

It is easy to see all the solutions of (5.6) are characteristic values of $\mathcal{B}_0$. Note that $F_1(\lambda)$ and $F_2(\lambda)$ respectively have unique real root $\widetilde{\lambda_i}$ and a series complex roots $\widetilde{\alpha_{ij}}, j = 1, 2, \cdots$. In addition, $F_i'(\lambda) < 0, i = 1, 2,$

$$\lim_{\lambda \to +\infty} F_i(\lambda) = 0$$

and

$$F_i(0) = R_{i0}.$$

Therefore, if $R_{i0} > 1$, and $\widetilde{\lambda_i} > \lambda^*$; otherwise, $\widetilde{\lambda_i} < \lambda^*$.

Theorem 5.1. *The spectral of $\mathcal{B}_0$ only consists of $\lambda^*, \alpha_{ij}, \widetilde{\lambda_i}, \widetilde{\alpha}_{ij}$, and*
(1) if $R_{10} < 1$ and $R_{20} < 1$, then $Re\widetilde{\alpha_{ij}} < \lambda^*$, and $Re\widetilde{\alpha}_{ij} < \widetilde{\lambda_i} < \lambda^*$;
(2) if $R_{i0} > 1, i = 1, 2$, then $\widetilde{\lambda_i} > \lambda^*$.

Secondly, we discuss the spectral of $\mathcal{B}_1$, when $R_{10} < 1$ and $R_{20} > 1$. We want to find nontrivial solution of

$$(\mathcal{B}_1 - \lambda)v = 0$$

where $v = (s, 0, j, p)^T$, i.e.

$$\begin{cases} \dfrac{ds}{da} = -\lambda s - \mu(a)s - \hat{x}k(a)S_1^*(a) - k(a)H_{21}^* s - (\hat{y} - H_{21}^*\hat{n})k(a)S_1^*, \\ \dfrac{di}{da} = -\lambda i - \mu(a) - \alpha(a)i - \gamma_1(a)i + \hat{x}k(a)S_1^*(a), \\ \dfrac{dj}{da} = -\lambda j - \mu(a)j - \gamma_2(a)j + \alpha(a)i + k(a)H_{21}^* s + (\hat{y} - H_{21}^*\hat{n})k(a)S_1^*(a), \\ \dfrac{dp}{da} = -\lambda p - \mu(a)p \end{cases} \tag{5.7}$$

with the boundary conditions

$$\begin{aligned} &s(0) = p(0) = \int_0^{a^+} \beta(a)p(a)da, \quad i(0) = j(0) = 0, \quad \hat{x} = \int_0^{a^+} h_1(a)i(a)da, \\ &\hat{y} = \int_0^{a^+} h_2(a)j(a)da, \quad \hat{n} = \int_0^{a^+} p(a)da, \quad \hat{z} = \hat{x} + \hat{y}. \end{aligned} \tag{5.8}$$

Denote λ^* is the real solution of the characteristic equation (5.3), $\alpha_{ij}, j = 1, 2, \cdots$ are the complex roots of the characteristic equation (5.3).

Now we discuss the two following cases:
(1) If $\lambda = \lambda^*$ or $\lambda_i = \alpha_{ij}$ for some j, then the fourth equation of (5.7) with (5.8) has nontrivial solution p, so system (5.7) has nontrivial solution $(p, 0, 0, p)$ at least, and λ^*, α_{ij} are the characteristic values of $\mathcal{B}_1$.
(2) If $\lambda \neq \lambda^*$ and $\lambda \neq \alpha_{ij}, j = 1, 2, \cdots$, then the fourth equation of (5.7) with

(5.8) has only trivial solution, i.e $p(a) = 0$, that is $s(0) = p(0) = 0, \hat{n} = 0$. Solving the first three equations, we get

$$\begin{aligned}
s(a) &= -b_0\pi(a)\hat{z}e^{-H_{21}^*\int_0^a k(\tau)d\tau}\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}d\sigma,\\
i(a) &= b_0\pi(a)\hat{x}\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}e^{-\int_\sigma^a(\alpha(\tau)+\gamma_1(\tau))d\tau}e^{-H_{21}^*\int_0^\sigma k(\tau)d\tau}d\sigma,\\
j(a) &= b_0\pi(a)\hat{y}\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}e^{-\int_\sigma^a \gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^\sigma k(\tau)d\tau}d\sigma\\
&\quad - b_0\pi(a)\hat{y}H_{21}^*\int_0^a k(\eta)e^{-\int_\eta^a \gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^\eta k(\tau)d\tau}\int_0^\eta k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}d\sigma d\eta\\
&\quad - b_0\pi(a)\hat{x}H_{21}^*\int_0^a k(\eta)e^{-\int_\eta^a \gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^\eta k(\tau)d\tau}\int_0^\eta k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}d\sigma d\eta\\
&\quad + b_0\pi(a)\hat{x}\int_0^a \alpha(\sigma)e^{-\int_\sigma^a \gamma_2(\tau)d\tau}\int_0^\sigma k(\xi)e^{\lambda(\xi-a)}e^{-\lambda^*\xi}e^{-\int_\xi^\sigma(\alpha(\tau)+\gamma_1(\tau))d\tau}\\
&\quad e^{-H_{21}^*\int_0^\xi k(\tau)d\tau}d\xi d\sigma.
\end{aligned} \tag{5.9}$$

Changing the integral order of $j(a)$, we obtain

$$\begin{aligned}
j(a) =& b_0\pi(a)\hat{y}\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}e^{-\int_\sigma^a \gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^\sigma k(\tau)d\tau}d\sigma - b_0\pi(a)\\
&\hat{x}H_{21}^*\int_0^a k(\eta)e^{-\int_\eta^a \gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^\eta k(\tau)d\tau}\int_0^\eta k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}d\sigma d\eta\\
&+ b_0\hat{x}\pi(a)H(\lambda)
\end{aligned} \tag{5.10}$$

where

$$\begin{aligned}
H(\lambda) =& - H_{21}^*\int_0^a k(\eta)e^{-\int_\eta^a \gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^\eta k(\tau)d\tau}\int_0^\eta k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}d\sigma d\eta\\
&+ \int_0^a \alpha(\sigma)e^{-\int_\sigma^a \gamma_2(\tau)d\tau}\int_0^\sigma k(\xi)e^{\lambda(\xi-a)}e^{-\lambda^*\xi}e^{-\int_\xi^\sigma(\alpha(\tau)+\gamma_1(\tau))d\tau}\\
&e^{-H_{21}^*\int_0^\xi k(\tau)d\tau}d\xi d\sigma.
\end{aligned}$$

Substitute (5.10) into (5.9), we obtain

$$\begin{aligned}
&\Bigg(b_0\int_0^{a^+} h_1(a)\pi(a)\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}e^{-\int_\sigma^a(\alpha(\tau)+\gamma_1(\tau))d\tau}e^{-H_{21}^*\int_0^\sigma k(\tau)d\tau}\\
&\qquad d\sigma da - 1\Bigg)\hat{x} = 0,\\
&\hat{x}b_0\int_0^{a^+} h_2(a)\pi(a)\int_0^a H(\lambda)d\sigma da + \Bigg(b_0\int_0^{a^+} h_2(a)\pi(a)\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}\\
&\quad\Big\{e^{-\int_\sigma^a \gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^\sigma k(\tau)d\tau}d\sigma da - H_{21}^*\int_\sigma^a k(\eta)e^{-\int_\eta^a k(\eta)\gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^a k(\tau)d\tau}\\
&\qquad d\eta d\sigma\Big\}da - 1\Bigg)\hat{y} = 0,
\end{aligned}$$

that is

$$\begin{aligned} F_1(\lambda) =& b_0\int_0^{a^+} h_1(a)\pi(a)\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}e^{-\int_\sigma^a(\alpha(\tau)+\gamma_1(\tau))d\tau} \\ & e^{-H_{21}^*\int_0^\sigma k(\tau)d\tau}d\sigma da = 1, \\ F_2(\lambda) =& b_0\int_0^{a^+} h_2(a)\pi(a)\int_0^a k(\sigma)e^{-\lambda(a-\sigma)}e^{-\lambda^*\sigma}[e^{-H_{21}^*\int_0^\sigma k(\tau)d\tau}e^{-\int_\sigma^a\gamma_2(\tau)d\tau} \\ & d\sigma da - H_{21}^*\int_\sigma^a k(\eta)e^{-\int_\eta^a\gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^a k(\tau)d\tau}d\eta d\sigma]da = 1. \end{aligned} \tag{5.11}$$

We assume

$$\frac{J_1^*(a^+)}{P^*(a^+)} \le e^{-\int_0^{a^+}\gamma_2(\tau)d\tau}.$$

We straightly calculate and obtain the following formula

$$\begin{aligned} & e^{-\int_\sigma^a\gamma_2(\tau)d\tau}e^{-H_{21}^*\int_\sigma^a k(\tau)d\tau} - H_{21}^*\int_\sigma^a k(\eta)e^{-\int_\eta^a\gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^\eta k(\tau)d\tau}d\eta \\ \ge & \int_0^\sigma \gamma_2(\eta)e^{-\int_\eta^a\gamma_2(\tau)d\tau}e^{-H_{21}^*\int_0^\eta k(\tau)d\tau}d\eta \ge 0 \end{aligned} \tag{5.12}$$

and

$$e^{-H_{12}^*\int k(\tau)d\tau} \ge \int_0^a \gamma_2(\sigma)e^{-\int_\sigma^a\gamma_2(\tau)d\tau}e^{-H_{12}^*\int_0^\sigma k(\tau)d\tau}. \tag{5.13}$$

From (5.12) and (5.13), then we have

$$F_2(\lambda^*) < b_0\int_0^{a^+} h_2(a)\pi(a)e^{-\lambda^* a}\int_0^a k(\sigma)e^{-H_{21}^*\int_0^\sigma k(\tau)d\tau}e^{-\int_\sigma^a\gamma_2(\tau)d\tau}d\sigma da = R_2(H_{21}^*) = 1.$$

It is easy to see all the solutions of (5.11) are characteristic values of $\mathcal{B}_1$, and $Re\widetilde{\alpha_{2j}} < \widetilde{\lambda_2} < \lambda^*$.

Note that

$$F_1(\lambda^*) = b_0\int_0^{a^+} h_1(a)\pi(a)\int_0^a k(\sigma)e^{-\lambda^* a}e^{-\int_\sigma^a(\alpha(\tau)+\gamma_1(\tau))d\tau}e^{-H_{21}^*\int_0^\sigma k(\tau)d\tau}d\sigma da = 1.$$

Hence, $F_1(\lambda)$ has unique real root $\widetilde{\lambda_1}$ and a series complex roots $\widetilde{\alpha_{1j}}, j = 1, 2, \cdots$. Since $F_1'(\lambda) < 0$,

$$\lim_{\lambda\to+\infty} F_1(\lambda) = 0$$

and

$$F_1(0) = R_{10}.$$

Therefore, if $R_{10} > 1$, and then $\widetilde{\lambda_1} > \lambda^*$; otherwise, $\widetilde{\lambda_1} < \lambda^*$.

Theorem 5.2. *The spectral of* $\mathcal{B}_1$ *consists of* $\lambda^*, \alpha_{ij}, \widetilde{\lambda}_i, \widetilde{\alpha}_{ij}$, *if* $R_{10} < 1, R_{20} > 1$, *and* $\frac{J_1^*(a^+)}{P^*(a^+)} \le e^{-\int_0^{a^+}\gamma_2(\tau)d\tau}$, *then* $Re\alpha_{ij} < \lambda^*$, *and* $Re\widetilde{\alpha_{ij}} < \widetilde{\lambda}_i < \lambda^*$.

Finally, we analysis the spectral of $\mathcal{B}_2$. We have to find nontrivial solution of

$$(\mathcal{B}_2 - \lambda)v = 0$$

where $v = (s, i, j, p)^T$, i.e.

$$\begin{cases} \frac{ds}{da} = -\lambda s - \mu(a)s - (\hat{x} - H_1^*\hat{n})k(a)S^*(a) - (\hat{y} - H_2^*\hat{n})k(a)S^*(a) \\ \qquad - k(a)H^* s, \\ \frac{di}{da} = -\lambda i - \mu(a) - \alpha(a)i - \gamma_1(a)i + (\hat{x} - H_1^*\hat{n})k(a)S^*(a) + k(a)H_1^* s, \\ \frac{dj}{da} = -\lambda j - \mu(a)j - \gamma_2(a)j + \alpha(a)i + (\hat{y} - H_2^*\hat{n})k(a)S^*(a) + k(a)H_2^* s, \\ \frac{dp}{da} = -\lambda p - \mu(a)p, \end{cases} \tag{5.14}$$

with the boundary conditions

$$\begin{aligned} &s(0) = p(0) = \int_0^{a^+} \beta(a)p(a)da, \ \ i(0) = j(0) = 0, \ \ \hat{x} = \int_0^{a^+} h_1(a)i(a)da, \\ &\hat{y} = \int_0^{a^+} h_2(a)j(a)da, \ \ \hat{n} = \int_0^{a^+} p(a)da, \ \ \hat{z} = \hat{x} + \hat{y}. \end{aligned} \tag{5.15}$$

We use the same method and obtain the following characteristic equation

$$F_3(\lambda)G_2(\lambda) - F_4(\lambda)G_1(\lambda) = 0 \tag{5.16}$$

where

$$\begin{aligned} F_3(\lambda) =& b_0 \int_0^{a^+} h_1(a)\pi(a) \int_0^a k(\sigma)e^{\lambda(\sigma-a)}e^{-\lambda^*\sigma}[e^{-\int_\sigma^a \gamma_1(\tau)+\alpha(\tau)d\tau}e^{-H^*\int_0^a k(\tau)d\tau} \\ &- H_1^* \int_\sigma^a k(\eta)e^{-\int_\eta^\sigma(\gamma_1(a)+\alpha(a))d\tau}e^{-H^*\int_0^\eta k(\tau)d\tau}d\eta]d\sigma da - 1, \\ F_4(\lambda) =& -b_0H_1^* \int_0^{a^+} h_1(a)\pi(a) \int_0^a k(\eta)e^{-\int_\eta^\sigma(\gamma_1(a)+\alpha(a))d\tau}e^{-H^*\int_0^\eta k(\tau)d\tau} \\ &\int_0^\eta k(\sigma)e^{\lambda(\sigma-a)}e^{-\lambda^*\sigma}d\sigma d\eta, \\ G_1(\lambda) =& b_0 \int_0^{a^+} h_2(a)\pi(a) \int_0^a \{\int_0^a \alpha(\xi)e^{-\int_\xi^a \gamma_2(\tau)d\tau} \int_0^\xi k(\sigma)e^{\lambda(\sigma-a)}e^{-\lambda^*\sigma} \\ &[e^{-\int_\sigma^\xi \gamma_1(\tau)+\alpha(\tau)d\tau}e^{-H^*\int_0^\sigma k(\tau)d\tau} - H_1^*\int_0^\sigma k(\eta)e^{-\int_\eta^\sigma(\gamma_1(a)+\alpha(a))d\tau} \\ &e^{-H^*\int_0^\eta k(\tau)d\tau}]d\eta d\sigma d\xi\} - H_2^*\int_0^a k(\eta)e^{-\int_\eta^a \gamma_2(\tau)d\tau}e^{-H^*\int_0^\eta k(\tau)d\tau}d\eta da, \\ G_2(\lambda) =& b_0 \int_0^{a^+} h_2(a)\pi(a) \int_0^a \{\int_0^a \alpha(\sigma)e^{\lambda(\sigma-a)}e^{-\lambda^*\sigma}[e^{-\int_\sigma^a \gamma_2(\tau)d\tau}e^{-H^*\int_0^\sigma k(\tau)d\tau} \\ &- H_2^*\int_\sigma^a k(\eta)e^{-\int_\eta^\sigma \gamma_2(\tau)d\tau}e^{-H^*\int_0^\eta k(\tau)d\tau}]d\eta d\sigma - H_1^*\int_0^\sigma \alpha(\xi)e^{-\int_\xi^\sigma \gamma_2(\tau)d\tau} \\ &\int_0^\xi k(\eta)e^{-\int_\eta^\xi \gamma_1(\tau)+\alpha(\tau)d\tau}e^{-H^*\int_0^\eta k(\tau)d\tau}\int_0^\eta k(\sigma)e^{\lambda(\sigma-a)}e^{-\lambda^*\sigma}d\sigma d\eta d\xi. \end{aligned}$$

Theorem 5.3. *If $R_{10} > 1$, and $R_{20} > 1$, the spectral of $\mathcal{B}_2$ consists of $\lambda^*, \alpha_{ij}, \widetilde{\lambda}_i$, $\widetilde{\alpha}_{ij}$, and the relations of the characteristic roots is determined by (5.16).*

6. The stability of the steady state

In section 5, we analysis the spectral of linear operator at the steady states. According to the reference [10] and Lemma 3.4 , 3.5, we just only prove that the semigroups generated by $S_{0\phi}, S_{1\phi}$ and S_ϕ are finally compact. In fact, we just only prove the semigroup by $S_{2\phi}$ is finally compact. The other situations are the special case of this situation. We define $H^* = H_1^* + H_2^*$. Note that $S_{2\phi} = \mathcal{B}_2 - < \phi, \mathcal{B}_2 > E_*$, we rewrite $S_{2\phi} = A + T$, where

$$A = \begin{pmatrix} -\frac{\partial}{\partial a} - \mu(a) - H^* k(a) & 0 & 0 & 0 \\ H_1^* k(a) & -\frac{\partial}{\partial a} - \mu(a) - \alpha(a) - \gamma_1(a) & 0 & 0 \\ H_2^* k(a) & \alpha(a) & -\frac{\partial}{\partial a} - \mu(a) - \gamma_2(a) & 0 \\ 0 & 0 & 0 & -\frac{\partial}{\partial a} - \mu(a) \end{pmatrix}$$

and

$$(Tv)(a) = \begin{pmatrix} -(\hat{x} - H_1^* \hat{n})k(a)S^*(a) - (\hat{y} - H_2^* \hat{n})k(a)S^*(a) - < \varphi, \mathcal{B}v > S^*(a) \\ (\hat{x} - H_1^* \hat{n})k(a)S^*(a) - < \varphi, \mathcal{B}v > I^*(a) \\ (\hat{x} - H_2^* \hat{n})k(a)S^*(a) - < \varphi, \mathcal{B}v > J^*(a) \\ - < \varphi, \mathcal{B}v > P^*(a) \end{pmatrix}.$$

Since the dimension of T is finite, T is a compact operator. We just prove the semigroup generated by A is finally compact. We note that the domains of A and $\mathcal{A}$ are consistent. When $H^* = 0$, the operators A and $\mathcal{A}$ are also consistent. We consider the solution of the following system

$$\begin{cases} \dfrac{\partial f_1(a,t)}{\partial t} + \dfrac{\partial f_1(a,t)}{\partial a} = -\mu(a) f_1(a,t) - H^* k(a) f_1(a,t), \\ \dfrac{\partial f_2(a,t)}{\partial t} + \dfrac{\partial f_2(a,t)}{\partial a} = H_1^* k(a) f_1(a,t) - \mu(a) f_2(a,t) - \alpha(a) f_2(a,t) \\ \qquad\qquad - \gamma_1(a) f_2(a,t), \\ \dfrac{\partial f_3(a,t)}{\partial t} + \dfrac{\partial f_3(a,t)}{\partial a} = \alpha(a) f_2(a,t) + H_2^* k(a) f_2(a,t) - \mu(a) f_3(a,t) \\ \qquad\qquad - \gamma_2(a) f_3(a,t), \\ \dfrac{\partial f_4(a,t)}{\partial t} + \dfrac{\partial f_4(a,t)}{\partial a} = -\mu(a) f_4(a,t) \end{cases} \tag{6.1}$$

with the boundary conditions

$$f_1(0,t) = f_4(0,t) = \int_0^{a^+} \beta(a) f_4(a,t) da,$$
$$f_2(0,t) = f_3(0,t) = 0.$$

Denote $\mathcal{B}(t) = \int_0^{a^+} \beta(a) f_4(a,t)$ the new born at time t. And we obtain the semigroup generated by A

$$(e^{At}f^0)(a) = \begin{pmatrix} \mathcal{B}(t-a)\pi(a)e^{-H^*\int_0^a k(\tau)d\tau} \\ H_1^*\mathcal{B}(t-a)\pi(a)\int_0^a k(a)e^{-H^*\int_0^\sigma k(\tau)d\tau}e^{-\int_\sigma^a(\gamma_1(\tau)+\alpha(\tau))d\tau}d\sigma \\ L \\ \mathcal{B}(t-a)\pi(a) \end{pmatrix} \quad (6.2)$$

where

$$L = H_2^*\mathcal{B}(t-a)\pi(a)\int_0^a k(\sigma)e^{-H^*\int_0^\sigma k(\tau)d\tau}e^{-\int_\sigma^a \gamma_2(\tau)d\tau} + H_1^*\mathcal{B}(t-a)\pi(a)$$
$$\int_0^a \alpha(\rho)e^{-\int_\rho^\sigma \gamma_2(\tau)d\tau}\int_0^\rho k(\sigma)e^{-H^*\int_0^\sigma k(\tau)d\tau}e^{-\int_\sigma^\rho(\alpha(\tau)+\gamma_1(\tau))d\tau}d\sigma d\rho.$$

Note that $\mathcal{B}$ have two following natures:
(1) $|\mathcal{B}(t)| \le \hat{\beta}e^{\hat{\beta}t}\|f_4\|_{L^1_\pi}$.
(2) When $t > 2a^+$, $\int_0^{a^+} |\mathcal{B}(t-a+h)-\mathcal{B}(t-a)|da \le \mathcal{O}_1(h)\|f_4^0\|_X$.

Due to the compact of the Frećhet-Kolmogorov in L^1, we must check the following lemma.

Lemma 6.1. $\int_0^{a^+} |x_i(a)-x_i(a-h)|da \to 0$*, as* $h \to 0$.

In the following we check every term in (6.2) to satisfy the Lemma 6.1. Firstly, when h is small enough, we check the first term in (6.2).

$$\begin{aligned}
&\int_0^{a^+} |\mathcal{B}(t-a)\pi(a)e^{-H^*\int_0^a k(\tau)d\tau} - B(t-a-h)\pi(a-h)e^{-H^*\int_0^{a-h} k(\tau)d\tau}|da \\
\le &\int_0^a |\mathcal{B}(t-a)||\pi(a)e^{-H^*\int_0^a k(\tau)d\tau} - e^{-H^*\int_0^{a-h} k(\tau)d\tau}da + \int_0^a |\mathcal{B}(t-a)| \\
&|\pi(a)-\pi(a-h)|e^{-H^*\int_0^{a-h} k(\tau)d\tau}da + \int_0^a |\mathcal{B}(t-a)-\mathcal{B}(t-a-h)|\pi(a-h) \\
&e^{-H^*\int_0^{a-h} k(\tau)d\tau}da \\
\le &(H^*\hat{k}h+\hat{\mu}h)\int_0^{a^+} |\mathcal{B}(t-a)|da + \int_0^{a^+} |\mathcal{B}(t-a)-\mathcal{B}(t-a-h)|da \\
\le &\mathcal{O}_2(h)\|f_X^0\|_X
\end{aligned}$$

where $\mathcal{O}_2(h) \to 0$, as $h \to 0$.

Secondly, we check the second term when h is small enough.

$$\begin{aligned}
&\int_0^{a^+} |H_1^*\mathcal{B}(t-a)\pi(a)\int_0^a k(\sigma)e^{-H^*\int_0^\sigma k(\tau)d\tau}e^{-\int_\sigma^a(\gamma_1(\tau)+\alpha(\tau))d\tau} - \int_0^{a^+}|H_1^*\\
&\mathcal{B}(t-a+h)\pi(a-h)\int_0^{a-h} k(\sigma)e^{-H^*\int_0^\sigma k(\tau)d\tau}e^{-\int_\sigma^{a-h}(\gamma_1(\tau)+\alpha(\tau))|d\tau}|d\sigma da\\
\leq& H^*\int_0^{a^+}|\mathcal{B}(t-a)\pi(a)||\int_0^a k(\sigma)e^{-H^*\int_0^\sigma k(\tau)d\tau}e^{-\int_\sigma^a(\gamma_1(\tau)+\alpha(\tau))d\tau}\\
&-\int_0^{a-h}k(\sigma)e^{-H^*\int_0^\sigma k(\tau)d\tau}e^{-\int_\sigma^a(\gamma_1(\tau)+\alpha(\tau))d\tau}|d\sigma da + H_1^*\int_0^{a^+}\mathcal{B}(t-a)\\
&|\pi(a)-\pi(a-h)|\int_0^{a-h}k(\sigma)e^{-H^*\int_0^\sigma k(\tau)d\tau}e^{-\int_\sigma^a(\gamma_1(\tau)+\alpha(\tau))d\tau}|d\sigma da\\
&+H_1^*\int_0^{a^+}|\mathcal{B}(t-a)-\mathcal{B}(t-a+h)|\pi(a-h)\int_0^{a-h}k(\sigma)e^{-H^*\int_0^\sigma k(\tau)d\tau}\\
&e^{-\int_\sigma^a(\gamma_1(\tau)+\alpha(\tau))d\tau}|d\sigma da\\
\leq& H_1^*(\hat{\gamma}_1+\hat{\alpha}+\hat{k})h\int_0^{a^+}|\mathcal{B}(t-a)|da + \hat{k}H_1^*\hat{\mu}a^+\int_0^{a^+}|\mathcal{B}(t-a)|da\\
&+\hat{k}H_1^*a^+|\mathcal{B}(t-a)-\mathcal{B}(t-a+h)|da\\
\leq&\mathcal{O}_3(h)\|f_X^0\|_X
\end{aligned}$$

where $\mathcal{O}_3(h)\to 0$, as $h\to 0$.

Finally we can prove that the following term satisfies the Lemma 6.1.

$$\begin{aligned}
&H_1^*\int_0^{a^+}|\mathcal{B}(t-a)\pi(a)\int_0^a\alpha(\rho)e^{-\int_\rho^a\gamma_2(\tau)d\tau}\int_0^\rho k(\sigma)e^{-\int_\sigma^\rho(\alpha(\tau)+\gamma_1(\tau))d\tau}\\
&e^{-H_1^*\int_0^\sigma k(\tau)d\tau}d\sigma d\rho - H_1^*\int_0^{a^+}\mathcal{B}(t-a)\pi(a)\int_0^a\alpha(\rho)e^{-\int_\rho^a\gamma_2(\tau)d\tau}\\
&\int_0^\rho k(\sigma)e^{-H_1^*\int_0^\sigma k(\tau)d\tau}e^{-\int_\sigma^\rho(\alpha(\tau)+\gamma_1(\tau)d\tau)}d\sigma d\rho|da\\
\leq& H_1^*(\hat{\alpha}(e^{\hat{\gamma_2 a^+}}+\hat{\gamma}_2)a^+h+\hat{\mu}\hat{\alpha}\hat{k}a^+)h\int_0^{a^+}|\mathcal{B}(t-a)|da+\hat{\mu}\hat{\alpha}\hat{k}a^+\\
&|\mathcal{B}(t-a)-\mathcal{B}(t-a+h)|da\\
\leq&\mathcal{O}_4(h)\|f_X^0\|_X
\end{aligned}$$

where $\mathcal{O}_4(h)\to 0$, as $h\to 0$.

Due to the compact of the Frećhet-Kolmogorov in L^1 and Lemma 6.1, the semigroup generated by A is finally compact when $t > 2a^+$. Therefore, we obtain the following theorem.

Theorem 6.2. (1) *If* $R_0 = \max\{R_{10}, R_{20}\} < 1$, *and then disease-free steady state* E_0 *is locally asymptotically stable.*
(2) *If* $R_{10} < 1$, $R_{20} > 1$, *and* $\frac{J_1^*(a^+)}{P^*(a^+)} \le e^{-\int_0^{a^+} \gamma_2(\tau)d\tau}$, *then the dominated strain two steady state* E_1^* *is locally asymptotically stable.*
(3) *If* $R_{10} > 1$, *and* $R_{20} > 1$, *the local stability of the endemic steady state* E_* *is determined by* (5.16).

7. Conclusion

In this paper we formulate an age-structured epidemiological model for the disease transmission of two strain with mutation, then the mathematical analysis of this model was performed. We obtain the explicit expression of the basic reproduction number according to the strains, and prove that the disease-free steady state is locally asymptotical stable if $R_0 = \max\{R_{10}, R_{20}\} < 1$, in this case, the disease always dies out; then we prove that dominated strain two steady state which is locally asymptotically stable if $R_{10} < 1$, $R_{20} > 1$ and $\frac{J_1^*(a^+)}{P^*(a^+)} \le e^{-\int_0^{a^+} \gamma_2(\tau)d\tau}$. We also obtain the existence of the endemic steady state and local stability determined by (5.16). Compared the ODE model, the reproductive number is more realistic and has more biological meaning.

In future, there are some problems that will be solved. The global stability of the steady states, or whether there are some kind of bifurcations for the model or not, are still open. If the input is impulsive birth, what results will occur. The simulations of the age structure models are still to be resolve. Furthermore, what effect will occur, if we introduce the delay in our model.

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