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APPROXIMATING COMMON FIXED POINTS FOR TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we first show the weak convergence of the modified Ishikawa iteration process with errors of two total asymptotically nonexpansive mappings, which generalizes the result due to Khan and Fukhar-ud-din [1]. Next, we show the strong convergence of the modified Ishikawa iteration process with errors of two total asymptotically nonexpansive mappings satisfying Condition ($\mathbf{A}'$), which generalizes the result due to Fukhar-ud-din and Khan [2].

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1. Introduction

Let C be a nonempty closed convex subset of a Banach space E and let T be a mapping of C into itself. Then T is said to be *asymptotically nonexpansive* [3] if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n\to\infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1}$$

for all $x, y \in C$ and $n \geq 1$. In particular, if $k_n = 1$ for all $n \geq 1$, T is said to be *nonexpansive*. T is said to be *uniformly λ-Lipschitzian* if there exists a constant $\lambda > 0$, such that

$$\|T^n x - T^n y\| \leq \lambda \|x - y\|$$

for all $x, y \in C$ and $n \geq 1$. T is said to be *asymptotically nonexpansive in the intermediate sense* [4] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{2}$$

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Note that if we define

$$\kappa_n := \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \vee 0,$$

where $a \vee b := \max\{a,b\}$, then $\kappa_n \to 0$ as $n \to \infty$, and (2) reduces to

$$\|T^n x - T^n y\| \le \|x-y\| + \kappa_n \tag{3}$$

for all $x, y \in C$ and $n \ge 1$. T is said to be *total asymptotically nonexpansive* (in brief, TAN) [5] if there exist two nonnegative real sequences $\{c_n\}$ and $\{d_n\}$ with $c_n, d_n \to 0$, and $\phi \in \Gamma(R^+)$ such that

$$\|T^n x - T^n y\| \le \|x-y\| + c_n\phi(\|x-y\|) + d_n, \tag{4}$$

for all $x, y \in C$ and $n \ge 1$, where $R^+ := [0,\infty)$ and $\phi \in \Gamma(R^+)$ if and only if ϕ is strictly increasing, continuous on R^+ and $\phi(0) = 0$. It is not hard to see that the property (4) with $c_n = 0$ for all $n \ge 1$ is equivalent to (3) with $d_n = \kappa_n$ and if we take $\phi(t) = t$ for all $t \ge 0$ and $d_n = 0$ for all $n \ge 1$ in (4), it is reduced to (1). It is not difficult to see that, if $F(T) \ne \emptyset$, then nonexpansive, asymptotically nonexpansive and asymptotically nonexpansive in the intermediate sense mappings all are the special cases of total asymptotically nonexpansive mapping. For two mappings S, T of C into itself, the following iteration scheme was introduced by Das and Debata [6]: $x_1 \in C$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S[(1-\beta_n)x_n + \beta_n T x_n] \tag{5}$$

for all $n \ge 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$. If $S = T$, then such an iteration scheme was introduced by Ishikawa [7]; see also Mann [8]. For two mappings S, T of C into itself, we consider a more general iterative scheme of the type (Kim et al.[9], cf., Xu [10]) emphasizing the randomness of errors as follows:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n x_n + \beta_n S y_n + \gamma_n u_n, \\ &y_n = \alpha'_n x_n + \beta'_n T x_n + \gamma'_n v_n, \end{aligned} \tag{6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$,$\{\gamma'_n\}$ are real sequences in [0,1] and $\{u_n\}$, $\{v_n\}$ are two sequences in C such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for all $n \ge 1$,

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$.

If $\gamma_n = \gamma'_n = 0$ for all $n \ge 1$, then (6) reduces to an iteration scheme (5). We also consider a more general iterative process of the type (Kim et al. [11]) emphasizing the randomness of errors as follows:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n x_n + \beta_n S^n y_n + \gamma_n u_n, \\ &y_n = \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n, \end{aligned} \tag{7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ are two sequences in C such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for all $n \geq 1$,

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$.

Recently, Khan and Fukhar-ud-din [1] proved the following result: Supposed that E is a uniformly convex Banach space satisfying Opial's condition and C is a nonempty bounded closed convex subset of E and $S, T : C \to C$ are nonexpansive mappings with a common fixed point. Suppose that the sequence $\{x_n\}$ defined by (6) satisfies $0 < a \leq \beta_n, \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0,1)$. Then $\{x_n\}$ converges weakly to a common fixed point of S and T. On the other hand, Fukhar-ud-din and Khan [2] proved the following result: Supposed that E is a uniformly convex Banach space and C is a nonempty closed convex subset of E and $S, T : C \to C$ are uniformly λ-Lipschitzian mappings satisfying Condition $(\mathbf{A}')$ with $\mathbf{F} = F(S) \cap F(T) \neq \emptyset$ and

$$\|S^n x - p\| \leq (1 + k_n)\|x - p\|, \qquad \|T^n x - p\| \leq (1 + k_n)\|x - p\|$$

for all $p \in \mathbf{F}$ and $n \geq 1$, where $\{k_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$. Suppose that the sequence $\{x_n\}$ defined by (7) satisfies $0 < a \leq \beta_n, \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0,1)$ and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in C. Then $\{x_n\}$ converges strongly to a common fixed point of S and T. Thus Theorem 3 of Qihou [12], Theorem 2 of Khan and Fukhar-ud-din [1], Theorem 2 of Senter and Dotson [13], Theorem 1 of Maiti and Ghosh [14], Theorem 2 of Schu [15] and Theorem 2 of Khan and Takahashi [16] are all special cases of the result due to Fukhar-ud-din and Khan [2].

In this paper, we first show that the iteration $\{x_n\}$ defined by (7) converges weakly to a common fixed point of S and T when E is a uniformly convex Banach space satisfying Opial's condition and $S, T : C \to C$ are total asymptotically nonexpansive mappings, which generalizes the result due to Khan and Fukhar-ud-din [1]. Next, we show that the iteration $\{x_n\}$ defined by (7) converges strongly to a common fixed point of S and T when E is a uniformly convex Banach space and $S, T : C \to C$ are total asymptotically nonexpansive mappings satisfying Condition $(\mathbf{A}')$, which generalizes the result due to Fukhar-ud-din and Khan [2].

2. Preliminaries

Throughout this paper we denote by E a real Banach space. Let C be a nonempty closed convex subset of E. A mapping $T : C \to E$ is said to be demiclosed at $y \in E$ if for any sequence $\{x_n\}$ in C, it follows from $x_n \rightharpoonup x$ and $Tx_n \to y$ that $x \in C$ and $T(x) = y$. $I - T$ is demiclosed at zero if for any sequence $\{x_n\}$ in C, the conditions $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ imply $x - Tx = 0$.

Recall that a Banach space E is said to be *uniformly convex* if the modulus of convexity $\delta_E = \delta_E(\epsilon)$, $0 < \epsilon \leq 2$, of E defined by

$$\delta_E(\epsilon) = \inf\Big\{1 - \frac{\|x+y\|}{2} : x, y \in E, \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\Big\}$$

satisfies the inequality $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0,2]$. If E is uniformly convex, then for each r, ϵ with $r \geq \epsilon > 0$, we have $\delta(\frac{\epsilon}{r}) > 0$ and

$$\Big\|\frac{x+y}{2}\Big\| \leq r\Big(1 - \delta(\frac{\epsilon}{r})\Big)$$

for every $x, y \in E$ with $\|x\| \leq r, \|y\| \leq r, \|x-y\| \geq \epsilon$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x. A Banach space E is said to satisfy *Opial's condition* [17] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. All Hilbert spaces and $l^p (1 < p < \infty)$ satisfy Opial's condition, while L^p with $1 < p \neq 2 < \infty$ do not. Two mappings $S, T : C \to C$ with $\mathbf{F} = F(S) \bigcap F(T) \neq \emptyset$, where C is a subset of E, are said to satisfy condition $(\mathbf{A}')$[2] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that either $\|x - Sx\| \geq f(d(x,\mathbf{F}))$ or $\|x - Tx\| \geq f(d(x,\mathbf{F}))$ for all $x \in C$, where $d(x,\mathbf{F}) = \inf_{z\in\mathbf{F}} \|x - z\|$.

3. Weak and strong convergence theorems

We first begin with the following:

Lemma 3.1 ([12]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$ and*

$$a_{n+1} \leq (1 + b_n)a_n + c_n$$

for all $n \geq 1$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 3.2 ([18]). *Let E be a uniformly convex Banach space. Let $x, y \in E$. If $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x-y\| \geq \epsilon > 0$, then $\|\lambda x + (1-\lambda)y\| \leq 1 - 2\lambda(1-\lambda)\delta(\epsilon)$ for λ with $0 \leq \lambda \leq 1$.*

Lemma 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $S, T : C \to C$ be two mappings with $F(S) \cap F(T) \neq \emptyset$ satisfying*

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n\phi(\|x - y\|) + d_n$$

and

$$\|S^n x - S^n y\| \leq \|x - y\| + c_n\phi(\|x - y\|) + d_n$$

for all $x, y \in C$ and $n \geq 1$, where $\phi \in \Gamma(R^+)$. Suppose that $\{c_n\}, \{d_n\}$ and ϕ satisfy the following two conditions:

(I) $\exists\alpha,\beta>0$ *such that* $\phi(t)\leq\alpha t$ *for all* $t\geq\beta$.

(II) $\sum_{n=1}^{\infty} c_n<\infty$, $\sum_{n=1}^{\infty} d_n<\infty$.

Suppose that the sequence $\{x_n\}$ *is defined by* (7) *and* $\{u_n\}$, $\{v_n\}$ *are two bounded sequences in* C. *Then* $\lim_{n\to\infty}\|x_n-z\|$ *exists, for any* $z\in F(S)\cap F(T)$.

Proof. For any $z\in F(S)\cap F(T)$, since $\{u_n\}$ and $\{v_n\}$ are bounded, let

$$M^0:=1\vee\phi(\beta)\vee\sup_{n\geq1}\|u_n-z\|\vee\sup_{n\geq1}\|v_n-z\|<\infty.$$

By (I) and the strict increasing of ϕ, we obtain

$$\phi(t)\leq\phi(\beta)+\alpha t,\quad t\geq0. \tag{8}$$

By using (8), we obtain

$$\begin{aligned}
&\|y_n-z\|\\
=\;&\|\alpha'_nx_n+\beta'_nT^nx_n+\gamma'_nv_n-z\|\\
\leq\;&\alpha'_n\|x_n-z\|+\beta'_n\|T^nx_n-z\|+\gamma'_n\|v_n-z\|\\
\leq\;&\alpha'_n\|x_n-z\|+\beta'_n\{\|x_n-z\|+c_n\phi(\|x_n-z\|)+d_n\}+\gamma'_nM^0\\
=\;&(1-\gamma'_n)\|x_n-z\|+\beta'_nc_n\phi(\|x_n-z\|)+\beta'_nd_n+\gamma'_nM^0\\
\leq\;&\|x_n-z\|+c_n[\phi(\beta)+\alpha\|x_n-z\|]+d_n+\gamma'_nM^0\\
\leq\;&(1+\alpha c_n)\|x_n-z\|+c_n\phi(\beta)+d_n+\gamma'_nM^0\\
\leq\;&(1+\alpha c_n)\|x_n-z\|+\lambda_nM^0,
\end{aligned}$$

where $\lambda_n=c_n+d_n+\gamma'_n$ and $\sum_{n=1}^{\infty}\lambda_n<\infty$. Since

$$\begin{aligned}
&\|S^ny_n-z\|\\
\leq\;&\|y_n-z\|+c_n\phi(\|y_n-z\|)+d_n\\
\leq\;&(1+\alpha c_n)\|x_n-z\|+\lambda_nM^0+c_n[\phi(\beta)+\alpha\|y_n-z\|]+d_n\\
\leq\;&(1+\alpha c_n)\|x_n-z\|+\alpha c_n\|y_n-z\|+(\lambda_n+c_n+d_n)M^0\\
\leq\;&(1+\alpha c_n)\|x_n-z\|+\alpha c_n[(1+\alpha c_n)\|x_n-z\|+\lambda_nM^0]+\\
&(\lambda_n+c_n+d_n)M^0\\
=\;&(1+\sigma_n)\|x_n-z\|+\nu_nM^0,
\end{aligned}$$

where $\sigma_n=2\alpha c_n+\alpha^2c_n^2$, $\nu_n=\alpha c_n\lambda_n+\lambda_n+c_n+d_n$, $\sum_{n=1}^{\infty}\sigma_n<\infty$ and $\sum_{n=1}^{\infty}\nu_n<\infty$, we obtain

$$\begin{aligned}
\|x_{n+1}-z\|\;&=\;\|\alpha_nx_n+\beta_nS^ny_n+\gamma_nu_n-z\|\\
&\leq\;\alpha_n\|x_n-z\|+\beta_n\|S^ny_n-z\|+\gamma_n\|u_n-z\|\\
&\leq\;\alpha_n\|x_n-z\|+\beta_n\{(1+\sigma_n)\|x_n-z\|+\nu_nM^0\}+\gamma_nM^0\\
&=\;(1-\gamma_n)\|x_n-z\|+\beta_n\sigma_n\|x_n-z\|+\beta_n\nu_nM^0+\gamma_nM^0
\end{aligned}$$

$$\leq \quad (1+\sigma_n)\|x_n - z\| + (\nu_n + \gamma_n)M^0.$$

By Lemma 3.1, we see that $\lim_{n\to\infty} \|x_n - z\|$ exists. □

Theorem 3.4. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $S, T : C \to C$ be uniformly continuous and two mappings with $F(S) \cap F(T) \neq \emptyset$ satisfying*

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n \phi(\|x - y\|) + d_n$$

and

$$\|S^n x - S^n y\| \leq \|x - y\| + c_n \phi(\|x - y\|) + d_n$$

for all $x, y \in C$ and $n \geq 1$, where $\phi \in \Gamma(R^+)$. Let $\{c_n\}, \{d_n\}$ and ϕ be as taken in Lemma 3.3. Suppose that for any x_1 in C, the sequence $\{x_n\}$ defined by (7) satisfies $0 < a \leq \beta_n, \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0,1)$ and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in C . Then $\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

Proof. Note that $w_n := \max\{\gamma'_n, \frac{\gamma_n}{a}\} \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} w_n < \infty$. For any $z \in F(S) \cap F(T)$, as in the proof of Lemma 3.3, $\{x_n\}$ and $\{y_n\}$ are bounded. Since $\{u_n\}$ and $\{v_n\}$ are bounded in C, let

$$M := 1 \vee \phi(\beta) \vee W < \infty,$$

where $W := \sup_{n\geq 1} \|u_n - z\| \vee \sup_{n\geq 1} \|x_n - z\| \vee \sup_{n\geq 1} \phi(\|x_n - z\|) \vee \sup_{n\geq 1} \phi(\|y_n - z\|) \vee \sup_{n\geq 1} \|v_n - z\| \vee \sup_{n\geq 1} \|x_n - u_n\| \vee \sup_{n\geq 1} \|x_n - v_n\|$. By Lemma 3.3, we see that $\lim_{n\to\infty} \|x_n - z\| (\equiv r)$ exists. Without loss of generality, we assume $r > 0$. As in the proof of Lemma 3.3, we obtain

$$\begin{aligned} \|S^n y_n - z\| &\leq (1+\sigma_n)\|x_n - z\| + \nu_n M \\ &\leq \|x_n - z\| + \sigma_n M + \nu_n M \\ &= \|x_n - z\| + \tau_n M, \end{aligned}$$

where $\tau_n = \sigma_n + \nu_n$ and $\displaystyle\sum_{n=1}^{\infty} \tau_n < \infty$. Thus

$$\begin{aligned} \|S^n y_n - z + \gamma_n(u_n - x_n)\| &\leq \|S^n y_n - z\| + \gamma_n \|u_n - x_n\| \\ &\leq \|x_n - z\| + \tau_n M + \gamma_n M \\ &= \|x_n - z\| + (\tau_n + \gamma_n)M, \end{aligned}$$

and hence

$$\begin{aligned} \|x_n - z + \gamma_n(u_n - x_n)\| &\leq \|x_n - z\| + \gamma_n \|u_n - x_n\| \\ &\leq \|x_n - z\| + \gamma_n M \\ &\leq \|x_n - z\| + (\tau_n + \gamma_n)M. \end{aligned}$$

By using Lemma 3.2, we obtain

$$\|x_{n+1} - z\|$$

$$
\begin{aligned}
&= \|\alpha_n x_n + \beta_n S^n y_n + \gamma_n u_n - z\| \\
&= \|\alpha_n (x_n - z) + \beta_n (S^n y_n - z) + \gamma_n (u_n - z)\| \\
&= \|\beta_n (S^n y_n - z) + \alpha_n (x_n - z) + \gamma_n (u_n - x_n + x_n - z) + \\
&\quad \beta_n\gamma_n (u_n - x_n) - \beta_n\gamma_n (u_n - x_n)\| \\
&= \|\beta_n (S^n y_n - z) + (1-\beta_n)(x_n - z) + \gamma_n (u_n - x_n) + \beta_n\gamma_n (u_n - x_n) - \\
&\quad \beta_n\gamma_n (u_n - x_n)\| \\
&= \|\beta_n (S^n y_n - z) + \beta_n\gamma_n (u_n - x_n) + (1-\beta_n)(x_n - z) + \\
&\quad (1-\beta_n)\gamma_n (u_n - x_n)\| \\
&= \|\beta_n (S^n y_n - z + \gamma_n (u_n - x_n)) + (1-\beta_n)(x_n - z + \gamma_n (u_n - x_n))\| \\
&\le \Big(\|x_n - z\| + \varepsilon_n M\Big)\Big[1 - 2\beta_n (1-\beta_n)\delta_E\Big(\frac{\|S^n y_n - x_n\|}{\|x_n - z\| + \varepsilon_n M}\Big)\Big],
\end{aligned}
$$

where $\varepsilon_n = \tau_n + \gamma_n$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Hence we obtain

$$
\begin{aligned}
& 2\beta_n (1-\beta_n)\Big(\|x_n - z\| + \varepsilon_n M\Big)\delta_E\Big(\frac{\|S^n y_n - x_n\|}{\|x_n - z\| + \varepsilon_n M}\Big) \\
\le\ & \|x_n - z\| - \|x_{n+1} - z\| + \varepsilon_n M.
\end{aligned}
$$

Since

$$
2a(1-b)\sum_{n=1}^{\infty}\Big(\|x_n - z\| + \varepsilon_n M\Big)\delta_E\Big(\frac{\|S^n y_n - x_n\|}{\|x_n - z\| + \varepsilon_n M}\Big) < \infty,
$$

and δ_E is strictly increasing and continuous, we obtain

$$
\lim_{n\to\infty} \|S^n y_n - x_n\| = 0. \tag{9}
$$

Since

$$
\begin{aligned}
& \|x_{n+1} - z\| \\
=\ & \|\alpha_n x_n + \beta_n S^n y_n + \gamma_n u_n - z\| \\
\le\ & \alpha_n \|x_n - z\| + \beta_n \|S^n y_n - z\| + \gamma_n \|u_n - z\| \\
\le\ & \alpha_n \|x_n - z\| + \beta_n \{\|y_n - z\| + c_n \phi(\|y_n - z\|) + d_n\} + M\gamma_n \\
=\ & \alpha_n \|x_n - z\| + \beta_n \|y_n - z\| + \beta_n c_n \phi(\|y_n - z\|) + \beta_n d_n + M\gamma_n \\
=\ & (1 - \beta_n - \gamma_n)\|x_n - z\| + \beta_n \|y_n - z\| + \beta_n c_n \phi(\|y_n - z\| + \beta_n d_n + M\gamma_n \\
\le\ & (1 - \beta_n)\|x_n - z\| + \beta_n \|y_n - z\| + \beta_n c_n M + \beta_n d_n + M\gamma_n
\end{aligned}
$$

and hence

$$
\begin{aligned}
\frac{\|x_{n+1} - z\| - \|x_n - z\|}{\beta_n} &\le \|y_n - z\| - \|x_n - z\| + c_n M + d_n + M\frac{\gamma_n}{a} \\
&\le \|y_n - z\| - \|x_n - z\| + c_n M + d_n + M w_n.
\end{aligned}
$$

So, we have

$$\begin{aligned}\|x_n - z\| - \|y_n - z\| &\leq \frac{\|x_n - z\| - \|x_{n+1} - z\|}{\beta_n} + c_n M + d_n + M w_n \qquad (10)\\ &\leq \frac{\|x_n - z\| - \|x_{n+1} - z\|}{a} + (c_n + d_n + w_n)M.\end{aligned}$$

Since

$$\begin{aligned}\|T^n x_n - z + \gamma_n'(v_n - x_n)\| &\leq \|x_n - z\| + c_n\phi(\|x_n - z\|) + d_n + \gamma_n' M\\ &\leq \|x_n - z\| + (c_n + d_n + w_n)M\end{aligned}$$

and

$$\|x_n - z + \gamma_n'(v_n - x_n)\| \leq \|x_n - z\| + (c_n + d_n + w_n)M,$$

we obtain

$$\begin{aligned}&\|y_n - z\| \qquad (11)\\ =\ & \|\alpha_n' x_n + \beta_n' T^n x_n + \gamma_n' v_n - z\|\\ =\ & \|\alpha_n'(x_n - z) + \beta_n'(T^n x_n - z) + \gamma_n'(v_n - z)\|\\ =\ & \|\beta_n'(T^n x_n - z) + (1 - \beta_n')(x_n - z) + \gamma_n'(v_n - x_n) + \beta_n'\gamma_n'(v_n - x_n)\\ & -\beta_n'\gamma_n'(v_n - x_n)\|\\ =\ & \Big\|\beta_n'(T^n x_n - z + \gamma_n'(v_n - x_n)) + (1 - \beta_n')(x_n - z + \gamma_n'(v_n - x_n))\|\\ \leq\ & (\|x_n - z\| + \zeta_n M)\Big[1 - 2\beta_n'(1 - \beta_n')\delta_E\Big(\frac{\|T^n x_n - x_n\|}{\|x_n - z\| + \zeta_n M}\Big)\Big],\end{aligned}$$

where $\zeta_n = c_n + d_n + w_n$ and $\sum_{n=1}^{\infty} \zeta_n < \infty$. By using (10) and (11), we obtain

$$\begin{aligned}& 2a(1-b)(\|x_n - z\| + \zeta_n M)\delta_E\Big(\frac{\|T^n x_n - x_n)\|}{\|x_n - z\| + \zeta_n M}\Big)\\ \leq\ & 2\beta_n'(1 - \beta_n')(\|x_n - z\| + \zeta_n M)\delta_E\Big(\frac{\|T^n x_n - x_n)\|}{\|x_n - z\| + \zeta_n M}\Big)\\ \leq\ & \|x_n - z\| - \|y_n - z\| + \zeta_n M\\ \leq\ & \frac{\|x_n - z\| - \|x_{n+1} - z\|}{a} + \zeta_n M + \zeta_n M\\ =\ & \frac{\|x_n - z\| - \|x_{n+1} - z\|}{a} + 2\zeta_n M.\end{aligned}$$

Since

$$2a(1-b)\sum_{n=1}^{\infty}(\|x_n - z\| + \zeta_n M)\delta_E\Big(\frac{\|T^n x_n - x_n\|}{\|x_n - z\| + \zeta_n M}\Big) < \infty,$$

we obtain

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0 \qquad (12)$$

similarly to the arguement above. Since

$$\begin{aligned}\|y_n - x_n\| &= \|\alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n - x_n\| \\ &\leq \beta'_n \|T^n x_n - x_n\| + \gamma'_n \|v_n - x_n\| \\ &\leq b\|T^n x_n - x_n\| + \gamma'_n M,\end{aligned}$$

and by using (12), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{13}$$

Since

$$\begin{aligned}&\|S^n x_n - x_n\| \\ \leq\ &\|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\| \\ \leq\ &\|x_n - y_n\| + c_n\phi(\|x_n - y_n\|) + d_n + \|S^n y_n - x_n\|,\end{aligned}$$

and by using (9) and (13), we obtain

$$\lim_{n\to\infty} \|S^n x_n - x_n\| = 0. \tag{14}$$

Since

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha_n x_n + \beta_n S^n y_n + \gamma_n u_n - x_n\| \\ &\leq \beta_n \|S^n y_n - x_n\| + \gamma_n \|u_n - x_n\| \\ &\leq b\|S^n y_n - x_n\| + \gamma_n M\end{aligned}$$

and by (9), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{15}$$

Since

$$\begin{aligned}&\|x_n - Tx_n\| \\ \leq\ &\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| \\ &+\|T^{n+1}x_n - Tx_n\| \\ \leq\ &\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|x_n - x_{n+1}\| + c_{n+1} \cdot \\ &\phi(\|x_n - x_{n+1}\|) + d_{n+1} + \|T^{n+1}x_n - Tx_n\| \\ =\ &2\|x_n - x_{n+1}\| + c_{n+1}\phi(\|x_n - x_{n+1}\|) + d_{n+1} + \\ &\|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_n - Tx_n\|\end{aligned}$$

and by the uniform continuity of T and (12) and (15), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

We also have $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ similarly to the arguement above. □

Lemma 3.5 ([19])**.** *Let E be a uniformly convex Banach space. Let C be a nonempty closed convex subset of E and let $T : C \to C$ be a continuous total* *asymptotically nonexpansive mapping. Then* $I - T$ *is demiclosed at zero in the sense that whenever* $x_n \rightharpoonup x$ *and*

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\|x_n - T^m x_n\| = 0$$

it follows that $x = Tx$.

Our Theorem 3.6 carries over Theorem 1 of Khan and Fukhar-ud-din [1] to total asymptotically nonexpansive mappings.

Theorem 3.6. *Let* E *be a uniformly convex Banach space satisfying Opial's condition. Let* $C, S, T, \{c_n\}, \{d_n\}$ *and* ϕ *be as taken in Theorem 3.4. Suppose that for any* x_1 *in* C*, the sequence* $\{x_n\}$ *defined by* (7) *satisfies* $0 < a \le \beta_n, \beta'_n \le b < 1$ *for all* $n \ge 1$ *and some* $a, b \in (0, 1)$ *and* $\{u_n\}$, $\{v_n\}$ *are two bounded sequences in* C *and* $F(S) \cap F(T) \ne \emptyset$. *Then* $\{x_n\}$ *converges weakly to a common fixed point of* S *and* T.

Proof. By Theorem 3.4, we obtain $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and so we have $\lim_{n\to\infty}\|x_n - T^m x_n\| = 0$ for all $m \in N$ by the uniform continuity of T. By Lemma 3.3, there exists $\lim\limits_{n\to\infty}\|x_n - z\|$ for $z \in F(S) \cap F(T)$ and thus $\{x_n\}$ is bounded. Let z_1 and z_2 be two weak subsequential limits of the sequence $\{x_n\}$. We claim that the conditions $x_{n_i} \rightharpoonup z_1$ and $x_{n_j} \rightharpoonup z_2$ imply $z_1 = z_2$. If not, by Opial's condition,

$$\begin{aligned}\lim_{n\to\infty}\|x_n - z_1\| &= \lim_{i\to\infty}\|x_{n_i} - z_1\| \\ &< \lim_{i\to\infty}\|x_{n_i} - z_2\| \\ &= \lim_{n\to\infty}\|x_n - z_2\|\end{aligned}$$

and by using similar method, we have

$$\lim_{n\to\infty}\|x_n - z_2\| < \lim_{n\to\infty}\|x_n - z_1\|.$$

This is a contradiction. Since $x_n \rightharpoonup z_1$ and $\lim_{n\to\infty}\|x_n - T^m x_n\| = 0$ for all $m \in N$ and by Lemma 3.5, $z_1 \in F(T)$. Similarly, $z_1 \in F(S)$. Hence $z_1 = z_2 \in F(S) \cap F(T)$ by the uniqueness of limits. The proof is completed. □

Our Theorem 3.7 carries over Theorem 2 of Fukhar-ud-din and Khan [2] to total asymptotically nonexpansive mappings.

Theorem 3.7. *Let* E *be a uniformly convex Banach space. Let* $C, S, T, \{c_n\}, \{d_n\}$ *and* ϕ *be as taken in Theorem 3.4. Let* $S, T : C \to C$ *be two mappings satisfying Condition* $(\mathbf{A}')$ *with* $\mathbf{F} = F(S) \cap F(T) \ne \emptyset$. *Suppose that for any* x_1 *in* C*, the sequence* $\{x_n\}$ *defined by* (7) *satisfies* $0 < a \le \beta_n, \beta'_n \le b < 1$ *for all* $n \ge 1$ *and some* $a, b \in (0, 1)$ *and* $\{u_n\}$, $\{v_n\}$ *are two bounded sequences in* C. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* S *and* T.

Proof. As in the proof of Lemma 3.3, we obtain

$$\|x_{n+1} - z\| \le \|x_n - z\| + h_n. \tag{16}$$

where $h_n = (\sigma_n + \nu_n + \gamma_n)M$ and $\sum_{n=1}^{\infty} h_n < \infty$. Thus

$$\inf_{z\in\mathbf{F}} \|x_{n+1} - z\| \leq \inf_{z\in\mathbf{F}} \|x_n - z\| + h_n.$$

By using Lemma 3.1, we see that $\lim_{n\to\infty} d(x_n, \mathbf{F})(\equiv k)$ exists. We first claim that $\lim_{n\to\infty} d(x_n, \mathbf{F}) = 0$. In fact, assume that $k = \lim_{n\to\infty} d(x_n, \mathbf{F}) > 0$. Then we can choose $n_0 \in N$ such that $0 < \frac{k}{2} < d(x_n, \mathbf{F})$ for all $n \geq n_0$. By using Condition $(\mathbf{A}')$ and Theorem 3.4, we obtain

$$0 < f(\frac{k}{2}) \leq f(d(x_n, \mathbf{F})) \leq \|x_n - Sx_n\| \to 0$$

or

$$0 < f(\frac{k}{2}) \leq f(d(x_n, \mathbf{F})) \leq \|x_n - Tx_n\| \to 0$$

as $n \to \infty$. This is a contradiction. So, we obtain $k = 0$. Next, we claim that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} d(x_n, \mathbf{F}) = 0$ and $\sum_{n=1}^{\infty} h_n < \infty$, there exists $n_0 \in N$ such that for all $n \geq n_0$, we obtain

$$d(x_n, \mathbf{F}) < \frac{\epsilon}{4} \quad \text{and} \quad \sum_{i=n_0}^{\infty} h_i < \frac{\epsilon}{4}. \tag{17}$$

Let $n, m \geq n_0$ and $p \in \mathbf{F}$. Then, by using (16), we obtain

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\ &\leq \|x_{n_0} - p\| + \sum_{i=n_0}^{n-1} h_i + \|x_{n_0} - p\| + \sum_{i=n_0}^{m-1} h_i \\ &\leq 2[\|x_{n_0} - p\| + \sum_{i=n_0}^{\infty} h_i]. \end{aligned}$$

Taking the infimum over all $p \in \mathbf{F}$ on both sides and by using (17), we obtain

$$\begin{aligned} \|x_n - x_m\| &\leq 2[d(x_{n_0}, \mathbf{F}) + \sum_{i=n_0}^{\infty} h_i] \\ &< 2(\frac{\epsilon}{4} + \frac{\epsilon}{4}) = \epsilon \end{aligned}$$

for all $n, m \geq n_0$. This implies that $\{x_n\}$ is a Cauchy sequence. Let $\lim_{n\to\infty} x_n = q$. Then $d(q, \mathbf{F}) = 0$. Since $\mathbf{F}$ is closed, we obtain $q \in \mathbf{F}$. Hence $\{x_n\}$ converges strongly to a common fixed point of S and T. □

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