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SYMMETRIC SOLUTIONS FOR A FOURTH-ORDER MULTI-POINT BOUNDARY VALUE PROBLEMS WITH ONE-DIMENSIONAL p-LAPLACIAN AT RESONANCE†

AIJUN YANG AND HELIN WANG*

ABSTRACT. We consider the fourth-order differential equation with one-dimensional p-Laplacian $(\phi_p(x''(t)))'' = f(t, x(t), x'(t), x''(t))$ *a.e.* $t \in [0, 1]$, subject to the boundary conditions $x''(0) = 0$, $(\phi_p(x''(t)))'|_{t=0} = 0$, $x(0) = \sum_{i=1}^{n} \mu_i x(\xi_i)$, $x(t) = x(1-t)$, $t \in [0, 1]$, where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < \frac{1}{2}$, $\mu_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, $\sum_{i=1}^{n} \mu_i = 1$ and $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is a L^1-Carathéodory function with $f(t, u, v, w) = f(1-t, u, -v, w)$ for $(t, u, v, w) \in [0, 1] \times \mathbb{R}^3$. We obtain the existence of at least one nonconstant symmetric solution by applying an extension of Mawhin's continuation theorem due to Ge. Furthermore, an example is given to illustrate the results.

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1. Introduction

In this paper, we are interested in the fourth-order symmetric multi-point BVP with the one-dimensional p-Laplacian

$$(\phi_p(x''(t)))'' = f(t, x(t), x'(t), x''(t)), \ a.e.\ t \in [0, 1], \tag{1.1}$$

$$x''(0) = 0, \ (\phi_p(x''(t)))'|_{t=0} = 0, \ x(0) = \sum_{i=1}^{n} \mu_i x(\xi_i), \tag{1.2}$$

$$x(t) = x(1-t) \ a.e.\ t \in [0, 1], \tag{1.3}$$

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where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < \frac{1}{2}$, $\mu_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, with

$$\sum_{i=1}^{n} \mu_i = 1. \tag{1.4}$$

Throughout we assume:

(A1) $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is symmetric on $[0,1]$, i.e.

$$f(t,u,v,w) = f(1-t,u,-v,w) \quad \text{for } t \in [0,1]$$

and satisfies the L^1-Carath$\acute{e}$odory conditions, and $f(t,b,0,0) \not\equiv 0$, $\forall b \in \mathbb{R}$;

(A2) $\sum_{i=1}^{n} \mu_i \xi_i (2q - (2\xi_i)^{2q-1}) \neq 0$.

Due to the condition (1.4), the differential operator on the left side of (1.1) is not invertible. In the literature, BVPs of this type are referred to as problems at resonance.

Boundary value problems with a p-Laplacian have received a lot of attention in recent years. They often occur in the study of the n-dimensional p-Laplacian equation, non-Newtonian fluid theory and the turbulent flow of a gas in porous medium. Many works have been carried out to discuss the existence of solutions or positive solutions, multiple solutions for the local or nonlocal BVPs [3,5,14,16,19].

Multi-point BVPs of ordinary differential equations arise in a variety of different areas of Applied Mathematics and Physics. The study of multi-point BVPs for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [6]. Since then many authors have studied more nonlinear multi-point BVPs [10-19]. The methods used therein mainly depend on the degree theory, fixed-point theorems, upper and lower techniques, and monotone iteration.

Recently, there is an increasing interest in considering some higher order BVPs, we refer the readers to [3-5,19] for details. However, as far as we know, the study of symmetric solutions for fourth-order p-Laplacian BVPs has rarely appeared.

Motivated by the papers mentioned above, we aim at studying the BVPs (1.1)-(1.3) at resonance. Due to the fact that the classical Mawhin's continuation theorem can't be directly used to discuss the BVP with nonlinear differential operator, in this paper, we investigate the multi-point BVP (1.1)-(1.3) by applying an extension of Mawhin's continuation theorem due to Ge [2]. Furthermore, an example is given to illustrate the result.

2. Preliminaries

For the convenience of readers, we present here some background definitions and lemmas.

Definition 2.1. $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is called a L^1-Carath$\acute{e}$odory function, if the following conditions hold:

(B1) for each $(u,v,w) \in \mathbb{R}^3$, the mapping $t \mapsto f(t,u,v,w)$ is Lebesgue measurable;

(B2) for a.e. $t \in [0,1]$, the mapping $(u,v,w) \mapsto f(t,u,v,w)$ is continuous on $\mathbb{R}^3$;
(B3) for each $r > 0$, there exists $\alpha_r \in L^1[0,1]$ such that for a.e. $t \in [0,1]$ and every (u,v,w) such that $\max\{|u|,|v|,|w|\} \le r$, we have $|f(t,u,v,w)| \le \alpha_r(t)$.

Proposition 2.1 ([7])**.** *ϕ_p satisfies the following properties*
(C1) *ϕ_p is continuous, monotonically increasing and invertible. Moreover, $\phi_p^{-1} = \phi_q$ with $p > 1$ a real constant satisfying $\frac{1}{p} + \frac{1}{q} = 1$;*
(C2) *for $\forall u, v \ge 0$, $\phi_p(u+v) \le \phi_p(u) + \phi_p(v)$, if $1 < p < 2$;*
$\phi_p(u+v) \le 2^{p-2}(\phi_p(u) + \phi_p(v))$, if $p \ge 2$.

Definition 2.2 ([2])**.** Let X and Z be two Banach spaces with norms $||\cdot||_X$ and $||\cdot||_Z$, respectively. A continuous operator $M : \mathrm{dom}M \to Z$ is said to be quasi-linear if
(D1) $\mathrm{Im}M$ is a closed subset of Z;
(D2) $\ker M = \{x \in \mathrm{dom}M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$.

Definition 2.3 ([7])**.** Let X be a Banach spaces and $X_1 \subset X$ a subspace. A linear operator $P : X \to X_1$ is said to be a projector provided that $P^2 = P$. The operator $Q : X \to X_1$ is said to be a semi-projector provided that $Q^2 = Q$ and $Q(\lambda x) = \lambda Qx$ for $x \in X,\ \lambda \in \mathbb{R}$.

Let $X_1 = \ker M$ and X_2 be the complementary space of X_1 in X, then $X = X_1 \oplus X_2$. On the other hand, suppose Z_1 is a subspace of Z and Z_2 is the complementary of Z_1 in Z, so that $Z = Z_1 \oplus Z_2$. Let $P : X \to X_1$ be a projector and $Q : Z \to Z_1$ be a semi-projector, and $\Omega \subset X$ an open and bounded set with the origin $\theta \in \Omega$, where θ is the origin of a linear space.

Suppose $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0,1]$ is a continuous operator. Denote N_1 by N. Let $\sum_\lambda = \{x \in \overline{\Omega} : Mx = N_\lambda x\}$.

Definition 2.4 ([2])**.** N_λ is said to be M-compact in $\overline{\Omega}$ if
(D3) there is a vector subspace Z_1 of Z with $\dim Z_1 = \dim X_1$ and an operator $R : \overline{\Omega} \times [0,1] \to X_2$ continuous and compact such that for $\lambda \in [0,1]$,

$$(I-Q)N_\lambda(\overline{\Omega}) \subset \mathrm{Im}M \subset (I-Q)Z, \tag{2.1}$$

$$QN_\lambda x = 0,\ \lambda \in (0,1) \iff QNx = 0, \tag{2.2}$$

$R(\cdot, 0)$ is the zero operator and

$$R(\cdot,\ \lambda)|_{\sum_\lambda} = (I-P)|_{\sum_\lambda}, \tag{2.3}$$

$$M[P + R(\cdot,\ \lambda)] = (I-Q)N_\lambda. \tag{2.4}$$

Theorem 2.1 ([2])**.** *Let X and Z be two Banach spaces with norms $||\cdot||_X$ and $||\cdot||_Z$, respectively, and $\Omega \subset X$ an open and bounded set. Suppose $M : \mathrm{dom}M \to Z$ is a quasi-linear operator and $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0,1]$ is M-compact. In addition, if*

(E1) *$Mx \neq N_\lambda x$, for $\lambda \in (0,1)$, $x \in \text{dom}M \cap \partial\Omega$;*
(E2) $\deg\{JQN, \Omega \cap \ker M, 0\} \neq 0$,
where $N = N_1$ and $J : Z_1 \to X_1$ is a homeomorphism with $J(\theta) = \theta$, then the abstract equation $Mx = Nx$ has at least one solution in $\overline{\Omega}$.

3. Related lemmas

Let $AC[0,1]$ denotes the space of absolutely continuous functions on the interval $[0,1]$. We work in the spaces

$$X = \Big\{x \in C^2[0,1] \ : \ x(t) = x(1-t) \text{ for } t \in [0,1] \text{ and } (\phi_p(x''(\cdot)))' \in AC[0,1], (\phi_p(x''(\cdot)))'' \in L^1[0,1]\Big\}$$

with the norm $||x||_X = \max\{||x||_\infty, ||x'||_\infty, ||x''||_\infty\}$, where $||x||_\infty = \sup\limits_{t\in[0,1]} |x(t)|$ and $Z = \{z \in L^1[0,1] : z(t) = z(1-t),\ t \in [0,1]\}$ with the usual Lebesgue norm denoted by $|| \cdot ||_1$.
Define $M : \text{dom}M \to Z$ by $Mx(t) = (\phi_p(x''(t)))''$ with

$$\text{dom}M = \Big\{x \in X \ : \ x''(0) = 0,\ (\phi_p(x''(t)))'\,|_{t=0} = 0,\ x(0) = \sum\nolimits_{i=1}^{n} \mu_i x(\xi_i)\Big\}.$$

For any open and bounded $\Omega \subset X$, we define $N_\lambda : \overline{\Omega} \to Z$ by $N_\lambda x(t) = \lambda f(t,\ x(t),\ x'(t),\ x''(t)),\ \ t \in [0,1]$. Then the BVP (1.1)-(1.3) can be written as $Mx = Nx$.

Lemma 3.1. *The operator $M : \text{dom}M \to Z$ is quasi-linear.*

Proof. It is clear that $X_1 = \ker M = \{x \in \text{dom}M : x = a \in \mathbb{R}\}$.
Let $x \in \text{dom}M$ and consider the equation $(\phi_p(x''(t)))'' = z(t)$ subject to (1.2) and (1.3), then $z \in Z$. It follows from (1.2) and the symmetric conditions that

$$x'(t) = -\int_t^{\frac{1}{2}} \phi_q\left(\int_0^s (s-\tau)z(\tau)d\tau\right) ds, \tag{3.1}$$

and then

$$x(t) = -\int_0^t \int_s^{\frac{1}{2}} \phi_q\left(\int_0^\tau (\tau-k)z(k)dk\right) d\tau ds + x(0). \tag{3.2}$$

In view of (1.3) and $\sum\limits_{i=1}^{n} \mu_i = 1$, we get

$$\sum_{i=1}^{n} \mu_i \int_0^{\xi_i} \int_t^{\frac{1}{2}} \phi_q\left(\int_0^s (s-\tau)z(\tau)d\tau\right) dsdt = 0. \tag{3.3}$$

Thus,

$$\text{Im}M \subset \left\{z \in Z : \sum_{i=1}^{n} \mu_i \int_0^{\xi_i} \int_t^{\frac{1}{2}} \phi_q\left(\int_0^s (s-\tau)z(\tau)d\tau\right) dsdt = 0\right\}. \tag{3.4}$$

Conversely, if (3.3) holds for $z \in Z$, we take $x \in \operatorname{dom} M$ as given by (3.2) and establish that it is symmetric and $(\phi_p(x''(\cdot)))'$ is absolutely continuous along with derivative, then $(\phi_p(x''(t)))'' = z(t)$ for $t \in [0,1]$ and (1.2) and (1.3) are satisfied. Together with (3.4), we have

$$\operatorname{Im} M = \left\{ z \in Z : \sum_{i=1}^{n} \mu_i \int_0^{\xi_i} \int_t^{\frac{1}{2}} \phi_q \left(\int_0^s (s-\tau) z(\tau) d\tau \right) ds dt = 0 \right\}. \tag{3.5}$$

So, $\dim \ker M = 1 < \infty$, $\operatorname{Im} M \subset Z$ is closed. Therefore, M is a quasi-linear operator. □

Lemma 3.2. *The operator $N_\lambda : \overline{\Omega} \to Z$ is M-compact in $\overline{\Omega}$.*

Proof. We recall the condition (A2) and define the continuous operator $Q : Z \to Z_1$ by

$$\begin{aligned} Qz(t) = & 2\phi_p \left(\frac{2^{2q} q(2q-1)}{\sum\limits_{i=1}^{n} \mu_i \xi_i \left(2q - (2\xi_i)^{2q-1}\right)} \right) \phi_p \times \\ & \left(\sum_{i=1}^{n} \mu_i \int_0^{\xi_i} \int_t^{\frac{1}{2}} \phi_q \left(\int_0^s (s-\tau) z(\tau) d\tau \right) ds dt \right). \end{aligned} \tag{3.6}$$

It is easy to check that $Q^2 z = Qz$ and $Q(\lambda z) = \lambda Q z$ for $z \in Z$, $\lambda \in \mathbb{R}$, that is, Q is a semi-projector and $\dim X_1 = \dim Z_1 = 1$. In addition, (3.5) and (3.6) imply that $\operatorname{Im} M = \ker Q$.

Let $\Omega \subset X$ be an open and bounded subset with $\theta \in \Omega$. For $\forall x \in \overline{\Omega}$, we have $Q[(I-Q)N_\lambda(x)] = 0$. So $(I-Q)N_\lambda(x) \in \ker Q = \operatorname{Im} M$. For $\forall z \in \operatorname{Im} M$, one gets $Qz = 0$. Thus, $z = z - Qz = (I-Q)z \in (I-Q)Z$. Therefore, (2.1) holds. Obviously, (2.2) is satisfied.

Define $R : \overline{\Omega} \times [0,1] \to X_2$ by

$$R(x,\lambda)(t) = -\int_0^t \int_s^{\frac{1}{2}} \phi_q \left(\int_0^\tau (\tau - k) \lambda \left(f(k, x(k), x'(k), x''(k)) - (Qf)(k) \right) dk \right) d\tau ds, \tag{3.7}$$

where X_2 is the complementary space of $X_1 = \ker M$ in X. Clearly, $R(\cdot, 0) = \theta$. Now we prove that $R : \overline{\Omega} \times [0,1] \to X_2$ is compact and continuous.

We first show that R is relatively compact for $\forall \lambda \in [0,1]$. Since $\Omega \subset X$ is a bounded set, then there exists $r > 0$ such that $\overline{\Omega} \subset \{x \in X : ||x||_X \le r\}$. Because the function f satisfies the L^1-Carathéodory conditions, there exists $\alpha_r \in L^1[0,1]$ such that for a.e. $t \in [0,1]$, $|f(t, x(t), x'(t), x''(t))| \le \alpha_r(t)$ for

$x \in \overline{\Omega}$. Then for any $x \in \overline{\Omega}$, $\lambda \in [0,1]$, we obtain

$$\begin{aligned}
|R(x,\lambda)(t)| \leq & \int_0^t \left| \int_s^{\frac{1}{2}} \phi_q \left(\int_0^\tau (\tau - k)\lambda(f(k,x(k),x'(k),x''(k)) - (Qf)(k))dk \right) d\tau \right| ds \\
\leq & \int_0^1 \phi_q \left(\int_0^1 |\alpha_r(s)|ds + \int_0^1 |(Qf)(s)|ds \right) dt \\
= & \phi_q(||\alpha_r||_1 + ||Qf||_1) := L, \\
|R'(x,\lambda)(t)| = & \left| \int_t^{\frac{1}{2}} \phi_q \left(\int_0^s (s-\tau)\lambda(f(\tau,x(\tau),x'(\tau),x''(\tau)) - (Qf)(\tau))d\tau \right) ds \right| \\
\leq & \int_0^1 \phi_q \left(\int_0^1 |\alpha_r(s)|ds + \int_0^1 |(Qf)(s)|ds \right) dt = L, \\
|R''(x,\lambda)(t)| = & \left| \phi_q \left(\int_0^t (t-s)\lambda(f(s,x(s),x'(s),x''(s)) - (Qf)(s))ds \right) \right| \\
\leq & \phi_q \left(\int_0^1 |\alpha_r(s)|ds + \int_0^1 |(Qf)(s)|ds \right) \\
= & \phi_q(||\alpha_r||_1 + ||Qf||_1) = L,
\end{aligned}$$

that is, $R(\cdot,\lambda)\overline{\Omega}$ is uniformly bounded. Meanwhile, for $\forall t_1, t_2 \in [0,1]$,

$$|R(x,\lambda)(t_2) - R(x,\lambda)(t_1)| = \left| \int_{t_1}^{t_2} R'(x,\lambda)(s)ds \right| \leq L|t_2 - t_1| \to 0, \text{ as } |t_2 - t_1| \to 0.$$

Similarly,

$$\left| R(x,\lambda)'(t_2) - R(x,\lambda)'(t_1) \right| = \left| \int_{t_1}^{t_2} R''(x,\lambda)(s)ds \right| \leq L|t_2 - t_1| \to 0, \text{ as } |t_2 - t_1| \to 0.$$

Also,

$$\begin{aligned}
& |\phi_p(R''(x,\lambda)(t_2)) - \phi_p(R''(x,\lambda)(t_1))| \\
= & \Big| \int_0^{t_2} (t_2 - s)\lambda(f(s,x(s),x'(s),x''(s)) - (Qf)(s))ds \\
& - \int_0^{t_1} (t_1 - s)\lambda(f(s,x(s),x'(s),x''(s)) - (Qf)(s))ds \Big| \\
\leq & \Big| \int_0^{t_2} (t_2 - t_1)\lambda(f(s,x(s),x'(s),x''(s)) - (Qf)(s))ds \Big| \\
& + \Big| \int_{t_1}^{t_2} (t_1 - s)\lambda(f(s,x(s),x'(s),x''(s)) - (Qf)(s))ds \Big| \\
\leq & \int_0^1 \left(\alpha_r(s) + |(Qf)(s)| \right) ds \cdot |t_2 - t_1| + \int_{t_1}^{t_2} (t_1 - s) \left(\alpha_r(s) + |(Qf)(s)| \right) ds, \\
\leq & (||\alpha_r||_1 + ||Qf||_1)|t_2 - t_1| + \int_{t_1}^{t_2} (\alpha_r(s) + |(Qf)(s)|)ds \to 0 \text{ as } |t_2 - t_1| \to 0.
\end{aligned}$$

In view of the continuity of ϕ_p, we have $|R''(x,\lambda)(t_2)-R''(x,\lambda)(t_1)| \to 0$, as $|t_2-t_1| \to 0$. So, $R(\cdot,\lambda)\overline{\Omega}$ is equicontinuous on [0,1]. Thus, Arzela-Ascoli Theorem implies that $R(\cdot,\lambda)\overline{\Omega}$ is relatively compact.

Since f is a L^1-Carathéodory function, the continuity of R on $\overline{\Omega}$ follows from the Lebesgue dominated convergence theorem.

Define $P: X \to X_1$ by $(Px)(t) = x(0)$ for $t \in [0,1]$. $\forall x \in \sum_\lambda$, we have $\lambda f(t, x(t), x'(t), x''(t)) = (\phi_p(x''(t)))'' \in \mathrm{Im}M = \ker Q$. So

$$\begin{aligned}R(x,\lambda)(t) &= -\int_0^t \int_s^{\frac{1}{2}} \phi_q\left(\int_0^\tau (\tau-k)\lambda\left(f(k,x(k),x'(k),x''(k)) - (Qf)(k)\right)dk\right) d\tau ds\\ &= -\int_0^t \int_s^{\frac{1}{2}} \phi_q\left(\int_0^\tau (\tau-k)(\phi_p(x''(k)))''dk\right) d\tau ds\\ &= \int_0^t x'(s)ds = x(t) - x(0) = [(I-P)x](t),\end{aligned}$$

which implies (2.3). $\forall\ x \in \overline{\Omega}$, we have

$$\begin{aligned}&M[Px + R(x,\lambda)](t)\\ =&M\left[x(0) - \int_0^t \int_s^{\frac{1}{2}} \phi_q\left(\int_0^\tau (\tau-k)\lambda(f(k,x(k),x'(k),x''(k)) - (Qf)(k))dk\right) d\tau ds\right]\\ =&\lambda[f(t,x(t),x'(t),x''(t)) - Qf(t,x(t),x'(t),x''(t))]\\ =&[((I-Q)N_\lambda)(x)](t),\end{aligned}$$

which yields (2.4). Therefore, N_λ is M-compact in $\overline{\Omega}$. □

4. Main results

Theorem 4.1. *Suppose that*

(H1) *there exists a constant $A > 0$ such that for $\forall x \in \mathrm{dom}M \setminus \ker M$ satisfying $|x(t)| > A$ for all $t \in [0,1]$, we have $QNx \neq 0$;*

(H2) *there exist functions $\alpha, \beta, \gamma, \rho \in L^1[0,1]$ such that for $\forall (x,y,z) \in \mathbb{R}^3$ and a.e. $t \in [0,1]$, we have*

$$|f(t,x,y,z)| \leq \alpha(t)|x|^{p-1} + \beta(t)|y|^{p-1} + \gamma(t)|z|^{p-1} + \rho(t), \tag{4.1}$$

we denote $\alpha_1 = ||\alpha||_1$, $\beta_1 = ||\beta||_1$, $\gamma_1 = ||\gamma||_1$, $\rho_1 = ||\rho||_1$;

(H3) *there exist a constant $B > 0$ such that for $\forall b \in \mathbb{R}$ with $|b| > B$, we have either*

$$b \cdot \frac{2^{2q}q(2q-1)}{\sum\limits_{i=1}^n \mu_i\xi_i(2q-(2\xi_i)^{2q-1})} \cdot \sum_{i=1}^n \mu_i \int_0^{\xi_i} \int_t^{\frac{1}{2}} \phi_q\left(\int_0^s (s-\tau)f(\tau,b,0,0)d\tau\right) dsdt < 0 \tag{4.2}$$

or

$$b \cdot \frac{2^{2q}q(2q-1)}{\sum\limits_{i=1}^n \mu_i\xi_i(2q-(2\xi_i)^{2q-1})} \cdot \sum_{i=1}^n \mu_i \int_0^{\xi_i} \int_t^{\frac{1}{2}} \phi_q\left(\int_0^s (s-\tau)f(\tau,b,0,0)d\tau\right) dsdt > 0; \tag{4.3}$$

(H4)

$$2^{q-3}(\alpha_1 + \beta_1 + 2^{p-1}\gamma_1)^{q-1} < 1 \text{ for } p < 2, \tag{4.4}$$

$$\frac{1}{2}(2^{p-2}\alpha_1+\beta_1+2^{p-1}\gamma_1)^{q-1}<1 \text{ for } p\geq 2. \tag{4.5}$$

Then the BVP(1.1)-(1.3) *has at least one nonconstant symmetric solution.*

Lemma 4.1. $U_1=\{x\in \mathrm{dom}M: Mx=N_\lambda x \text{ for some } \lambda\in(0,1)\}$ *is bounded.*

Proof. For $\forall x\in U_1$, we have $N_\lambda x=Mx\in \mathrm{Im}M=\ker Q$ and then $QNx=0$. It follows from (H1) that there exists $t_0\in[0,1]$ such that $|x(t_0)|\leq A$. Now, $|x(t)|=|x(t_0)+\int_{t_0}^t x'(s)ds|\leq A+||x'||_\infty$, that is, $||x||_\infty\leq A+||x'||_\infty$. Since x is symmetric on $[0,1]$, then

$$|x'(t)|=|x'(\frac{1}{2})+\int_{\frac{1}{2}}^t x''(s)ds|=|\int_{\frac{1}{2}}^t x''(s)ds|\leq\frac{1}{2}||x''||_\infty.$$

that is, $||x'||_\infty\leq\frac{1}{2}||x''||_\infty$. And then $||x||_\infty\leq A+\frac{1}{2}||x''||_\infty$.
Also,

$$x''(t)=\phi_q(\int_0^t(t-s)\lambda f(s,x(s),x'(s),x''(s))ds).$$

(I) For $1<p<2$, from (H2) and Proposition 2.1, one gets

$$\begin{aligned}||x''||_\infty=&\sup_{t\in[0,1]}|\phi_q(\int_0^t(t-s)\lambda f(s,x(s),x'(s),x''(s))ds)|\\ \leq&\phi_q(\int_0^1(\alpha(t)|x(t)|^{p-1}+\beta(t)|x'(t)|^{p-1}+\gamma(t)|x''(t)|^{p-1}+\rho(t))dt)\\ \leq&\phi_q[\alpha_1||x||_\infty^{p-1}+\beta_1||x'||_\infty^{p-1}+\gamma_1||x''||_\infty^{p-1}+\rho_1]\\ \leq&\phi_q[\alpha_1(A^{p-1}+\frac{1}{2^{p-1}}||x''||_\infty^{p-1})+\frac{1}{2^{p-1}}\beta_1||x''||_\infty^{p-1}+\gamma_1||x''||_\infty^{p-1}+\rho_1]\\ =&\phi_q[(\alpha_1+\beta_1+2^{p-1}\gamma_1)(\frac{||x''||_\infty}{2})^{p-1}+(\alpha_1A^{p-1}+\rho_1)]\\ \leq&2^{q-3}(\alpha_1+\beta_1+2^{p-1}\gamma_1)^{q-1}||x''||_\infty+2^{q-2}(\alpha_1A^{p-1}+\rho_1)^{q-1}.\end{aligned}$$

Noticing (H4), one arrives at

$$||x''||_\infty\leq\frac{2^{q-2}(\alpha_1A^{p-1}+\rho_1)^{q-1}}{1-2^{q-3}(\alpha_1+\beta_1+2^{p-1}\gamma_1)^{q-1}}:=L_1, \tag{4.6}$$

which yields $||x'||_\infty\leq\frac{1}{2}L_1$ and $||x||_\infty\leq A+\frac{1}{2}L_1$. Let $L_2=\max\{L_1,A+\frac{1}{2}L_1\}$.

(II) For $p \geq 2$, similarly, we have

$$\begin{aligned}
||x''||_\infty =& \sup_{t\in[0,1]} |\phi_q(\int_0^t (t-s)\lambda f(s,x(s),x'(s),x''(s))ds)| \\
\leq& \phi_q[\alpha_1||x||_\infty^{p-1} + \beta_1||x'||_\infty^{p-1} + \gamma_1||x''||_\infty^{p-1} + \rho_1] \\
\leq& \phi_q[2^{p-2}\alpha_1(A^{p-1} + \frac{1}{2^{p-1}}||x''||_\infty^{p-1}) + \frac{1}{2^{p-1}}\beta_1||x''||_\infty^{p-1} + ||x''||_\infty^{p-1} + \rho_1] \\
=& \phi_q[(2^{p-2}\alpha_1 + \beta_1 + 2^{p-1}\gamma_1)(\frac{||x''||_\infty}{2})^{p-1} + (2^{p-2}\alpha_1 A^{p-1} + \rho_1)] \\
\leq& \frac{1}{2}(2^{p-2}\alpha_1 + \beta_1 + 2^{p-1}\gamma_1)^{q-1}||x''||_\infty + (2^{p-2}\alpha_1 A^{p-1} + \rho_1)^{q-1}.
\end{aligned}$$

From (H4), we have

$$||x''||_\infty \leq \frac{(2^{p-2}\alpha_1 A^{p-1} + \rho_1)^{q-1}}{1 - \frac{1}{2}(2^{p-2}\alpha_1 + \beta_1 + 2^{p-1}\gamma_1)^{q-1}||x''||_\infty} := M_1,$$

which leads to $||x'||_\infty \leq \frac{1}{2}M_1$ and $||x||_\infty \leq A + \frac{1}{2}M_1$.
Let $M_2 = \max\{M_1, A + \frac{1}{2}M_1\}$.
Thus, $||x||_X \leq \max\{L_2, M_2\}$, i.e. U_1 is bounded. □

Lemma 4.2. *If $U_2 = \{x \in \ker M : -\lambda x + (1-\lambda)JQNx = 0, \lambda \in [0,1]\}$, where $J : \mathrm{Im}Q \to \ker M$ is a homomorphism, then U_2 is bounded.*

Proof. Define $J : \mathrm{Im}Q \to \ker M$ by $J(b) = b$. Then for $\forall b \in U_2$,

$$\begin{aligned}
\lambda b =& 2(1-\lambda)\phi_p\left(\frac{2^{2q}q(2q-1)}{\sum\limits_{i=1}^n \mu_i\xi_i(2q-(2\xi_i)^{2q-1})}\right)\phi_p \times \\
&\left(\sum_{i=1}^n \mu_i \int_0^{\xi_i}\int_t^{\frac{1}{2}} \phi_q\left(\int_0^s (s-\tau)f(\tau,b,0,0)d\tau\right)dsdt\right).
\end{aligned}$$

If $\lambda = 1$, then $b = 0$. In the case $\lambda \in [0,1)$, if $|b| > B$, then by (4.2), we have

$$\begin{aligned}
0 \leq \lambda b^2 =& 2(1-\lambda)b\phi_p\left(\frac{2^{2q}q(2q-1)}{\sum\limits_{i=1}^n \mu_i\xi_i(2q-(2\xi_i)^{2q-1})}\right)\phi_p \times \\
&\left(\sum_{i=1}^n \mu_i \int_0^{\xi_i}\int_t^{\frac{1}{2}} \phi_q\left(\int_0^s (s-\tau)f(\tau,b,0,0)d\tau\right)dsdt\right) < 0,
\end{aligned}$$

which is a contradiction. Thus, $||x||_X = |b| \leq B$ for $\forall x \in U_2$, that is, U_2 is bounded. □

Proof of Theorem 4.1. Let $U = \{x \in \mathrm{dom}M : ||x||_X < \max\{L_2, M_2, B\} + 1\}$, then $U \supset \overline{U}_1 \cup \overline{U}_2$ be a bounded and open set, then from Lemmas 4.1 and

4.2, we have
(i) $Mx \neq N_\lambda x$ for $\forall(x,\lambda) \in [\,\mathrm{dom}M \cap \partial U] \times (0,1)$;
(ii) Let $H(x,\lambda) = -\lambda x + (1-\lambda)JQNx$, J is defined as in Lemma 4.2. and we can see that $H(x,\lambda) \neq 0$, $\forall x \in \mathrm{dom}M \cap \partial U$. Therefore,

$$\begin{aligned}\deg\{JQN\mid_{\overline{U}\cap\ker M},\ U\cap\ker M,\ 0\} &= \deg\{H(\cdot,\ 0),\ U\cap\ker M, 0\}\\ &= \deg\{H(\cdot,\ 1),\ U\cap\ker M, 0\}\\ &= \deg\{-I,\ U\cap\ker M,\ 0\}\\ &\neq 0.\end{aligned}$$

Theorem 2.1 yields that $Mx = Nx$ has at least one symmetric solution $x^* \in \mathrm{dom}M \cap \overline{U}$. Observe that $x^*(t)$ is not a constant. Otherwise, suppose $x^* \equiv 0$, then from (1.1) we have $f(t,b,0,0) \equiv 0$, which contradicts (A1). The proof is completed. □

Remark 4.1. When the second part of condition (H3) holds, if we choose $\tilde{U}_2 = \{x \in \ker M : \lambda x + (1-\lambda)JQNx = 0, \lambda \in [0,1]\}$ and take homomorphism $\tilde{H}(x,\lambda) = \lambda x + (1-\lambda)JQNx$. Then by a similar argument, we can complete the proof.

Example 4.1. Consider

$$\begin{cases} (\phi_3(x''(t)))'' = f(t,x(t),x'(t),x''(t)),\ a.e.\ t\in[0,1],\\ x''(0) = 0, (\phi_p(x''(0)))' = 0\\ x(0) = 2x(\dfrac{1}{6}) - x(\dfrac{1}{4}),\ x(t) = x(1-t). \end{cases} \tag{4.7}$$

Corresponding to the BVP (1.1)-(1.3), we have $p=3$, $q=\frac{3}{2}$, $\mu_1 = 2$, $\mu_2 = -1$, $\xi_1 = \frac{1}{6}$, $\xi_2 = \frac{1}{4}$ and

$$f(t,u,v,w) = 2t(1-t)e^{t(1-t)} + \frac{1}{2}t(1-t)u^2 + \left(t - t^2 + \frac{1}{12}\right)v^2 + t^2(1-t)^2w^2.$$

We can easily verify that (A1)-(A2) hold. Let $\alpha(t) = \frac{1}{2}t(1-t)$, $\beta(t) = t - t^2 + \frac{1}{12}$, $\gamma(t) = t^2(1-t)^2$, $\rho(t) = 2t(1-t)e^{t(1-t)}$, then $\alpha_1 = \frac{1}{12}$, $\beta_1 = \frac{1}{4}$, $\gamma_1 = \frac{1}{30}$. Also, we can check that (H1)-(H4) are all satisfied. Thus, BVP (4.7) has a nonconstant symmetric solution, by using Theorem 4.1.

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Aijun Yang received M.Sc. from Hebei Normal University and Ph.D at Beijing Institute of Technology. Since 2010 she has been at Zhejiang University of Technology. Her research interests include nonlinear analysis and boundary value problems of differential and difference equations.

College of Science, Zhejiang University of Technology, Hangzhou, Zhejiang, 310023, China.
e-mail: `yangaij2004@163.com`

Helin Wang received M.Sc. from Yanshan University, and Ph.D. from Shanghai Institute of Optics and Fine Mechanics, Chinese Academy of Sciences. Since 2010 he has been at Zhejiang University of Technology. His research interests are fiber sensor and laser technology.

College of Science, Zhejiang University of Technology, Hangzhou, Zhejiang, 310023, China.
e-mail: `whl982032@163.com`