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POSITIVE SOLUTIONS FOR MULTI-POINT BOUNDARY VALUE PROBLEM OF FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS†

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Abstract. In this paper, we establish some sufficient conditions for the existence of positive solutions for a class of multi-point boundary value problem for fractional functional differential equations involving the Caputo fractional derivative. Our results are based on two fixed point theorems. Two examples are also provided to illustrate our main results.

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1. Introduction

The purpose of this paper is to investigate the existence of positive solutions for the boundary value problems, for fractional order functional differential equations:

$$\begin{cases} \mathbf{D}_{0+}^{\alpha} x(t) = f(t, x(\theta(t))), \quad t \in (0,1), \\ x(t) + a x'(t) = \phi(t), \quad t \in [-r, 0], \\ x(1) + b x'(1) + \sum_{i=1}^{m-2} c_i \mathbf{D}_{0+}^{\beta_i} x(\xi_i) = 0, \end{cases} \tag{1}$$

where $1 < \alpha < 2$, $0 \le \beta_i \le \alpha - 1$, $i = 1, 2, \cdots, m-2$, $r \ge 0$ are real numbers and $\mathbf{D}_{0+}^{\alpha}$, $\mathbf{D}_{0+}^{\beta_i}$ are the Caputo fractional derivatives, $f : [0,1] \times [0, +\infty) \to [0, +\infty)$ is a given function, $\theta(t) \le t$ with $\sup_{0 \le t \le 1} \theta(t) > 0$ and $\phi \in C([-r, 0], (-\infty, 0])$ with $\phi(0) = 0$, $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \xi_{m-1} = 1$, $a \ge 1$, $b \ge 0$, $c_i \ge 0$ satisfy $1 + \sum_{i=1}^{m-2} c_i \frac{1}{\Gamma(2-\beta_i)} {\xi_i}^{1-\beta_i} + b - a > 0$.

In recent years, the theory of fractional differential equations has played an important role in different research areas, such as engineering, physics, chemistry, signal analysis, etc. Applied problems require definitions of fractional

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derivatives allowing the utilization of physically interpretable boundary conditions. Caputo's fractional derivative satisfies these demands. There has been a significant development in the study on existence of solution, positive solution of differential equations involving fractional derivatives(see, for example, [1, 2, 3, 8, 9, 10, 5], and references therein).

In [11], Zhang considered the boundary value problem of fractional order

$$\begin{cases} \mathbf{D}_{0+}^{\alpha}x(t) = f(t,x(t)), \quad 1 < \alpha \leq 2, \quad t \in (0,1), \\ x(0) + x'(0) = 0, \quad x(1) + x'(1) = 0, \end{cases} \tag{2}$$

where $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function. In [11], the existence of positive solutions were studied via cone-theoretic techniques. Inspired by the work of Zhang's paper, the aim of the present paper is to establish some simple criteria for the existence of positive solutions of the probelm(1). It is worth noting that (2) is the special case of (1). Tools used in this paper are two fixed point theorems on cone. This paper is organized as follows. In section 2, we present some preliminary results and lemmas needed in the following sections. Section 3 will be concerned with the existence results of positive solutions for problem (1). The last section is devoted to examples illustrating the applicability of problem (1).

2. Preliminaries and Lemmas

Definition 2.1 ([6])**.** The fractional integral of order α for a function $f \in L^1[a,b]$ is defined by

$$I_{a+}^{\alpha}f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds, \quad \alpha > 0.$$

When $a = 0$, we write $I^{\alpha}f(t)$.

Definition 2.2 ([6])**.** For a function $f : [a,b] \to \mathbb{R}$, the Caputo derivative of fractional order $\alpha > 0$ is defined by

$$\mathbf{D}_{a+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds, \quad n = [\alpha]+1.$$

Definition 2.3 ([6])**.** For a function $f : [a,b] \to \mathbb{R}$, the Riemann-Liouville derivative of fractional order $\alpha > 0$ is defined by

$$D_{a+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}f(s)ds, \quad n = [\alpha]+1.$$

Definition 2.4. The map β is said to be a nonnegative continuous concave functional on cone P provided that $\beta : P \to [0,+\infty)$ is continuous and

$$\beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\beta(y), \quad x,y \in P, \quad t \in [0,1].$$

Lemma 2.1 ([6])**.** *Let $\alpha > 0$, then*

$$I_{a+}^{\alpha}\mathbf{D}_{a+}^{\alpha}x(t) = x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{k!}(t-a)^k, \ \textit{for some} \ c_k \in \mathbb{R}, \ n = [\alpha]+1.$$

Theorem 2.2 ([4]). *Suppose that E is a Banach space, $P \subset E$ is a cone. Let Ω_1, Ω_2 be two bounded open sets in E and $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. Let $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be completely continuous. Suppose that one of the following two conditions holds:*

(A1) $\|Tx\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1, \|Tx\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$;
(A2) $\|Tx\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1, \|Tx\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$.

Then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 2.3 ([7]). *Let P be a cone in a real Banach space E, $P_c = \{x \in P | \|x\| < c\}$, β is a nonnegative continuous concave functional on P such that $\beta(x) \leq \|x\|$, for all $x \in \overline{P}_c$ and $P(\beta, b, d) = \{x \in P | b \leq \beta(x), \|x\| \leq d\}$. Suppose that $T : \overline{P}_c \to \overline{P}_c$ is completely continuous and there exist positive constants $0 < a < b < d \leq c$ such that*

(I) $\{x \in P(\beta, b, d) \mid \beta(x) > b\} \neq \emptyset$ *and* $\beta(Tx) > b$ *for* $x \in P(\beta, b, d)$;
(II) $\|Tx\| < a$ *for* $x \in \overline{P}_a$;
(III) $\beta(Tx) > b$ *for* $x \in P(\beta, b, c)$ *with* $\|Tx\| > d$.

Then T has at least three fixed points x_1, x_2, x_3 satisfying $\|x_1\| < a, b < \beta(x_2)$, and $a < \|x_3\|$ with $\beta(x_3) < b$.

Lemma 2.4. *Assume that $\sigma \in L^1[0,1]$, $1 < \alpha < 2$, $0 \leq \beta \leq \alpha - 1$, $a \geq 0$ are constants, then*

$$\mathbf{D}^{\beta}_{a+} \left(I^{\alpha}_{a+} \sigma(t) \right) = I^{\alpha-\beta}_{a+} \sigma(t) \tag{3}$$

and

$$\mathbf{D}^{\beta}_{a+} t = \frac{(t-a)^{1-\beta}}{\Gamma(2-\beta)}. \tag{4}$$

Proof. Since $(I^{\alpha}_{a+}\sigma(t))' = D^1_{a+} I^{\alpha}_{a+} \sigma(t) = D^1_{a+} I^1_{a+} I^{\alpha-1}_{a+} \sigma(t) = I^{\alpha-1}_{a+}\sigma(t)$, one has

$$\begin{aligned}
\mathbf{D}^{\beta}_{a+} \left(I^{\alpha}_{a+} \sigma(t) \right) &= \frac{1}{\Gamma(1-\beta)} \int_a^t (t-s)^{-\beta} \frac{1}{\Gamma(\alpha-1)} \int_a^s (s-r)^{\alpha-2} \sigma(r) dr ds \\
&= \frac{1}{\Gamma(1-\beta)\Gamma(\alpha-1)} \int_a^t \sigma(r) dr \int_r^t (t-s)^{-\beta} (s-r)^{\alpha-2} ds \\
&= \frac{\int_a^t (t-r)^{\alpha-\beta-1} \sigma(r) dr \int_0^1 x^{\alpha-2} (1-x)^{-\beta} dx}{\Gamma(1-\beta)\Gamma(\alpha-1)} \\
&= \frac{B(\alpha-1, 1-\beta) \int_a^t (t-r)^{\alpha-\beta-1} \sigma(r) dr}{\Gamma(1-\beta)\Gamma(\alpha-1)} \\
&= I^{\alpha-\beta}_{a+} \sigma(t),
\end{aligned} \tag{5}$$

and

$$\mathbf{D}^{\beta}_{a+} t = \frac{1}{\Gamma(1-\beta)} \int_a^t (t-s)^{-\beta} ds = \frac{(t-a)^{1-\beta}}{\Gamma(2-\beta)}. \tag{6}$$

The proof is complete. □

Lemma 2.5. *For a given $\sigma \in C[0,1]$, then problem*

$$\begin{cases} \mathbf{D}_{0+}^{\alpha} x(t) = \sigma(t), \quad 1 < \alpha < 2, \quad t \in (0,1), \\ x(0) + a x'(0) = 0, \quad x(1) + b x'(1) + \sum_{i=1}^{m-2} c_i \mathbf{D}^{\beta_i} x(\xi_i) = 0, \end{cases} \tag{7}$$

has a unique solution x given by

$$x(t) = \int_0^1 G(t,s)\sigma(s)ds, \tag{8}$$

where

$$G(t,s) = \frac{1}{\Delta} \begin{cases} g_{i0}(t,s) = \frac{\Delta(t-s)^{\alpha-1} + (a-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}(a-t)}{\Gamma(\alpha-1)} \\ \quad + \sum_{j=i+1}^{m-2} \frac{c_j(\xi_j - s)^{\alpha-\beta_j-1}(a-t)}{\Gamma(\alpha-\beta_j)}, \\ \quad \xi_i \le s \le \min\{t,\ \xi_{i+1}\},\ i = 0, 1, \cdots, m-3, \\ g_{i1}(t,s) = \frac{(a-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}(a-t)}{\Gamma(\alpha-1)} \\ \quad + \sum_{j=i+1}^{m-2} \frac{c_j(\xi_j - s)^{\alpha-\beta_j-1}(a-t)}{\Gamma(\alpha-\beta_j)}, \\ \quad \max\{t,\ \xi_i\} \le s \le \xi_{i+1},\ i = 0, 1, \cdots, m-3, \\ g_{m-20}(t,s) = \frac{\Delta(t-s)^{\alpha-1} + (a-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}(a-t)}{\Gamma(\alpha-1)}, \\ \quad \xi_{m-2} \le s \le t, \\ g_{m-21}(t,s) = \frac{(a-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}(a-t)}{\Gamma(\alpha-1)}, \\ \quad \max\{t,\ \xi_{m-2}\} \le s < 1, \end{cases} \tag{9}$$

and $\Delta = 1 + \sum_{i=1}^{m-2} c_i \frac{1}{\Gamma(2-\beta_i)} {\xi_i}^{1-\beta_i} + b - a$.

Proof. Assume that x satisfies problem (7), then Lemma 2.1 implies

$$I^{\alpha}\sigma(t) = x(t) - x(0) - x'(0)t.$$

Using the boundary conditions, we show that

$$\begin{cases} x(0) + a x'(0) = 0, \\ x(0) + \left(1 + b + \sum_{i=1}^{m-2} c_i \frac{{\xi_i}^{1-\beta_i}}{\Gamma(2-\beta_i)}\right) x'(0) = -I^{\alpha}\sigma(1) - bI^{\alpha-1}\sigma(1) \\ \qquad\qquad - \sum_{i=1}^{m-2} c_i I^{\alpha-\beta_i}\sigma(\xi_i). \end{cases} \tag{10}$$

Hence

$$\begin{aligned} x(t) =& I^{\alpha}\sigma(t) + \frac{a-t}{\Delta}\left(I^{\alpha}\sigma(1) + bI^{\alpha-1}\sigma(1) + \sum_{i=1}^{m-2} c_i I^{\alpha-\beta_i}\sigma(\xi_i)\right) \\ =& \int_0^1 G(t,s)\sigma(s)ds. \end{aligned} \tag{11}$$

The proof is complete. □

Lemma 2.6. *Function $G(t,s)$ in Lemma 2.5 satisfies the following properties:*

(P1) *$G(t,s)$ is continuous in $[0,1] \times [0,1)$ and $G(t,s) > 0$ for any $t, s \in (0,1)$;*

(P2) *there exists a positive function* $\gamma(s) \in C(0,1)$ *such that*

$$\min_{0\leq t\leq \xi_1} G(t,s) \geq \gamma(s) \max_{0\leq t\leq 1} G(t,s), \quad s\in[0,1)$$

and

$$\gamma(s) \geq \frac{1}{1+\Delta}\frac{a-\xi_1}{a}.$$

Proof. (P1) is obvious from (9). We only prove (P2) is true, from (9), we have

$$\begin{aligned}
\min_{0\leq t\leq \xi_1} G(t,s) =& \frac{1}{\Delta}\begin{cases} \min_{0\leq t\leq \xi_1}\{g_{i0}(t,s), g_{i1}(t,s)\}, \quad \xi_i \leq s < \xi_{i+1}, \\ \quad i=0,1,\cdots,m-3, \\ \min_{0\leq t\leq \xi_1}\{g_{m-20}(t,s), g_{m-21}(t,s)\}, \quad \xi_{m-2}\leq s<1, \end{cases} \\
\geq& \frac{1}{\Delta}\begin{cases} \left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \sum_{j=i+1}^{m-2} c_j \frac{(\xi_j-s)^{\alpha-\beta_j-1}}{\Gamma(\alpha-\beta_j)}\right)(a-\xi_1), \\ \xi_i\leq s<\xi_{i+1}, \quad i=0,1,\cdots,m-3, \\ \left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)(a-\xi_1), \quad \xi_{m-2}\leq s<1, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\max_{0\leq t\leq 1} G(t,s) =& \frac{1}{\Delta}\begin{cases} \max_{0\leq t\leq 1}\{g_{i0}(t,s), g_{i1}(t,s)\}, \quad \xi_i \leq s < \xi_{i+1}, \\ \quad i=0,1,\cdots,m-3, \\ \max_{0\leq t\leq 1}\{g_{m-20}(t,s), g_{m-21}(t,s)\}, \quad \xi_{m-2}\leq s<1, \end{cases} \\
\leq& \frac{1}{\Delta}\begin{cases} \left(\frac{\Delta(1-s)^{\alpha-1}+(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \sum_{j=i+1}^{m-2} c_j \right. \\ \left.\times \frac{(\xi_j-s)^{\alpha-\beta_j-1}}{\Gamma(\alpha-\beta_j)}\right)a, \xi_i\leq s<\xi_{i+1}, \quad i=0,1,\cdots,m-3, \\ \left(\frac{\Delta(1-s)^{\alpha-1}+(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)a, \quad \xi_{m-2}\leq s<1, \end{cases} \\
=& M(s).
\end{aligned}$$

Let

$$\gamma(s) = \begin{cases} \frac{\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \sum_{j=i+1}^{m-2} c_j \frac{(\xi_j-s)^{\alpha-\beta_j-1}}{\Gamma(\alpha-\beta_j)}}{\frac{\Delta(1-s)^{\alpha-1}+(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \sum_{j=i+1}^{m-2} c_j \frac{(\xi_j-s)^{\alpha-\beta_j-1}}{\Gamma(\alpha-\beta_j)}}\frac{a-\xi_1}{a}, \\ \xi_i\leq s<\xi_{i+1}, \quad i=0,1,\cdots,m-3, \\ \frac{\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}}{\frac{\Delta(1-s)^{\alpha-1}+(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}}\frac{a-\xi_1}{a}, \quad \xi_{m-2}\leq s<1. \end{cases} \tag{12}$$

It is easy to see that

$$\min_{0\leq t\leq \xi_1} G(t,s) \geq \gamma(s) \max_{0\leq t\leq 1} G(t,s), \quad s\in[0,1),$$

and

$$\gamma(s) \geq \frac{a-\xi_1}{a(1+\Delta)} := \gamma. \tag{13}$$

The proof is complete. □

Let

$$\overline{\phi}(t)=\begin{cases}0, & t\in[0,1],\\ \frac{e^{\frac{-t}{a}}}{a}\int_0^t e^{\frac{s}{a}}\phi(s)ds, & t\in[-r,0].\end{cases}\tag{14}$$

If $x(t)$ satisfies problem (1), let $y(t)=x(t)-\overline{\phi}(t)$, $t\in[-r,1]$, then $y(\theta(t))=x(\theta(t))-\overline{\phi}(\theta(t))$, $t\in[0,1]$. Thus y satisfies the equation

$$y(t)=\begin{cases}\int_0^1 G(t,s)f(s,y(\theta(s))+\overline{\phi}(\theta(s)))ds, & t\in[0,1],\\ e^{\frac{-t}{a}}\int_0^1 G(0,s)f(s,y(\theta(s))+\overline{\phi}(\theta(s)))ds, & t\in[-r,0].\end{cases}\tag{15}$$

Consider the Banach space $E=C[-r,1]$ with the norm $\|x\|=\sup_{t\in[-r,1]}|x(t)|$. Define an operator T by

$$Ty(t)=\begin{cases}\int_0^1 G(t,s)f(s,y(\theta(s))+\overline{\phi}(\theta(s)))ds, & t\in[0,1],\\ e^{\frac{-t}{a}}\int_0^1 G(0,s)f(s,y(\theta(s))+\overline{\phi}(\theta(s)))ds, & t\in[-r,0].\end{cases}\tag{16}$$

Let $\lambda=e^{\frac{-r}{a}}\gamma$, we define the cone $P\subset E$ by

$$P=\left\{x\in E|x\geq 0,\min_{0\leq t\leq\xi_1}x(t)\geq\lambda\|x\|\right\}$$

and $\Lambda=\{t\in[0,1]|0\leq\theta(t)\leq\xi_1\}$.

3. Main results

For convenience, set

$$\lambda_1=\frac{1}{\int_\Lambda G(0,s)ds},\quad \lambda_2=\frac{1}{e^{\frac{r}{a}}\int_0^1 M(s)ds},\quad \lambda_3=\frac{1}{\min_{t\in\Lambda}\int_\Lambda G(t,s)ds}.$$

Lemma 3.1. *Assume that f satisfies the following conditions:*

(I) *$f(t,u)$ is measurable with respect to t on $[0,1]$;*
(II) *$f(t,u)$ is continuous with respect to u on $[0,+\infty)$;*
(III) *there exist three real numbers $\alpha_1,\alpha_2\in[1,\alpha)$, $\mu\geq 0$ and two functions $p_1\in L^{\frac{1}{\alpha_1-1}}[0,1]$, $p_2\in L^{\frac{1}{\alpha_2-1}}[0,1]$ (If $\alpha_1=1$, means $p_1\in L^\infty[0,1]$, similar to α_2), such that*

$$f(t,u)\leq p_1(t)+p_2(t)u^\mu,\quad t\in[0,1],\ u\in[0,+\infty).$$

Then $T:P\to P$ is a completely continuous operator.

Proof. The proof will be given in three steps.

Step 1: $T : P \to P$.
For any $y \in P$, by (16), we see that $Ty \geq 0$. Moreover,

$$\begin{aligned}\min_{0\leq t\leq \xi_1} Ty(t) &= \min_{0\leq t\leq \xi_1}\int_0^1 G(t,s)f(s,y(\theta(s))+\overline{\phi}(\theta(s)))ds\\ &\geq \gamma \max_{0\leq t\leq 1}\int_0^1 G(t,s)f(s,y(\theta(s))+\overline{\phi}(\theta(s)))ds\\ &\geq \gamma\int_0^1 G(0,s)f(s,y(\theta(s))+\overline{\phi}(\theta(s)))ds\\ &\geq \max_{-r\leq t\leq 0} e^{-\frac{t+r}{a}}\gamma\int_0^1 G(0,s)f(s,y(\theta(s))+\overline{\phi}(\theta(s)))ds,\end{aligned} \tag{17}$$

(17) implies that $\min_{0\leq t\leq \xi_1} Ty(t) \geq \gamma \max_{0\leq t\leq 1} Ty(t)$ and $\min_{0\leq t\leq \xi_1} Ty(t) \geq \gamma e^{-\frac{r}{a}} \max_{-r\leq t\leq 0} Ty(t)$. Hence, we obtain

$$\min_{0\leq t\leq \xi_1} Ty(t) \geq \lambda \|Ty\|.$$

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in P. Then

$$\begin{aligned}&|T(y_n(t)) - T(y(t))|\\ &\leq \max_{0\leq t\leq 1} e^{\frac{r}{a}}\int_0^1 G(t,s)|f(s,y_n(\theta(s))+\overline{\phi}(\theta(s))) - f(s,y(\theta(s))+\overline{\phi}(\theta(s)))|ds\\ &\leq \sup_{0\leq t\leq 1}|f(t,y_n(\theta(t))+\overline{\phi}(\theta(t))) - f(t,y(\theta(t))+\overline{\phi}(\theta(t)))|e^{\frac{r}{a}}\int_0^1 M(s)ds.\end{aligned} \tag{18}$$

In view of condition (II) and $M \in L^1[0,1]$, (18) means that $\|Ty_n - Ty\| \to 0$ as $n \to 0$.

Step 2: T maps bounded sets into bounded sets in P.
For each $y \in B_r = \{y \in P | \|y\| \leq r\}$, then we have

$$\begin{aligned}|Ty(t)| &\leq \max_{0\leq t\leq 1} e^{\frac{r}{a}}\int_0^1 G(t,s)|f(s,y(\theta(s))+\overline{\phi}(\theta(s)))|ds\\ &\leq e^{\frac{r}{a}}\int_0^1 M(s)[p_1(s)+p_2(s)(r+\|\overline{\phi}\|)^{\mu}]ds.\end{aligned} \tag{19}$$

Note that $\frac{1}{2-\alpha_i} > 1$, together with Minkowski's inequality, we know that $M \in L^{\frac{1}{2-\alpha_i}}[0,1]$, $i = 1,2$. Hence, (19) means that

$$\begin{aligned}\|Ty\| \leq e^{\frac{r}{a}}\Big(&\|M\|_{L^{\frac{1}{2-\alpha_1}}[0,1]}\|p_1\|_{L^{\frac{1}{\alpha_1-1}}[0,1]} + \|M\|_{L^{\frac{1}{2-\alpha_2}}[0,1]}\|p_2\|_{L^{\frac{1}{\alpha_2-1}}[0,1]}\\ &\times (r+\|\overline{\phi}\|)^{\mu}\Big).\end{aligned}$$

Step 3: T maps bounded sets into equicontinuous sets of P.
Let $y \in B_r$, $t, \tau \in [-r,1]$ with $t < \tau$.

Case 1. $t, \tau \in [-r, 0]$. Then for $\tau - t \to 0$,

$$|Ty(\tau) - Ty(t)| \le \left(e^{-\frac{t}{a}} - e^{-\frac{\tau}{a}}\right) \int_0^1 G(0,s) f(s, y(\theta(s)) + \overline{\phi}(\theta(s)))ds \to 0.$$

Case 2. $t, \tau \in [0, 1]$

$$\begin{aligned}|Ty(\tau) - Ty(t)| &\le \int_0^1 |G(\tau,s) - G(t,s)| f(s, y(\theta(s)) + \overline{\phi}(\theta(s)))ds \\ &\le \sum_{i=0}^{m-3} \int_{\xi_i}^{\xi_{i+1}} |G(\tau,s) - G(t,s)| [p_1(s) + p_2(s)(r + \|\overline{\phi}\|)^\mu] ds \\ &\quad + \int_{\xi_{m-2}}^1 |G(\tau,s) - G(t,s)| [p_1(s) + p_2(s)(r + \|\overline{\phi}\|)^\mu] ds.\end{aligned}$$

In view of $G(t,s)$ is uniformly continuous on $[0,1] \times [\xi_i, \xi_{i+1}]$, $i = 0, 1, \cdots, m-3$, hence

$$\sum_{i=0}^{m-3} \int_{\xi_i}^{\xi_{i+1}} |G(\tau,s) - G(t,s)| [p_1(s) + p_2(s)(r + \|\overline{\phi}\|)^\mu] ds \to 0, \quad \tau - t \to 0. \tag{20}$$

Subcase 1. $\xi_i \le t < \tau \le \xi_{i+1}$, $i = 0, 1, \cdots, m-3$.

$$\begin{aligned}&\int_{\xi_{m-2}}^1 |G(\tau,s) - G(t,s)| [p_1(s) + p_2(s)(r + \|\overline{\phi}\|)^\mu] ds \\ \le &(\tau - t) \frac{1}{\Delta} \int_{\xi_{m-2}}^1 \left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) [p_1(s) + p_2(s)(r + \|\overline{\phi}\|)^\mu] ds \\ \le &(\tau - t) \frac{1}{\Delta} \left[\sum_{i=1}^{2} \frac{(r + \|\overline{\phi}\|)^{\mu(i-1)}}{\Gamma(\alpha)} \left(\frac{2 - \alpha_i}{\alpha - \alpha_i + 1}\right)^{2-\alpha_i} \|p_i\|_{L^{\frac{1}{\alpha_i - 1}}[0,1]} \right. \\ &\left. + \sum_{i=1}^{2} \frac{b(r + \|\overline{\phi}\|)^{\mu(i-1)}}{\Gamma(\alpha-1)} \left(\frac{2 - \alpha_i}{\alpha - \alpha_i}\right)^{2-\alpha_i} \|p_i\|_{L^{\frac{1}{\alpha_i - 1}}[0,1]}\right] \to 0, \quad \tau - t \to 0.\end{aligned} \tag{21}$$

Subcase 2. $\xi_{m-2} \le t < \tau \le 1$.

$$\begin{aligned}&\int_{\xi_{m-2}}^1 |G(\tau,s) - G(t,s)| [p_1(s) + p_2(s)(r + \|\overline{\phi}\|)^\mu] ds \\ &\le (\tau - t) \frac{1}{\Delta} \int_{\xi_{m-2}}^1 \left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{b(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) [p_1(s) + p_2(s)(r + \|\overline{\phi}\|)^\mu] ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{\xi_{m-2}}^t [(\tau - s)^{\alpha-1} - (t-s)^{\alpha-1}] [p_1(s) + p_2(s)(r + \|\overline{\phi}\|)^\mu] ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_t^\tau (\tau - s)^{\alpha-1} [p_1(s) + p_2(s)(r + \|\overline{\phi}\|)^\mu] ds,\end{aligned} \tag{22}$$

notice that $(t-s)^{\alpha-1}$ is uniformly continuous on $[\xi_{m-2}, 1] \times [\xi_{m-2}, 1]$, then

$$\int_{\xi_{m-2}}^t [(\tau - s)^{\alpha-1} - (t-s)^{\alpha-1}] [p_1(s) + p_2(s)(r + \|\overline{\phi}\|)^\mu] ds \to 0, \quad \tau - t \to 0. \tag{23}$$

Similar to subcase 1, (22) and (23) imply that

$$\int_{\xi_{m-2}}^{1} |G(\tau,s)-G(t,s)|[p_1(s)+p_2(s)(r+\|\overline{\phi}\|)^{\mu}]ds \to 0, \quad \tau - t \to 0. \tag{24}$$

Subcase 3. $-r < t \le 0 \le \tau < \xi_1$ or $\xi_i < t \le \xi_{i+1} \le \tau < \xi_{i+2}$, $i = 0, 1, \cdots, m-3$. In this case, we can obtain (24) from case I and case II directly.

As a consequence of steps 1 to 3, together with the Arzelá-Ascoli theorem, we conclude that $T : P \to P$ is a completely continuous operator. □

Theorem 3.2. *Assume that conditions* (I)-(III) *in Lemma 3.1 hold. If f satisfies the following conditions:*

(H1) $\lim_{u\to 0+} \inf_{t\in\Lambda} \frac{f(t,u)}{u} = +\infty$, $\lim_{u\to +\infty} \inf_{t\in\Lambda} \frac{f(t,u)}{u} = +\infty$;
(H2) *there exist two positive numbers $\rho > 0$ and $L \in (0, \lambda_2)$ such that $f(t,u) \le L\rho$, $(t,u) \in [0,1] \times [0, \rho + \|\overline{\phi}\|]$,*

then problem (1) *has at least two positive solutions x_1, x_2.*

Proof. At first, from $\lim_{u\to 0+} \inf_{t\in\Lambda} \frac{f(t,u)}{u} = +\infty$, we know that $\forall L_1 \in [\frac{\lambda_1}{\lambda}, +\infty)$, $\exists \rho_1 \in (0, \lambda\rho)$ such that

$$f(t,u) \ge L_1 u, \quad (t,u) \in \Lambda \times (0, \rho_1]. \tag{25}$$

Let

$$\Omega_1 = \{y | y \in E, \|y\| < \rho_1\}.$$

For $y \in P \cap \partial\Omega_1 \subset P$, we have $\min\limits_{t\in[0,\xi_1]} y(t) \ge \lambda\|y\|$. Hence, If $y \in P \cap \partial\Omega_1$, we have

$$\begin{aligned} Ty(0) &= \int_0^1 G(0,s) f(s, y(\theta(s)) + \overline{\phi}(\theta((s)))ds \\ &\ge \int_\Lambda G(0,s) f(s, y(\theta(s)) + \overline{\phi}(\theta((s)))ds \\ &= \int_\Lambda G(0,s) f(s, y(\theta(s)))ds \\ &\ge \int_\Lambda G(0,s) L_1 y(\theta(s))ds \\ &\ge L_1 \lambda \|y\| \int_\Lambda G(0,s)ds \ge \|y\|. \end{aligned} \tag{26}$$

Thus

$$\|Ty\| \ge \|y\|, \quad y \in P \cap \partial\Omega_1. \tag{27}$$

Secondly, from $\lim_{u\to +\infty} \inf_{t\in\Lambda} \frac{f(t,u)}{u} = +\infty$, we know that $\forall L_2 \in [\frac{\lambda_1}{\lambda}, +\infty)$, $\exists \rho_2 \in (\frac{\rho}{\lambda}, +\infty)$, such that

$$f(t,u) \ge L_2 u, \quad (t,u) \in \Lambda \times [\lambda\rho_2, +\infty). \tag{28}$$

Set

$$\Omega_2 = \{y | y \in E, \|y\| < \rho_2\}. \tag{29}$$

Similar to the process of (26), we get

$$\|Ty\| \geq \|y\|, \quad y \in P \cap \partial\Omega_2. \tag{30}$$

Thirdly, set

$$\Omega = \{y | y \in E, \|y\| < \rho\}. \tag{31}$$

If $y \in P \cap \partial\Omega$, then $0 \leq y(\theta(s)) + \overline{\phi}(\theta(s)) \leq \rho + \|\overline{\phi}\|$, $s \in [0,1]$. By (H2), we have

$$\begin{aligned} Ty(t) &\leq e^{\frac{r}{a}} \int_0^1 M(s) f\left(s, y(\theta(s)) + \overline{\phi}(\theta(s))\right) ds \\ &\leq e^{\frac{r}{a}} L\|y\| \int_0^1 M(s) ds < \|y\|, \end{aligned} \tag{32}$$

hence

$$\|Ty\| < \|y\|, \quad y \in P \cap \partial\Omega. \tag{33}$$

According to (27), (30), (33) and Theorem 2.2, one see that T have two fixed points $y_1 \in P \cap (\overline{\Omega} \backslash \Omega_1)$ and $y_2 \in P \cap (\overline{\Omega}_2 \backslash \Omega)$. Then problem (1) has at least two positive solutions $x_1 = y_1 + \overline{\phi}$, $x_2 = y_2 + \overline{\phi}$ satisfy

$$\|x_1\| = \max_{t \in [-r,1]} (y_1(t) + \overline{\phi}(t)) < \rho + \|\overline{\phi}\|, \quad \max_{t \in [0,1]} x_1(t) = \max_{t \in [0,1]} y_1(t) < \rho$$

and

$$\|x_2\| = \max_{t \in [-r,1]} (y_2(t) + \overline{\phi}(t)) \geq \|y_2\| > \rho.$$

The proof is complete. □

Theorem 3.3. *Assume that conditions* (I)-(III) *in Lemma 3.1 hold and* $\phi(t) \equiv 0$, $t \in [-r, 0]$. *If* f *satisfies the following conditions:*

(H3) $\lim_{u \to 0+} \sup_{t \in [0,1]} \frac{f(t,u)}{u} = 0$, $\lim_{u \to +\infty} \sup_{t \in [0,1]} \frac{f(t,u)}{u} = 0$;

(H4) *there exist two positive numbers* $\tilde{\rho} > 0$ *and* $\tilde{L} \in (\lambda_1, \infty)$ *such that* $f(t,u) \geq \tilde{L}\tilde{\rho}$, $(t,u) \in \Lambda \times [\lambda\tilde{\rho}, \tilde{\rho}]$,

then problem (1) *has at least two positive solutions.*

Proof. Firstly, from $\lim_{u \to 0+} \sup_{t \in [0,1]} \frac{f(t,u)}{u} = 0$, we know that $\forall \varepsilon \in (0, \lambda_2]$, $\exists \tilde{\rho}_1 \in (0, \lambda\tilde{\rho})$ such that

$$f(t,u) \leq \varepsilon u, \quad (t,u) \in [0,1] \times (0, \tilde{\rho}_1]. \tag{34}$$

Let

$$\Omega_1 = \{y | y \in E, \|y\| < \tilde{\rho}_1\}.$$

Replace L with ε in (32), similarly, we have

$$\|Ty\| \leq \varepsilon \|y\| e^{\frac{r}{a}} \int_0^1 M(s) ds \leq \|y\|, \quad y \in P \cap \partial\tilde{\Omega}_1. \tag{35}$$

Secondly, from $\lim_{u \to +\infty} \sup_{t \in [0,1]} \frac{f(t,u)}{u} = 0$, we see that $\forall \tilde{\varepsilon} \in (0, \lambda_2]$, $\exists \tilde{\rho}_2 \in (\frac{\tilde{\rho}}{\lambda}, +\infty)$, such that

$$f(t,u) \leq \tilde{\varepsilon} u, \quad u \in [\lambda\tilde{\rho}_2, +\infty). \tag{36}$$

Let

$$\tilde{\Omega}_2 = \{y | y \in E, \|y\| < \tilde{\rho}_2\}. \tag{37}$$

Similar to (35), we can get

$$\|Ty\| \leq \tilde{\varepsilon}\|y\| e^{\frac{r}{a}} \int_0^1 M(s)ds \leq \|y\|, \quad y \in P \cap \partial\tilde{\Omega}_2. \tag{38}$$

Thirdly, set

$$\tilde{\Omega} = \{y | y \in E, \|y\| < \tilde{\rho}\}. \tag{39}$$

If $y \in P \cap \partial\tilde{\Omega}$, by (H4), substitute $\tilde{L}$ for $L_1\lambda$ in (26), we have

$$\begin{aligned} Ty(0) &\geq \tilde{L}\|y\| \int_\Lambda G(0,s)ds \\ &> \|y\|, \end{aligned} \tag{40}$$

hence

$$\|Ty\| > \|y\|, \quad y \in P \cap \partial\tilde{\Omega}. \tag{41}$$

From (35), (38), (41) and Theorem 2.2, we know that T have two fixed points $y_1 \in P \cap (\overline{\tilde{\Omega}} \backslash \tilde{\Omega}_1)$ and $y_2 \in P \cap (\overline{\tilde{\Omega}_2} \backslash \tilde{\Omega})$. Then problem (1) has at least two positive solutions $x_1 = y_1$, $x_2 = y_2$ satisfy $0 < \max_{t\in[0,1]} x_1(t) < \tilde{\rho} < \max_{t\in[0,1]} x_2(t)$. The proof is complete. □

Theorem 3.4. *Assume that conditions* (I)-(III) *in Lemma 3.1 hold and* $\phi(t) \equiv 0$, $t \in [-r, 0]$. *If* f *satisfies one of the following conditions:*

(H5) $\lim_{u\to 0+} \sup_{t\in[0,1]} \frac{f(t,u)}{u} \leq \lambda_2$, $\lim_{u\to +\infty} \inf_{t\in\Lambda} \frac{f(t,u)}{u} \geq \frac{\lambda_1}{\lambda}$;

(H6) $\lim_{u\to 0+} \sup_{t\in[0,1]} \frac{f(t,u)}{u} \geq \frac{\lambda_1}{\lambda}$, $\lim_{u\to +\infty} \inf_{t\in\Lambda} \frac{f(t,u)}{u} \leq \lambda_2$.

Then problem (1) *has at least one positive solution.*

Proof. We only prove the Theorem in the case of (H5), the proof for (H6) is similar.

From $\lim_{x\to 0+} \sup_{t\in[0,1]} \frac{f(t,u)}{u} \leq \lambda_2$, we can choose sufficiently small $\rho_1 > 0$, such that

$$f(t,u) \leq \lambda_2 u, \quad (t,u) \in [0,1] \times (0, \rho_1]. \tag{42}$$

Set

$$\Omega_{\rho_1} = \{y | y \in E, \|y\| < \rho_1\}. \tag{43}$$

Replace L with λ_2 in (32), similarly, we have

$$\|Ty\| \leq \|y\|, \quad y \in P \cap \partial\Omega_{\rho_1}. \tag{44}$$

On the other hand, By $\lim_{u\to +\infty} \inf_{t\in\Lambda} \frac{f(t,u)}{u} \geq \frac{\lambda_1}{\lambda}$, we can choose sufficiently large $\rho_2 > \frac{\rho_1}{\lambda}$, such that

$$f(t,u) \geq \frac{\lambda_1}{\lambda} u, \quad (t,u) \in \Lambda \times [\lambda\rho_2, \infty). \tag{45}$$

Set

$$\Omega_{\rho_2} = \{y | y \in E, \|y\| < \rho_2\}. \tag{46}$$

If $y \in P \cap \partial\Omega_{\rho_2} \subset P$, we have $\min_{t\in[0,\xi_1]} y(t) \geq \lambda\|y\|$. Hence, for $y \in P \cap \partial\Omega_{\rho_2}$, replace L_1 with $\frac{\lambda_1}{\lambda}$ in (26), similarly, we have

$$\|Ty\| \geq \|y\|, \quad y \in P \cap \partial\Omega_{\rho_2}. \tag{47}$$

Using (44), (47) and Theorem 2.2, we see that T has at least one fixed point $y \in P \cap (\overline{\Omega}_{\rho_2} \backslash \Omega\rho_1)$, namely, $x(t) = y(t)$ is a positive solution of problem (1). The proof is complete. □

Theorem 3.5. *Assume that conditions* (I)-(III) *in Lemma 3.1 hold and there exist positive constants* c, d, h *such that* $c + \|\overline{\phi}\| < \lambda d < d < \frac{\lambda_2}{\lambda_3} h$ *and*

(H1)′ $f(t,u) < \lambda_2 c$, $(t,u) \in [0,1] \times [0, c + \|\overline{\phi}\|]$;
(H2)′ $f(t,u) \leq \lambda_2 h$, $(t,u) \in [0,1] \times [0, h + \|\overline{\phi}\|]$;
(H3)′ $f(t,u) > \lambda_3 d$, $(t,u) \in \Lambda \times [\lambda d, h + \|\overline{\phi}\|]$.

Then problem (1) *has at least three positive solutions.*

Proof. We define the nonnegative continuous concave functional β by $\beta(y) = \min_{t\in\Lambda} y(t)$.

Taking $y \in \overline{P}_h$, we have $\|y\| \leq h$. Then $0 \leq y(\theta(s)) + \overline{\phi}(\theta(s)) \leq \|y\| + \|\overline{\phi}\| \leq h + \|\overline{\phi}\|$, for $s \in [0,1]$. So

$$\begin{aligned} \|Ty\| &\leq e^{\frac{r}{a}} \int_0^1 M(s) f(s, y(\theta(s)) + \overline{\phi}(\theta(s)))ds \\ &\leq \lambda_2 h e^{\frac{r}{a}} \int_0^1 M(s)ds = h. \end{aligned} \tag{48}$$

Hence, (48) together with Lemma 3.1 imply that $T : \overline{P}_h \to \overline{P}_h$ is completely continuous. Similar to (48), from (H1)′, we can get that $\|Ty\| < c$, for $\|y\| \leq c$.

It is obvious that $\{y \in P(\beta, d, h) | \beta(y) > d\} \neq \emptyset$. We choose $y \in P(\beta, d, h)$, then $\lambda d \leq \lambda\beta(y) \leq \lambda\|y\| \leq y(\theta(s)) + \overline{\phi}(\theta(s)) \leq h + \|\overline{\phi}\|$, $s \in \Lambda$. Hence

$$\begin{aligned} \beta(Ty) &= \min_{t\in\Lambda} \int_0^1 G(t,s) f(s, y(\theta(s)) + \overline{\theta}((s)))ds \\ &> \lambda_3 d \min_{t\in\Lambda} \int_\Lambda G(t,s)ds = d. \end{aligned} \tag{49}$$

From Theorem 2.3, T has at least three fixed points y_1, y_2, y_3. Namely, problem (1) has at least three positive solutions x_1, x_2, x_3 satisfy $x_i = y_i + \overline{\phi}$, $i = 1,2,3$ and

$$\max_{t\in[0,1]} x_1(t) < c, \quad d < \min_{t\in\Lambda} x_2(t), \quad c < \max_{t\in[0,1]} x_3(t), \quad \min_{t\in\Lambda} x_3(t) < d.$$

The proof is complete. □

4. Examples

Example 4.1. Consider the problem

$$\begin{cases} \mathbf{D}_{0+}^{\frac{3}{2}}x(t) = \dfrac{1}{40}\left(1+\sin t + \dfrac{(1+t)x^2(t-\frac{1}{2})e^{x(t-\frac{1}{2})}}{1+e^{x(t-\frac{1}{2})}}\right), & t \in (0,1), \\ x(0)+2x'(0) = -t^2, & t\in[-1,0], \\ x(1)+2x'(1)+3\mathbf{D}_{0+}^{\frac{1}{2}}x(0.5)=0, \end{cases} \tag{50}$$

where $f(t,x) = \frac{1}{40}\left(1+\sin t+\frac{(1+t)x^2e^x}{1+e^x}\right)$, $(t,x)\in[0,1]\times[0,+\infty)$, $a=b=2$, $c_1=3$, $\alpha=\frac{3}{2}$, $\beta_1=\frac{1}{2}$, $\xi_1=0.5$, $\phi(t)=-t^2$, $t\in[-1,0]$.

It is easy to see that $\Lambda=[0.5,1]$, $\|\overline{\phi}\|\approx 0.18977$. From $\lim\limits_{x\to 0+}\inf\limits_{t\in[0.5,1]}\frac{f(t,x)}{x} = \lim\limits_{x\to+\infty}\inf\limits_{t\in[0.5,1]}\frac{f(t,x)}{x}=+\infty$ and f satisfies Lemma 3.1, so condition (H1) in Theorem 3.2 holds.

Since $\lambda\approx 0.103535$, $\lambda_2\approx 0.145737$, let $\rho=1$, $L=0.14$, then $f(t,x)\le 0.14 = L\rho$, $(t,x)\in[0,1]\times[0,1.18977]$. Hence, condition (H2) in Theorem 3.2 holds too. From Theorem 3.2, problem (50) has at least two positive solutions.

Example 4.2. Consider the problem

$$\begin{cases} \mathbf{D}_{0+}^{\frac{3}{2}}x(t) = f\left(t, x\left(t-\dfrac{1}{2}\right)\right), & t\in(0,1), \\ x(0)+2x'(0)=-t^2, & t\in[-1,0], \\ x(1)+2x'(1)+3\mathbf{D}_{0+}^{\frac{1}{2}}x(0.5)=0, \end{cases} \tag{51}$$

where

$$f(t,u) = \frac{t}{200} + \begin{cases} \dfrac{u^3}{4}, & (t,u)\in[0,1]\times[0,0.3], \\ 3.9+389.325(u-0.31), & (t,u)\in[0,1]\times(0.3,0.31], \\ 4.1+\dfrac{2}{30.69}(u-31), & (t,u)\in[0,1]\times(0.31,31], \\ u-31+4.1|\cos(u-31)|, & (t,u)\in[0,1]\times(31,+\infty). \end{cases}$$

Choosing $c=0.05$, $d=3$, $h=30$, from $\lambda_3\approx 1.28260$, it is easy to verify that all conditions in Theorem 3.5 hold, then problem (51) has at least three positive solutions.

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