

## POLYNOMIAL CONVERGENCE OF PRIMAL-DUAL ALGORITHMS FOR SDLCP BASED ON THE M-Z FAMILY OF DIRECTIONS<sup>†</sup>

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ABSTRACT. We establish the polynomial convergence of a new class of path-following methods for SDLCP whose search directions belong to the class of directions introduced by Monteiro [3]. We show that the polynomial iteration-complexity bounds of the well known algorithms for linear programming, namely the short-step path-following algorithm of Kojima et al. and Monteiro and Alder, carry over to the context of SDLCP.

AMS Mathematics Subject Classification : 90C33, 65G20, 65G50.

*Key words and phrases* : interior-point algorithm, polynomial complexity, path-following methods, semidefinite linear complementarity problems.

### 1. Introduction

Several authors have discussed generalizations of interior-point algorithms for linear programming (LP) to the context of semidefinite programming (SDP). The landmark work in this direction is due to Nesterov and Nemirovskii [1], where a general approach for using interior-point algorithms for solving convex programs is proposed, based on the notion of self-concordant functions. Since then many authors have proposed interior-point algorithms for solving the SDP problems and SDLCP, including Kojima, Shida and Shindoh [2], Monteiro [3,4], Monteiro and Zhang [5,6], and Zhang [7].

### 2. Notation and terminology

The set of all symmetric  $n \times n$  matrices is denoted by  $S^n$ . For  $Q \in S^n$ ,  $Q \succeq 0$  means  $Q$  is positive semidefinite and  $Q \succ 0$  means  $Q$  is positive definite. The inner product between them in the vector space  $R^{m \times n}$  is defined as  $P \bullet Q \equiv \text{Tr } P^T Q$ . The Euclidean norm and its associated operator norm are both denoted by

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Received March 8, 2011. Revised June 4, 2011. Accepted June 16, 2011.

<sup>†</sup>This work was supported by Natural Science Foundation of Chongqing Municipal Education Commission(No.KJ091104 and KJ101108).

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$\|\circ\|$ ; The Frobenius norm of  $Q \in R^{n \times n}$  is  $\|Q\|_F \equiv (Q \bullet Q)^{1/2}$ . For  $Q, R \in R^{n \times n}$ ,  $S_+^n$  and  $S_{++}^n$  denote the set of all matrices in  $S^n$  which are positive semidefinite and positive definite, respectively.

### 3. The SDLCP problem and preliminary discussion

This subsection describes the SDLCP problem and the corresponding assumptions. It also contains some notations and terminology that are used throughout our presentation. Semidefinite linear complementarity problems (SDLCP) determines a matrix pair  $(X, S) \in S^n \times S^n$  satisfying

$$(X, S) \in \mathcal{F}, \quad X \succeq 0, \quad Y \succeq 0, \quad X \bullet Y = 0. \quad (1)$$

Here  $\mathcal{F}$  is an  $n(n+1)/2$ -dimensional affine subspace of  $S^n \times S^n$ . We call  $(X, S) \in \mathcal{F}$  with  $X \succeq 0$  and  $Y \succeq 0$  a feasible solution of the SDLCP (1) and  $(X, S) \in \mathcal{F}$  with  $X \succ 0$  and  $Y \succ 0$  an interior feasible solution of the SDLCP (1) denoted by  $\mathcal{F}_+$  and  $\mathcal{F}_{++}$ , respectively.

Throughout our presentation, we assume that

[A1]  $\mathcal{F}$  is monotone, that is  $(X_1 - X_2) \bullet (S_1 - S_2) \geq 0$  for any  $(X_1, S_1) \in \mathcal{F}$  and  $(X_2, S_2) \in \mathcal{F}$ .

[A2]  $\mathcal{F}_{++}$  is nonempty.

Under assumptions [A1] and [A2], it is known that problem (1) has at least one solution. Since for  $(X, S) \in S_+^n \times S_+^n$ , we have  $X \bullet Y = 0$  if and only if  $XY = 0$ , problem (1) is equivalent to find a pair  $(X, S)$  such that

$$(X, S) \in \mathcal{F}_+, \quad XS = 0.$$

It has been shown by Kojima, Shindoh and Hara [11] that the perturbed system

$$(X, S) \in \mathcal{F}_+, \quad XS = \mu I. \quad (2)$$

has a unique solution in  $\mathcal{F}_+$ , denoted by  $(X_\mu, S_\mu)$ , for every  $\mu > 0$ , and  $\lim_{\mu \rightarrow 0} (X_\mu, S_\mu)$  exists and is a solution of (1). The set  $\{(X_\mu, S_\mu) : \mu > 0\}$  is called the central path associated with (1) and plays a fundamental role in the development of interior point algorithms for solving SDP and SDCLP. Using the square root  $X^{1/2}$ , (2) can also be alternatively expressed in the following symmetric form:

$$(X, S) \in \mathcal{F}_+, \quad X^{1/2}SX^{1/2} = \mu I \quad \left( \text{or, } S^{1/2}XS^{1/2} = \mu I \right).$$

The path-following algorithms studied in this paper are all based on the following centrality measures of a point for  $(X, S) \in \mathcal{F}_+$ :

$$N_F(\mu, \gamma) = \left\| X^{1/2}SX^{1/2} - \mu I \right\|_F \leq \gamma \mu.$$

Path following algorithms for solving (1) are based on the idea of approximately tracing the central path. Application of Newton method for computing the solution of (2) with  $\mu = \hat{\mu}$  leads to the Newton search direction  $(\widehat{\Delta X}, \widehat{\Delta S})$

which solves the linear system

$$X\widehat{\Delta S} + \widehat{\Delta X}S = \hat{\mu}I - XS, \quad \left( X + \widehat{\Delta X}, S + \widehat{\Delta S} \right) \in \mathcal{F}. \quad (3)$$

This system does not always have a solution. To overcome this bottleneck, if we adapt the M-Z search directions to the monotone SDLCP, we can describe it as a solution of the system of equations:

$$X^{-1/2}(X\Delta S + \Delta XS)X^{1/2} + X^{1/2}(\Delta SX + S\Delta X)X^{-1/2} = 2(\hat{\mu}I - X^{1/2}SX^{1/2}). \quad (4)$$

Here  $(X, S) \in \mathcal{F}_{++}$  denotes an iterate and  $\mu = X \bullet S/n$ . It was shown in paper [8] that the system (4) of equations above has the unique solution  $(\Delta X, \Delta S) \in S^n \times S^n$ .

**Theorem 3.1.** *System (4) has a unique solution.*

We let throughout this section that  $(X, S) \in \mathcal{F}_{++}$  and that  $(\Delta X, \Delta S)$  is a solution of system (4) with  $\hat{\mu} = \sigma\mu$  for some  $\mu > 0$  and  $\sigma \in [0, 1]$ . Moreover, we define for every  $\alpha \in \mathbb{R}$ ,

$$X(\alpha) \equiv X + \alpha\Delta X, \quad S(\alpha) \equiv S + \alpha\Delta S, \quad (5)$$

$$\mu(\alpha) \equiv (1 - \alpha + \sigma\alpha)\mu. \quad (6)$$

**Lemma 3.2.** *For every  $\alpha \in \mathbb{R}$ , we have*

$$X(\alpha)S(\alpha) - \mu(\alpha)I = (1 - \alpha)(XS - \mu I) + \alpha(XS - \sigma\mu I) + \alpha(X\Delta S + \Delta XS) + \alpha^2\Delta X\Delta S. \quad (7)$$

*Proof.* Follows immediately from (5), (6) and (4) with  $\hat{\mu} = \sigma\mu$ .  $\square$

For a nonsingular matrix  $P \in \mathbb{R}^{n \times n}$ , consider the following operator  $H_P : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  defined as

$$H_P(M) \equiv \frac{1}{2} [PMP^{-1} + (PMP^{-1})^T], \quad \forall M \in \mathbb{R}^{n \times n}.$$

**Lemma 3.3.** *For every  $\alpha \in [0, 1]$ , we have*

$$\|H_{X^{-1/2}}[X(\alpha)S(\alpha) - \mu(\alpha)I]\|_F \leq (1 - \alpha) \|X^{1/2}SX^{1/2} - \mu I\|_F + \alpha^2\delta_x\delta_s/2\mu, \quad (8)$$

where

$$\delta_x = \mu \|X^{-1/2}\Delta XX^{-1/2}\|_F, \quad \delta_s = \|X^{1/2}\Delta SX^{1/2}\|_F. \quad (9)$$

*Proof.* Using (7), we can obtain

$$\begin{aligned} & 2H_{X^{-1/2}}[X(\alpha)S(\alpha) - \mu(\alpha)I] \\ = & 2(1 - \alpha) \left( X^{1/2}SX^{1/2} - \mu I \right) + 2\alpha \left( X^{1/2}SX^{1/2} - \sigma\mu I \right) \\ & + \alpha \left[ X^{-1/2}(X\Delta S + \Delta XS)X^{1/2} + X^{1/2}(\Delta SX + S\Delta X)X^{-1/2} \right] \\ & + \alpha^2 \left( X^{-1/2}\Delta X\Delta SX^{1/2} + X^{1/2}\Delta S\Delta XX^{-1/2} \right) \\ = & 2(1 - \alpha) \left( X^{1/2}SX^{1/2} - \mu I \right) + \alpha^2 \left( X^{-1/2}\Delta X\Delta SX^{1/2} + X^{1/2}\Delta S\Delta XX^{-1/2} \right). \end{aligned}$$

Then take Frobenius norm on both sides, we can prove the (8) holds.  $\square$

**Lemma 3.4.** *Let  $(X, S) \in \mathcal{F}_{++}$  be such that  $\|X^{1/2}SX^{1/2} - \mu I\| \leq \mu\gamma$  for some  $\gamma \in [0, 1)$  and  $\mu > 0$ . Suppose that  $(\Delta X, \Delta S) \in S^{n \times n} \times S^{n \times n}$  is a solution of (4) for  $\mathcal{W} \in R^{n \times n}$ , where  $\mathcal{W} = \sigma\mu I - X^{1/2}SX^{1/2}$ . Let  $\delta_x = \mu \|X^{-1/2}\Delta XX^{-1/2}\|_F$  and  $\delta_s = \|X^{1/2}\Delta SX^{1/2}\|_F$ . Then,*

$$\delta_x \delta_s \leq \frac{1}{2} (\delta_x^2 + \delta_s^2) \leq \frac{\|\mathcal{W}\|_F^2}{2(1-\gamma)^2}.$$

*Proof.* We let  $\mathcal{W} = H_{X^{-1/2}}[X\Delta S + \Delta XS]$ . Using (4) and simple algebraic manipulation, we can obtain

$$\begin{aligned} \mathcal{W} &= X^{1/2}\Delta SX^{1/2} + \mu X^{-1/2}\Delta XX^{-1/2} + \frac{1}{2}X^{-1/2}\Delta XX^{-1/2}(X^{1/2}SX^{1/2} - \mu I) \\ &\quad + \frac{1}{2}(X^{1/2}SX^{1/2} - \mu I)X^{-1/2}\Delta XX^{-1/2}, \end{aligned}$$

from which it follows that

$$\begin{aligned} \|\mathcal{W}\|_F &\geq \left\| X^{1/2}\Delta SX^{1/2} + \mu X^{-1/2}\Delta XX^{-1/2} \right\|_F \\ &\quad - \left\| X^{1/2}SX^{1/2} - \mu I \right\| \left\| X^{-1/2}\Delta XX^{-1/2} \right\|_F \\ &\geq \left( \left\| X^{1/2}\Delta SX^{1/2} \right\|_F^2 + \mu^2 \left\| X^{-1/2}\Delta XX^{-1/2} \right\|_F^2 \right)^{1/2} - \gamma\mu\delta_x/\mu \\ &\geq \sqrt{\delta_x^2 + \delta_s^2} - \gamma\delta_x \geq (1-\gamma)\sqrt{\delta_x^2 + \delta_s^2}, \end{aligned}$$

where the second inequality follows from the assumption that  $\|X^{1/2}SX^{1/2} - \mu I\| \leq \mu\gamma$  and the fact that  $(X^{-1/2}\Delta XX^{-1/2}) \bullet (X^{1/2}\Delta SX^{1/2}) = \Delta X \bullet \Delta S \geq 0$ , due to the monotonicity of  $\mathcal{F}$ . The result now follows trivially from the last inequality.  $\square$

**Lemma 3.5.** *Suppose that  $(X, S) \in N_F(\mu, \gamma)$  for some  $\gamma \in (0, 1)$  and let  $(\Delta X, \Delta S) \in S^{n \times n} \times S^{n \times n}$  be the solution of (4). Then,*

$$\|H_{X^{-1/2}}[X(\alpha)S(\alpha) - \mu(\alpha)I]\|_F \leq \left\{ (1-\alpha)\gamma + \alpha^2 \frac{n(1-\sigma)^2 + \gamma^2}{4(1-\gamma)^2} \right\} \mu.$$

*Proof.* Follows immediately from (8), the assumption that  $(X, S) \in N_F(\mu, \gamma)$  and Lemma 2.3, we can obtain

$$\begin{aligned} &\|H_{X^{-1/2}}[X(\alpha)S(\alpha) - \mu(\alpha)I]\|_F \\ &\leq \left\{ (1-\alpha)\gamma + \alpha^2 \frac{\|\sigma\mu I - X^{1/2}SX^{1/2}\|_F^2}{4(1-\gamma)^2\mu^2} \right\} \mu \\ &= \left\{ (1-\alpha)\gamma + \alpha^2 \frac{\|(\sigma-1)\mu I\|_F^2 + \|\mu I - X^{1/2}SX^{1/2}\|_F^2}{4(1-\gamma)^2\mu^2} \right\} \mu \\ &\leq \left\{ (1-\alpha)\gamma + \alpha^2 \frac{n(\sigma-1)^2 + \gamma^2}{4(1-\gamma)^2} \right\} \mu. \end{aligned}$$

The equality holds from the fact  $(X^{1/2}SX^{1/2} - \mu I) \bullet I = 0$ , then we complete the proof.  $\square$

#### 4. The short-step path following algorithm

In this section, we analyze the polynomial convergence of a short-step path-following algorithm based on the  $\|\circ\|_F$  norm neighborhood and the M-Z family of search directions.

##### Algorithm-I

Choose constants  $\gamma, \delta \in (0, 1)$  satisfying the conditions of and let  $\sigma = 1 - \delta/n$ . Let  $(X^0, S^0) \in \mathcal{F}_{++}$  and  $\mu_0 = X^0 \bullet S^0/n$  be such that  $(X^0, S^0) \in N_F(\mu, \gamma)$  and set  $k = 0$ .

**Repeat until**  $\mu\mu_k \leq \varepsilon\mu_0$ , do

step1. Compute the solution  $(\Delta X^k, \Delta S^k)$  of system (4) with  $(X, S) = (X^k, S^k)$  and  $\hat{\mu} = \sigma\mu_k$ ;

step2. Set a  $(X^{k+1}, S^{k+1}) \equiv (X^k, S^k) + (\Delta X^k, \Delta S^k)$  and  $\mu_{k+1} = \sigma\mu_k$ ;

step3. Increment  $k$  by 1.

**End**

We start by stating two technical results. The first one is due to Monteiro (see Lemma 2.1 of [4]) and plays a crucial role in our analysis.

**Lemma 4.1.** *Suppose that  $(X, S) \in S_{++}^n \times S_{++}^n$  and  $\mathcal{M} \in R^{n \times n}$  is a nonsingular matrix. Then, for every  $\mu \in R$ , we have  $\|X^{1/2}SX^{1/2} - \mu I\|_F \leq \|H_{\mathcal{M}}(XS - \mu I)\|_F$ , with equality holding if  $\mathcal{M}XS\mathcal{M}^{-1} \in S^n$ .*

**Lemma 4.2.** *Suppose  $V, Q \in R^{n \times n}$  be given, and  $\mathcal{M}$  is nonsingular which satisfying*

$$\|H_{\mathcal{M}}(V) - I\| < 1, \quad (10)$$

then, the matrix  $V$  is nonsingular.

*Proof.* Define  $M \equiv \mathcal{M}V\mathcal{M}^{-1}/2$ . Condition (10) implies that  $M + M^T \succ 0$ , and this clearly implies that  $M$  is nonsingular. Hence,  $V$  is also nonsingular.  $\square$

When the constant  $\Gamma$  defined in (11) is such that  $\Gamma \leq \gamma$ , the theorem below implies that the sequence  $\{(X^k, S^k)\}$  generated by Algorithm-I is contained in the neighborhood  $N_F(\mu^k, \gamma)$ .

**Theorem 4.3.** *Suppose  $\gamma \in (0, 1)$  and  $\delta \in [0, \sqrt{n}]$  be constants satisfying*

$$\Gamma \equiv \frac{\gamma^2 + \delta^2}{4(1 - \gamma)^2} \left(1 - \frac{\delta}{\sqrt{n}}\right)^{-1} \leq 1. \quad (11)$$

*Suppose that  $(X, S) \in N_F(\mu, \gamma)$  for some  $\mu > 0$ , and that  $(\Delta X, \Delta S)$  denote the solution of system (4) with  $\hat{\mu} = \sigma\mu$  and  $\sigma = 1 - \delta/\sqrt{n}$ . Then,*

(1)  $(\tilde{X}, \tilde{S}) = (X + \Delta X, S + \Delta S) \in N_F(\sigma\mu, \Gamma)$ ; (2)  $\tilde{X} \bullet \tilde{S} = (1 - \delta/\sqrt{n})X \bullet S$ .

*Proof.* It follows from Lemma 2.5, the definition of  $\sigma$  and [11] that for every  $\alpha \in [0, 1]$ ,

$$\begin{aligned} \|H_{X^{-1/2}} [X(\alpha)S(\alpha) - \mu(\alpha)I]\|_F &\leq \left\{ (1-\alpha)\gamma + \alpha^2 \frac{n(\sigma-1)^2 + \gamma^2}{4(1-\gamma)^2} \right\} \mu. \\ &\leq \left\{ (1-\alpha)\gamma + \alpha \frac{\delta^2 + \gamma^2}{4(1-\gamma)^2} \right\} \mu. \\ &= \{(1-\alpha)\gamma + \alpha\Gamma(1 - \delta/\sqrt{n})\} \mu \\ &= \{(1-\alpha)\gamma + \sigma\Gamma\alpha\} \mu, \end{aligned}$$

and hence, in view of (6) and (11), we have

$$\left\| H_{X^{-1/2}} \left[ \frac{X(\alpha)S(\alpha)}{\mu(\alpha)} \right] - I \right\|_F \leq \frac{(1-\alpha)\gamma + \sigma\Gamma\alpha}{1-\alpha + \sigma\alpha} \leq \max\{\gamma, \Gamma\} < 1.$$

By Lemma 3.2, this implies that  $X(\alpha)S(\alpha)$  is nonsingular for every  $\alpha \in (0, 1]$ . Hence,  $X(\alpha)$  and  $S(\alpha)$  are also nonsingular for every  $\alpha \in (0, 1]$ . Using the fact that  $(X, S) \in \mathcal{F}_{++}$ ,  $(X + \Delta X, S + \Delta S) \in \mathcal{F}$  and a simple continuity argument, we see  $(X(\alpha), S(\alpha)) \in \mathcal{F}_{++} \subseteq S_{++}^n \times S_{++}^n$  for every  $\alpha \in (0, 1]$ . Applying Lemma 3.1 with  $(X, S) = (X(\alpha), S(\alpha))$  and  $\mathcal{M} = X^{-1/2}$ , we conclude that for every  $\alpha \in (0, 1]$ ,

$$\begin{aligned} \left\| X(\alpha)^{1/2} S(\alpha) X(\alpha)^{1/2} - \mu(\alpha)I \right\|_F &\leq \|H_{X^{-1/2}} [X(\alpha)S(\alpha) - \mu(\alpha)I]\|_F. \\ &\leq \left\| X^{-1/2} X(\alpha) S(\alpha) X^{1/2} - \mu(\alpha)I \right\|_F. \\ &\leq \{(1-\alpha)\gamma + \sigma\Gamma\alpha\} \mu. \end{aligned}$$

Setting  $\alpha = 1$  in the last relation and using the fact that  $(X(1), S(1)) \in \mathcal{F}_{++}$  together with (5) and (6), we conclude that  $(X(1), S(1)) \equiv (X + \Delta X, S + \Delta S) \in N_F(\sigma\mu, \Gamma)$ . Statement (2) follows from (6) with  $\alpha = 1$  and the definition of  $\sigma$ .  $\square$

**Theorem 4.4.** *Suppose that  $\gamma$  and  $\delta$  are constants in  $(0, 1)$  satisfying (11). Then, every iterate  $(X^k, S^k)$  generated by Algorithm-I is in  $N_F(\mu^k, \gamma)$  and satisfies*

$$X^k \bullet S^k \leq (1 - \delta/\sqrt{n})^k (X^0 \bullet S^0). \quad (12)$$

Moreover, Algorithm-I terminates in at most  $\mathcal{O}(\sqrt{n} \log \varepsilon^{-1})$  iterations.

*Proof.* The proof that every iterate  $(X^k, S^k)$  is in  $N_F(\mu^k, \gamma)$  follows immediately from Theorem 3.3 and a simple argument. Relation (12) follows from the fact that  $\mu^k = \sigma^k \mu^0$ .  $\square$

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