ON SUFFICIENCY AND DUALITY IN MULTIOBJECTIVE SUBSET PROGRAMMING PROBLEMS INVOLVING GENERALIZED d-TYPE I UNIVEX FUNCTIONS[†]

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ABSTRACT. In this paper, we introduce new classes of generalized convex n-set functions called d-weak strictly pseudo-quasi type-I univex, d-strong pseudo-quasi type-I univex and d-weak strictly pseudo type-I univex functions and focus our study on multiobjective subset programming problem. Sufficient optimality conditions are obtained under the assumptions of aforesaid functions. Duality results are also established for Mond-Weir and general Mond-Weir type dual problems in which the involved functions satisfy appropriate generalized d-type-I univexity conditions.

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1. Introduction

The analysis of optimization problems with set functions has been the subject of many papers and have various interesting applications in fluid flow (Begis and Glowinski [5]), electrical insulator design (Céa *et al.* [6]), regional design (Corley and Roberts [7, 8]), statistics (Dantzig and Wald [11] and Neyman and Pearson [20]).

The concept of optimizing n-set functions was initially developed by Morris [24] whose results are confined only to set functions of a single set. Corley [10] gave the concept of derivative of a real-valued n-set function and generalized the results of Morris [24] to n-set functions and established optimality conditions and Lagrangian duality.

Several authors have shown their interest to established sufficient optimality conditions for multiobjective subset programming problem by applying extra assumptions, such as convexity, generalized convexity, invexity and generalized invexity [4, 9, 15-18, 21, 27]. Furthermore, by using the sufficient optimality conditions, various dual models have been formulated and duality results have been established.

Hanson and Mond [12] defined two new classes of functions, called type I and type II function. The concept of type I and type II functions was further generalized to pseudo type I and quasi type I functions by Rueda and Hanson [25], and to pseudo-quasi type I, quasi-pseudo type I and strictly quasi-pseudo type I functions by Kaul *et al.* [14]. Later, Aghezzaf and Hachimi [1] introduced generalized type I vector-valued functions, which are different from those defined in Kaul *et al.* [14].

Ye [26] introduced *d*-invexity by replacing derivative with directional derivative, and developed necessary conditions for weak efficiency for a nondifferentiable multiobjective programming problem assuming the directional derivatives of objective and active constraints functions to be convex. Preda *et al.* [22] established optimality and duality results for a multiobjective programming problem involving *n*-set functions under the assumptions of *d*-type I, *d*-pseudo type I and *d*-quasi type I functions. Mishra *et al.* [19] introduced four types of generalized convexity for an *n*-set function and discuss optimality and duality for a multiobjective programming problem involving *n*-set functions. Ahmad and Sharma [2] established sufficient optimality conditions for a multiobjective subset programming problem under generalized (F, α, ρ, d)-type I functions.

Bector *et al.* [3] introduced some classes of univex functions by relaxing the definition of an invex function. Optimality and duality results are also obtained for a nonlinear multiobjective programming problem in [3].

Jayswal and Kumar [13] introduced new classes of generalized convex functions called *d-V*-type-I univex and obtained Karush-Kuhn-Tucker-type sufficient optimality conditions and Mond-Weir type duality results for nondifferentiable multiobjective programming problem. Recently, Preda *et al.* [23] introduced generalized $(\rho, \acute{\rho})$-*V*-univex functions and focus his study on optimality conditions and generalized Mond-Weir duality for multiobjective programming involving *n*-set functions which satisfy appropriate generalized univexity *V*-type-I conditions.

We now consider the following nonlinear multiobjective subset programming problem:

(P)   minimize $F(S) = \big[F_1(S), F_2(S), ..., F_p(S)\big]$
        subject to $G_j(S) \leqq 0, \ j \in M, \ S = (S_1, S_2, ..., S_n) \in A^n,$

where A^n is the *n*-fold product of σ-algebra A of subsets of a given set X, $F_i, i \in P = \{1, 2, ..., p\}$ and $G_j, j \in M = \{1, 2, ..., m\}$ are real-valued functions

defined on A^n. Let $X_0 = \{S \in A^n : G_j(S) \leqq 0, j \in M\}$ be the set of all feasible solutions to (P).

In this paper, we introduce new classes of generalized d-weak strictly pseudo-quasi type-I univex, d-strong pseudo-quasi type-I univex and d-weak strictly pseudo type-I univex functions in Section 2. Based upon these functions, sufficient optimality conditions are discussed for the multiobjective subset programming problem (P) in Section 3. Duality results are also obtained in the setting of Mond-Weir and general Mond-Weir type dual in Section 4 and 5 respectively. Furthermore, the results obtained in this paper extend and generalized the results of Jayswal and Kumar [13] to the class of n-set functions and the results of Mishra *et al.* [19] to the class of n-set univex functions.

2. Preliminaries

The following conventions for vectors in R^n will be followed throughout this paper:

$$x \geqq y \Leftrightarrow x_k \geqq y_k, \ k = 1, 2, \ldots, n;$$
$$x \geq y \Leftrightarrow x_k \geq y_k, \ k = 1, 2, \ldots, n \text{ and } x \neq y;$$
$$x > y \Leftrightarrow x_k > y_k, \ k = 1, 2, \ldots, n.$$

We write $x \in R^n_+$ iff $x \geqq 0$. Let (X, A, μ) be a finite atomless measure space with $L_1(X, A, \mu)$ separable and let d be the pseudometric on A^n defined by

$$d(S, T) = \Big[\sum_{k=1}^n \mu^2(S_k \Delta T_k) \Big]^{1/2}, S = (S_1, S_2, ..., S_n) \in A^n, T = (T_1, T_2, ..., T_n) \in A^n,$$

where Δ denotes symmetric difference; thus, (A^n, d) is a pseudometric space. For $h \in L_1(X, A, \mu)$ and $Z \in A$ with characteristic function $\chi_Z \in L_\infty(X, A, \mu)$, the integral $\int_Z h d\mu$ will be denoted by $\langle h, \chi_Z \rangle$.

We next define the notions of differentiability for n-set functions. This was originally introduced by Morris [24] for set functions, and subsequently extended by Corley [10] to n-set functions.

A function $\phi : A \to R$ is said to be differentiable at $S^0 \in A$ if there exist $D\phi(S^0) \in L_1(X, A, \mu)$, called the derivative of ϕ at S^0 and $\psi : A \times A \to R$ such that for each $S \in A$,

$$\phi(S) = \phi(S^0) + \big\langle D\phi(S^0), I_S - I_{S^0} \big\rangle + \psi(S, S^0),$$

where $\psi(S, S^0)$ is $o(d(S, S^0))$, that is, $\lim\limits_{d(S,S^0) \to 0} \frac{\psi(S,S^0)}{d(S,S^0)} = 0$.

A function $F : A^n \to R$ is said to have a partial derivative at $S^0 = (S_1^0, S_2^0, ..., S_n^0)$ with respect to its pth argument if the function

$$\phi(S_k) = F\left(S_1^0, ..., S_{k-1}^0, S_k, S_{k+1}^0, ..., S_n^0\right)$$

has derivative $D\phi(S_k^0)$ and we define $D_k F(S^0) = D\phi(S_k^0)$. If $D_k F(S^0)$, $k = 1, 2, ..., n$, all exist, then we put $DF(S^0) = (D_1 F(S^0), D_2 F(S^0), ..., D_n F(S^0))$.

A function $F : A^n \to R$ is said to be differentiable at S^0 if there exist $DF(S^0)$ and $\psi : A^n \times A^n \to R$ such that

$$F(S) = F(S^0) + \sum_{k=1}^{n} \left\langle D_k F(S^0), I_{S_k} - I_{S_k^0} \right\rangle + \psi(S, S^0),$$

where $\psi(S, S^0)$ is $o(d(S, S^0))$ for all $S \in A^n$.

Definition 2.1. A feasible solution S^0 to (P) is said to be an *efficient solution* to (P), if there exists no other feasible solution S to (P) such that

$$F(S) \leq F(S^0).$$

Definition 2.2. A feasible solution S^0 to (P) is said to be a *weakly efficient solution* to (P), if there exists no other feasible solution S ($S \neq S^0$) to (P) such that

$$F(S) < F(S^0).$$

Along the lines of Jayswal and Kumar [13], we now define the following classes of n-set functions, called d-weak strictly pseudo-quasi type-I univex, d-strong pseudo-quasi type-I univex and d-weak strictly pseudo type-I univex functions.

Definition 2.3. We say that the pair of functions (F, G) is *d-weak strictly pseudo-quasi type*-I *univex* at $S^0 \in A^n$ with respect to b_0, b_1, ϕ_0, ϕ_1,
$\gamma = (\gamma_1, \gamma_2, ..., \gamma_p)$, $\delta = (\delta_1, \delta_2, ..., \delta_p)$ and $\eta = (\eta_1, \eta_2, ..., \eta_n)$, if there exist $\eta :$
$A^n \times A^n \to R^n, \gamma_i : A^n \times A^n \to R_+ \setminus \{0\}$, $i = 1, 2, ..., p$, $\delta_j : A^n \times A^n \to R_+ \setminus \{0\}$, $j = 1, 2, ..., m$, nonnegative functions b_0 and b_1, also defined on $A^n \times A^n$, and $\phi_0 : R \to R$, $\phi_1 : R \to R$, such that for all $S \in X_0$, the implications

$$b_0(S, S^0)\phi_0 \Big[\sum_{i=1}^{p} \gamma_i(S, S^0)F_i(S) - \sum_{i=1}^{p} \gamma_i(S, S^0)F_i(S^0) \Big] \leq 0$$

$$\Rightarrow \sum_{i=1}^{p}\sum_{k=1}^{n} \eta_k(S, S^0) \left\langle D_k F_i(S^0), I_{S_k} - I_{S_k^0} \right\rangle < 0,$$

$$-b_1(S, S^0)\phi_1 \Big[\sum_{j=1}^{m} \delta_j(S, S^0)G_j(S^0) \Big] \leqq 0$$

$$\Rightarrow \sum_{j=1}^{m}\sum_{k=1}^{n} \eta_k(S, S^0) \left\langle D_k G_j(S^0), I_{S_k} - I_{S_k^0} \right\rangle \leqq 0,$$

both hold.

Definition 2.4. We say that the pair of functions (F, G) is *d-strong pseudo-quasi type*-I *univex* at $S^0 \in A^n$ with respect to $b_0, b_1, \phi_0, \phi_1, \gamma = (\gamma_1, \gamma_2, ..., \gamma_p), \delta = (\delta_1, \delta_2, ..., \delta_p)$ and $\eta = (\eta_1, \eta_2, ..., \eta_n)$, if there exist $\eta : A^n \times A^n \to R^n, \gamma_i :$
$A^n \times A^n \to R_+ \setminus \{0\}$, $i = 1, 2, ..., p$, $\delta_j : A^n \times A^n \to R_+ \setminus \{0\}$, $j = 1, 2, ..., m$,

nonnegative functions b_0 and b_1, also defined on $A^n \times A^n$, and $\phi_0 : R \to R$, $\phi_1 : R \to R$, such that for all $S \in X_0$, the implications

$$b_0(S, S^0)\phi_0\Big[\sum_{i=1}^p \gamma_i(S, S^0)F_i(S) - \sum_{i=1}^p \gamma_i(S, S^0)F_i(S^0)\Big] \leq 0$$

$$\Rightarrow \sum_{i=1}^p \sum_{k=1}^n \eta_k(S, S^0)\left\langle D_k F_i(S^0), I_{S_k} - I_{S_k^0}\right\rangle \leq 0,$$

$$-b_1(S, S^0)\phi_1\Big[\sum_{j=1}^m \delta_j(S, S^0)G_j(S^0)\Big] \leqq 0$$

$$\Rightarrow \sum_{j=1}^m \sum_{k=1}^n \eta_k(S, S^0)\left\langle D_k G_j(S^0), I_{S_k} - I_{S_k^0}\right\rangle \leqq 0,$$

both hold.

Definition 2.5. We say that the pair of functions (F, G) is *d-weak strictly pseudo type*-I *univex* at $S^0 \in A^n$ with respect to $b_0, b_1, \phi_0, \phi_1, \gamma = (\gamma_1, \gamma_2, ..., \gamma_p), \delta = (\delta_1, \delta_2, ..., \delta_p)$ and $\eta = (\eta_1, \eta_2, ..., \eta_n)$, if there exist $\eta : A^n \times A^n \to R^n, \gamma_i : A^n \times A^n \to R_+ \setminus \{0\}$, $i = 1, 2, ..., p$, $\delta_j : A^n \times A^n \to R_+ \setminus \{0\}$, $j = 1, 2, ..., m$, nonnegative functions b_0 and b_1, also defined on $A^n \times A^n$, and $\phi_0 : R \to R$, $\phi_1 : R \to R$, such that for all $S \in X_0$, the implications

$$b_0(S, S^0)\phi_0\Big[\sum_{i=1}^p \gamma_i(S, S^0)F_i(S) - \sum_{i=1}^p \gamma_i(S, S^0)F_i(S^0)\Big] \leq 0$$

$$\Rightarrow \sum_{i=1}^p \sum_{k=1}^n \eta_k(S, S^0)\left\langle D_k F_i(S^0), I_{S_k} - I_{S_k^0}\right\rangle < 0,$$

$$-b_1(S, S^0)\phi_1\Big[\sum_{j=1}^m \delta_j(S, S^0)G_j(S^0)\Big] \leqq 0$$

$$\Rightarrow \sum_{j=1}^m \sum_{k=1}^n \eta_k(S, S^0)\left\langle D_k G_j(S^0), I_{S_k} - I_{S_k^0}\right\rangle < 0,$$

both hold.

Remark 2.1. The above definitions extend at n-set functions the concept of weak strictly pseudo-quasi-d-V-type-I univex, strong pseudo-quasi-d-V-type-I univex and weak strictly pseudo-d-V-type-I univex of Jayswal and Kumar [13] as well as extended at univexity the concept of d-weak strictly-pseudoquasi-type-I, d-strong-pseudoquasi-type-I and d-weak strictly pseudo-type-I of Mishra *et al.* [19].

3. Sufficient optimality conditions

The following theorem gives sufficient optimality conditions for a weakly efficient solution to (P) under the assumptions of generalized d-type-I univexity introduced in Section 2.

Theorem 3.1 (Sufficient optimality conditions). *Let S^0 be a feasible solution for* (P). *Assume that there exist* $\lambda_i^0 \geqq 0, i \in P, \sum_{i=1}^{p} \lambda_i^0 = 1$ *and* $\mu_j^0 \geqq 0, j \in M_0 = \{j \in M : G_j(S^0) = 0\}$, *such that for all* $S \in A^n$

$$\left\langle D_k(\lambda^0 F)(S^0) + D_k(\mu^0 G)(S^0), I_{S_k} - I_{S_k^0} \right\rangle \geqq 0. \tag{1}$$

Moreover, we assume that any one of the following conditions are satisfied:
(S1) $\lambda > 0$ *and* $(F, \mu G)$ *is d-strong pseudo-quasi type-I univex at S^0 with respect to* $b_0, b_1, \phi_0, \phi_1, \gamma = (\gamma_1, \gamma_2, ..., \gamma_p), \delta = (\delta_1, \delta_2, ..., \delta_p)$ *and* $\eta = (\eta_1, \eta_2, ..., \eta_n)$;
(S2) $(F, \mu G)$ *is d-weak strictly pseudo-quasi type-I univex at S^0 with respect to* $b_0, b_1, \phi_0, \phi_1, \gamma = (\gamma_1, \gamma_2, ..., \gamma_p), \delta = (\delta_1, \delta_2, ..., \delta_p)$ *and* $\eta = (\eta_1, \eta_2, ..., \eta_n)$;
(S3) $(F, \mu G)$ *is d-weak strictly pseudo type-I univex at S^0 with respect to* $b_0, b_1, \phi_0, \phi_1, \gamma = (\gamma_1, \gamma_2, ..., \gamma_p), \delta = (\delta_1, \delta_2, ..., \delta_p)$ *and* $\eta = (\eta_1, \eta_2, ..., \eta_n)$;
with η satisfying $\eta^T \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one $k \in \{1, 2, ..., n\}$.
Further, suppose that for $r \in R$, we have

$$r \leq 0 \Rightarrow \phi_0(r) \leq 0, \ r \leqq 0 \Rightarrow \phi_1(r) \leqq 0 \tag{2}$$

and

$$b_0(S, S^0) > 0, \ b_1(S, S^0) \geqq 0, \forall S \in X_0. \tag{3}$$

Then S^0 is a weakly efficient solution for (P).

Proof. Suppose contrary to the result that S^0 is not a weakly efficient solution of (P). Then there exists a feasible solution $S(S \neq S^0)$ such that

$$F_i(S) < F_i(S^0) \text{ for any } i \in P.$$

If hypothesis (S1) holds then, from $\lambda_i^0 \geqq 0, i \in P$, with $\sum_{i=1}^{p} \lambda_i^0 = 1$ and the positivity of $\gamma_i(S, S^0)$, we get

$$\sum_{i=1}^{p} \gamma_i(S, S^0) \lambda_i^0 F_i(S) < \sum_{i=1}^{p} \gamma_i(S, S^0) \lambda_i^0 F_i(S^0).$$

By (2) and (3) and the above inequality, we have

$$b_0(S, S^0) \phi_0 \left[\sum_{i=1}^{p} \gamma_i(S, S^0) \lambda_i^0 F_i(S) - \sum_{i=1}^{p} \gamma_i(S, S^0) \lambda_i^0 F_i(S^0) \right] < 0. \tag{4}$$

By the feasibility of S^0 and $G_j(S^0) = 0, \forall j \in M_0$, we have

$$-\sum_{j \in M_0} \delta_j(S, S^0) \mu_j^0 G_j(S^0) \leqq 0.$$

By (2) and (3) and the above inequality, we have

$$-b_1(S,S^0)\phi_1\Big[\sum_{j\in M_0}\delta_j(S,S^0)\mu_j^0 G_j(S^0)\Big] \leqq 0. \qquad (5)$$

By inequality (4) and (5) and hypothesis (S1), we have

$$\sum_{i=1}^{p}\sum_{k=1}^{n}\lambda_i^0\eta_k(S,S^0)\left\langle D_k F_i(S^0), I_{S_k} - I_{S_k^0}\right\rangle < 0$$

$$\text{and} \quad \sum_{j\in M_0}\sum_{k=1}^{n}\mu_j^0\eta_k(S,S^0)\left\langle D_k G_j(S^0), I_{S_k} - I_{S_k^0}\right\rangle \leqq 0.$$

The above inequalities together yield

$$\sum_{i=1}^{p}\sum_{k=1}^{n}\lambda_i^0\eta_k(S,S^0)\left\langle D_k F_i(S^0), I_{S_k} - I_{S_k^0}\right\rangle + \sum_{j\in M_0}\sum_{k=1}^{n}\mu_j^0\eta_k(S,S^0)\left\langle D_k G_j(S^0), I_{S_k} - I_{S_k^0}\right\rangle < 0.$$

That is,

$$\sum_{k=1}^{n}\eta_k(S,S^0)\left\langle D_k(\lambda^0 F)(S^0) + D_k(\mu^0 G)(S^0), I_{S_k} - I_{S_k^0}\right\rangle < 0.$$

From the assumption that $\eta^T\alpha < 0 \Rightarrow \alpha_k < 0$ for at least one $k \in \{1,2,...,n\}$, we obtain

$$\left\langle D_k(\lambda^0 F)(S^0) + D_k(\mu^0 G)(S^0), I_{S_k} - I_{S_k^0}\right\rangle < 0,$$

which contradicts (1).

By hypothesis (S2) from (4), (5) and similar to proof of hypothesis (S1), we get

$$\left\langle D_k(\lambda^0 F)(S^0) + D_k(\mu^0 G)(S^0), I_{S_k} - I_{S_k^0}\right\rangle < 0,$$

again a contradiction to (1).

By hypothesis (S3) from (4) and (5), we get

$$\sum_{i=1}^{p}\sum_{k=1}^{n}\lambda_i^0\eta_k(S,S^0)\left\langle D_k F_i(S^0), I_{S_k} - I_{S_k^0}\right\rangle < 0$$

$$\text{and} \quad \sum_{j\in M_0}\sum_{k=1}^{n}\mu_j^0\eta_k(S,S^0)\left\langle D_k G_j(S^0), I_{S_k} - I_{S_k^0}\right\rangle < 0.$$

By these two inequalities, we get

$$\sum_{i=1}^{p}\sum_{k=1}^{n}\lambda_i^0\eta_k(S,S^0)\left\langle D_k F_i(S^0), I_{S_k} - I_{S_k^0}\right\rangle$$

$$+ \sum_{j\in M_0}\sum_{k=1}^{n}\mu_j^0\eta_k(S,S^0)\left\langle D_k G_j(S^0), I_{S_k} - I_{S_k^0}\right\rangle < 0.$$

Rest of the proof is similar to hypothesis (S1). This completes the proof. $\square$

The following results from Zalmai [28] will be needed in the sequel.

Definition 3.1. A feasible solution S^0 is said to be a *regular feasible solution* if there exists $\hat{S} \in A^n$ such that

$$G_j(S^0) + \sum_{k=1}^{n} \left\langle D_k G_j(S^0), I_{\hat{S}_k} - I_{S_k^0} \right\rangle < 0, \ j \in M.$$

Lemma 3.1 ([28], Theorem 3.2). *Let S^0 be a regular efficient (or weakly efficient) solution to* (P) *and assume that $F_i, i \in P$ and $G_j, j \in M$ are differentiable at S^0. Then there exist $\lambda \in R_+^p$, $\sum_{i=1}^{p} \lambda_i = 1$, and $\mu \in R_+^m$ such that*

$$\sum_{k=1}^{n} \left\langle \sum_{i=1}^{p} \lambda_i D_k F_i(S^0) + \sum_{j=1}^{m} \mu_j D_k G_j(S^0), I_{S_k} - I_{S_k^0} \right\rangle \geqq 0, \ for \ all \ S \in A^n,$$

$$\mu_j G_j(S^0) = 0, \ j \in M.$$

4. Mond-Weir Duality

In this section, we associate the problem (P) with the following Mond-Weir dual problem (MD):

(MD)   maximize $F(T)$

     subject to

$$\langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle \geqq 0, \ \forall \ S \in A^n, \tag{6}$$

$$\sum_{j=1}^{m} \mu_j G_j(T) \geqq 0, \tag{7}$$

$$\lambda_i \geqq 0, \ i \in P \text{ and } \sum_{i=1}^{p} \lambda_i = 1, \tag{8}$$

$$\mu_j \geqq 0, \ j \in M \text{ and } T \in A^n. \tag{9}$$

Theorem 4.1 (Weak duality). *Let S and (T, λ, μ) be feasible solutions to* (P) *and* (MD), *respectively. Moreover, we assume that any one of the following conditions holds:*
(WD1) $\lambda > 0$ *and* $(F, \mu G)$ *is d-strong pseudo-quasi type-I univex at T with respect to $b_0, b_1, \phi_0, \phi_1, \gamma = (\gamma_1, \gamma_2, ..., \gamma_p), \delta = (\delta_1, \delta_2, ..., \delta_p)$ and $\eta = (\eta_1, \eta_2, ..., \eta_n)$;*
(WD2) $(F, \mu G)$ *is d-weak strictly pseudo-quasi type-I univex at T with respect to $b_0, b_1, \phi_0, \phi_1, \gamma = (\gamma_1, \gamma_2, ..., \gamma_p), \delta = (\delta_1, \delta_2, ..., \delta_p)$ and $\eta = (\eta_1, \eta_2, ..., \eta_n)$;*
(WD3) $(F, \mu G)$ *is d-weak strictly pseudo type-I univex at T with respect to $b_0, b_1, \phi_0, \phi_1, \gamma = (\gamma_1, \gamma_2, ..., \gamma_p), \delta = (\delta_1, \delta_2, ..., \delta_p)$ and $\eta = (\eta_1, \eta_2, ..., \eta_n)$;*
with η satisfying $\eta^T \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one $k \in \{1, 2, ..., n\}$.
Further, suppose that for $r \in R$, we have

$$r \leq 0 \Rightarrow \phi_0(r) \leq 0, \ r \leqq 0 \Rightarrow \phi_1(r) \leqq 0 \tag{10}$$

and

$$b_0(S,T) > 0, \ b_1(S,T) \geqq 0, \forall S \in X_0. \tag{11}$$

Then the following cannot hold:

$$F(S) \leq F(T).$$

Proof. Suppose contrary to the result that, i.e.

$$F(S) \leq F(T). \tag{12}$$

Since $\gamma_i(S,T) > 0$, (12) imply that

$$\sum_{i=1}^{p} \gamma_i(S,T) F_i(S) \leq \sum_{i=1}^{p} \gamma_i(S,T) F_i(T).$$

By (10) and (11) and the above inequality, we have

$$b_0(S,T)\phi_0\Big[\sum_{i=1}^{p} \gamma_i(S,T) F_i(S) - \sum_{i=1}^{p} \gamma_i(S,T) F_i(T)\Big] \leq 0. \tag{13}$$

By the feasibility of (T, λ, μ) for (MD), we have

$$-\sum_{j=1}^{m} \mu_j G_j(T) \leqq 0. \tag{14}$$

Since $\delta_j(S,T) > 0$, (14) implies that

$$-\sum_{j=1}^{m} \delta_j(S,T) \mu_j G_j(T) \leqq 0.$$

By (10) and (11) and the above inequality, we have

$$-b_1(S,T)\phi_1\Big[\sum_{j=1}^{m} \delta_j(S,T) \mu_j G_j(T)\Big] \leqq 0. \tag{15}$$

By inequality (13) and (15) and hypothesis (WD1), we have

$$\sum_{i=1}^{p} \sum_{k=1}^{n} \eta_k(S,T) \langle D_k F_i(T), I_{S_k} - I_{T_k} \rangle \leq 0$$

and $$\sum_{j=1}^{m} \sum_{k=1}^{n} \mu_j \eta_k(S,T) \langle D_k G_j(T), I_{S_k} - I_{T_k} \rangle \leqq 0.$$

Since $\lambda > 0$, the above two inequalities imply

$$\sum_{k=1}^{n} \eta_k(S,T) \langle D_k(\lambda F)(T), I_{S_k} - I_{T_k} \rangle < 0$$

and $$\sum_{k=1}^{n} \eta_k(S,T) \langle D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle \leqq 0.$$

The above two inequalities together yield

$$\sum_{k=1}^{n} \eta_k(S,T) \langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle < 0.$$

From the assumption that $\eta^T \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one $k \in \{1,2,...,n\}$, we obtain

$$\langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle < 0, \tag{16}$$

which contradicts (6).

By hypothesis (WD2) from (13) and (15) imply

$$\sum_{i=1}^{p}\sum_{k=1}^{n} \lambda_i \eta_k(S,T) \langle D_k F_i(T), I_{S_k} - I_{T_k} \rangle < 0$$

$$\text{and} \quad \sum_{j=1}^{m}\sum_{k=1}^{n} \mu_j \eta_k(S,T) \langle D_k G_j(T), I_{S_k} - I_{T_k} \rangle \leqq 0.$$

Since $\lambda \geqq 0$, and similar to proof of hypothesis (WD1), the above two inequalities imply (16), again a contradiction to (6).

By hypothesis (WD3) from (13) and (15) imply

$$\sum_{i=1}^{p}\sum_{k=1}^{n} \lambda_i \eta_k(S,T) \langle D_k F_i(T), I_{S_k} - I_{T_k} \rangle < 0$$

$$\text{and} \quad \sum_{j=1}^{m}\sum_{k=1}^{n} \mu_j \eta_k(S,T) \langle D_k G_j(T), I_{S_k} - I_{T_k} \rangle < 0.$$

Since $\lambda \geqq 0$, and similar to proof of hypothesis (WD1), the above two inequalities imply (16), again a contradiction to (6). This completes the proof. $\square$

Theorem 4.2 (Strong duality). *Let S^0 be a regular weakly efficient solution to (P). Then there exist $\lambda^0 \in R^p$, $\sum_{i=1}^{p}\lambda_i^0 = 1$ and $\mu^0 \in R^m$, such that (S^0, λ^0, μ^0) is a feasible solution to (MD) and the values of the objective functions of (P) and (MD) are equal at S^0 and (S^0, λ^0, μ^0), respectively. Furthermore, if the conditions of weak duality Theorem 4.1 also hold, for each feasible solution (T, λ, μ) to (MD), then (S^0, λ^0, μ^0) is a weakly efficient solution to (MD).*

Proof. Using Lemma 3.1 we obtain that there exist $\lambda_i^0 \geqq 0, i \in P$ with $\sum_{i=1}^{p}\lambda_i^0 = 1$ and $\mu_j^0 \geqq 0, j \in M$ such that (S^0, λ^0, μ^0) is feasible for (MD) and the values of the objective functions of (P) and (MD) are equal. The last part follows directly from Theorem 4.1. $\square$

5. Generalized Mond-Weir Duality

In this section, we associate the problem (P) with the following generalized Mond-Weir dual problem (GMD):

(GMD) maximize $F(T) + \sum_{j \in J_0} \mu_j G_j(T)e$

subject to

$$\langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k}\rangle \geqq 0, \ \forall \ S \in A^n, \tag{17}$$

$$\sum_{j \in J_\alpha} \mu_j G_j(T) \geqq 0 \text{ for } 1 \leqq \alpha \leqq r, \tag{18}$$

$$\lambda \geqq 0, \ \mu \geqq 0 \text{ and } \sum_{i=1}^p \lambda_i = 1, \tag{19}$$

where $e = (1, 1, ..., 1) \in R^p$ and $J_\alpha, \ 0 \leqq \alpha \leqq r$ is a partition of M, with $J_s \cap J_t = \phi$ for $s \neq t$ and $\cup_{s=0}^r J_s = M$.

Theorem 5.1 (Weak duality). *Let S and (T, λ, μ) be feasible solutions to (P) and (GMD), respectively. Moreover, we assume that any one of the following conditions holds:*
(GWD1) $\lambda > 0$ *and* $\left(F(.) + \sum_{j \in J_0} \mu_j G_j(.)e, \sum_{j \in J_\alpha} \mu_j G_j(.)\right)$ *is d-strong pseudo-quasi type-I univex at T with respect to* $b_0, b_1, \phi_0, \phi_1, \gamma = (\gamma_1, \gamma_2, ..., \gamma_p), \delta = (\delta_1, \delta_2, ..., \delta_p)$ *and* $\eta = (\eta_1, \eta_2, ..., \eta_n)$ *for any* $\alpha, \ 1 \leqq \alpha \leqq r$;
(GWD2) $\left(F(.) + \sum_{j \in J_0} \mu_j G_j(.)e, \sum_{j \in J_\alpha} \mu_j G_j(.)\right)$ *is d-weak strictly pseudo-quasi type-I univex at T with respect to* $b_0, b_1, \phi_0, \phi_1, \gamma = (\gamma_1, \gamma_2, ..., \gamma_p), \delta = (\delta_1, \delta_2, ..., \delta_p)$ *and* $\eta = (\eta_1, \eta_2, ..., \eta_n)$ *for any* $\alpha, \ 1 \leqq \alpha \leqq r$;
(GWD3) $\left(F(.) + \sum_{j \in J_0} \mu_j G_j(.)e, \sum_{j \in J_\alpha} \mu_j G_j(.)\right)$ *is d-weak strictly pseudo type-I nivex at T with respect to* $b_0, b_1, \phi_0, \phi_1, \gamma = (\gamma_1, \gamma_2, ..., \gamma_p), \delta = (\delta_1, \delta_2, ..., \delta_p)$ *and* $\eta = (\eta_1, \eta_2, ..., \eta_n)$ *for any* $\alpha, \ 1 \leqq \alpha \leqq r$;
with η satisfying $\eta^T \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one $k \in \{1, 2, ..., n\}$.
Further, suppose that for $r \in R$, we have

$$r \leq 0 \Rightarrow \phi_0(r) \leq 0, \ r \leqq 0 \Rightarrow \phi_1(r) \leqq 0 \tag{20}$$

and

$$b_0(S, T) > 0, \ b_1(S, T) \geqq 0, \forall S \in X_0. \tag{21}$$

Then the following cannot hold:

$$F(S) \leq F(T) + \sum_{j \in J_0} \mu_j G_j(T)e.$$

Proof. Suppose contrary to the result that, i.e.

$$F(S) \leq F(T) + \sum_{j \in J_0} \mu_j G_j(T)e.$$

By the feasibility of S and $\mu \geqq 0$, we have

$$F(S) + \sum_{j \in J_0} \mu_j G_j(S)e \leq F(T) + \sum_{j \in J_0} \mu_j G_j(T)e. \tag{22}$$

Since $\gamma_i(S, T) > 0$, (22) imply that

$$\sum_{i=1}^{p} \gamma_i(S, T)\Big(F_i(S) + \sum_{j \in J_0} \mu_j G_j(S)\Big) \leq \sum_{i=1}^{p} \gamma_i(S, T)\Big(F_i(T) + \sum_{j \in J_0} \mu_j G_j(T)\Big).$$

By (20) and (21) and the above inequality, we have

$$b_0(S, T)\phi_0\Big[\sum_{i=1}^{p} \gamma_i(S, T)\Big(F_i(S) + \sum_{j \in J_0} \mu_j G_j(S)\Big)$$

$$-\sum_{i=1}^{p} \gamma_i(S, T)\Big(F_i(T) + \sum_{j \in J_0} \mu_j G_j(T)\Big)\Big] \leq 0. \tag{23}$$

By the feasibility of (T, λ, μ) for (GMD), we have

$$-\sum_{j \in J_\alpha} \mu_j G_j(T) \leqq 0, \text{ for } 1 \leqq \alpha \leqq r. \tag{24}$$

Since $\delta_j(S, T) > 0$, (24) implies that

$$-\sum_{j \in J_\alpha} \delta_j(S, T)\mu_j G_j(T) \leqq 0.$$

By (20) and (21) and the above inequality, we have

$$-b_1(S, T)\phi_1\Big[\sum_{j \in J_\alpha} \delta_j(S, T)\mu_j G_j(T)\Big] \leqq 0. \tag{25}$$

By inequality (23) and (25) and hypothesis (GWD1), we have

$$\sum_{i=1}^{p}\sum_{k=1}^{n} \eta_k(S, T)\left\langle D_k\Big(F_i(T) + \sum_{j \in J_0} \mu_j G_j(T)\Big), I_{S_k} - I_{T_k}\right\rangle \leq 0$$

and $\quad \displaystyle\sum_{j \in J_\alpha}\sum_{k=1}^{n} \mu_j \eta_k(S, T)\langle D_k G_j(T), I_{S_k} - I_{T_k}\rangle \leqq 0, \quad 1 \leqq \alpha \leqq r.$

Since $\lambda > 0$, the above two inequalities together imply

$$\sum_{k=1}^{n} \eta_k(S, T)\left\langle D_k(\lambda F)(T) + \sum_{\alpha=0}^{r}(\mu_{j_\alpha} G_{j_\alpha})(T), I_{S_k} - I_{T_k}\right\rangle < 0. \tag{26}$$

Since $J_0, J_1, ..., J_r$ are partitions of M, (26) is equivalent to

$$\sum_{k=1}^{n} \eta_k(S, T)\langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k}\rangle < 0.$$

From the assumption that $\eta^T \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one $k \in \{1, 2, ..., n\}$, we obtain

$$\langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k}\rangle < 0, \tag{27}$$

which contradicts (17).

By hypothesis (GWD2) from (23) and (25) imply

$$\sum_{i=1}^{p}\sum_{k=1}^{n}\eta_k(S,T)\left\langle D_k\Big(F_i(T)+\sum_{j\in J_0}\mu_j G_j(T)\Big), I_{S_k}-I_{T_k}\right\rangle < 0$$

$$\text{and } \sum_{j\in J_\alpha}\sum_{k=1}^{n}\mu_j\eta_k(S,T)\left\langle D_k G_j(T), I_{S_k}-I_{T_k}\right\rangle \leqq 0, \quad 1\leqq\alpha\leqq r.$$

Since $\lambda\geqq 0$, and similar to proof of hypothesis (GWD1), the above two inequalities imply (27), again a contradiction to (17).

By hypothesis (GWD3) from (23) and (25) imply

$$\sum_{i=1}^{p}\sum_{k=1}^{n}\eta_k(S,T)\left\langle D_k\Big(F_i(T)+\sum_{j\in J_0}\mu_j G_j(T)\Big), I_{S_k}-I_{T_k}\right\rangle < 0$$

$$\text{and } \sum_{j\in J_\alpha}\sum_{k=1}^{n}\mu_j\eta_k(S,T)\left\langle D_k G_j(T), I_{S_k}-I_{T_k}\right\rangle < 0, \quad 1\leqq\alpha\leqq r.$$

Since $\lambda\geqq 0$, and similar to proof of hypothesis (GWD1), the above two inequalities imply (27), again a contradiction to (17). This completes the proof. $\square$

Theorem 5.2 (Strong duality)**.** *Let S^0 be a regular weakly efficient solution to* (P)*. Then there exist $\lambda^0 \in R^p$, $\sum_{i=1}^{p}\lambda_i^0 = 1$ and $\mu^0 \in R^m$, such that (S^0, λ^0, μ^0) is a feasible solution to* (GMD) *and $\mu_{J_0}G_{J_0}(S^0) = 0$, and the values of the objective functions of* (P) *and* (GMD) *are equal at S^0 and (S^0, λ^0, μ^0), respectively. Furthermore, if the conditions of weak duality Theorem 5.1 also hold, for each feasible solution (T, λ, μ) to* (GMD)*, then (S^0, λ^0, μ^0) is a weakly efficient solution to* (GMD)*.*

Proof. The proof of this theorem follows the lines of the proof of Theorem 4.2 in the light of Theorem 5.1. $\square$

6. Conclusion

The generalizations of invexity for multiobjective subset programming problem have been subject of many papers. In this paper, we introduced new classes of generalized d-type-I univex functions and established sufficient optimality conditions under various generalized d-type I univexity assumptions. Furthermore, the duality theorems in the setting of Mond-Weir and general Mond-Weir type dual are also presented. The obtained results in this paper extend and generalized the previously known results for multiobjective subset programming problem in the literature (for instance the papers [13] and [19]).

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