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AN EFFICIENT ALGORITHM FOR FINDING OPTIMAL CAR-DRIVING STRATEGY

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Abstract. In this paper, the problem of determining the optimal car-deriving strategy has been examined. In order to find the optimal driving strategy, we have modified a method based on measure theory. Further, we demonstrate that the modified method is an efficient and practical algorithm for dealing with optimal control problems in a canonical formulation.

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1. Introduction

Optimal control problems subject to canonical constraints from the applied viewpoint are an important class because most of the real-world and practical problems can be cast in the canonical form. Examples may be found in [9, 10, 16, 15] and the references cited therein. Since finding analytical solutions for these complex problems is not possible, implementing numerical methods are unavoidable. There are plenty of computational methods proposed for solving optimal control problems. In [15] the control parametrization technique and in [5, 9, 10, 15] the control parametrization enhancing technique are used to solve optimal control problems constrained by canonical constraints.
Recently, an efficient computational method based on measure theory is developed for solving optimal control problems [1, 6], time optimal control problems [13, 6] and optimal shape design problems [2, 3, 4]. Although this approach has been successfully used for solving many type of optimal control problems, we note that no results exist on the optimal control problems constrained by canonical formulation.

Most of the existing methods, for instance, the gradient-based algorithms [11], finite difference methods [8] and finite elements methods [17] bear an important

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drawback which is that the results depend heavily on the initial partition and the differentiability of the objective functional.

The advantages of the modified measure theoretical (MMT) approach lie in the fact that: 1. It allows us to carry out optimization in any coarse and nonlinear model space. 2. The algorithm is not iterative and no initial guess is required. 3. It is computationally efficient and flexible enough to accommodate general control optimization problems. 4. The algorithm works without the need for the differentiability of the objective function and is able to generate global minima which is numerically close to what one could reasonably call the global infimum of the control optimization problem.

This paper is organized as follows. Firstly, we describe the canonical optimal control problem and summarize its requirements. Then, MMT approch is presented in detail. Finally, the optimal car-driving strategy is computed using MMT approach.

2. The canonical optimal control problem

Consider a process described by the system of nonlinear differential equations as follows:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}), \quad \forall t \in [0, T], \tag{1}$$

with the initial and final conditions given by

$$\mathbf{x}(0) = \mathbf{x}^0, \tag{2}$$

$$\mathbf{x}(T) = \mathbf{x}^T, \tag{3}$$

where $\mathbf{x}(t) = [x_1(t), ..., x_n(t)]^\top \in R^n$, $\mathbf{u}(t) = [u_1(t), ..., u_m(t)]^\top \in R^m$ are the state and control vectors, respectively, and $\mathbf{f}(t, \mathbf{x}, \mathbf{u}) = [f_1(t, \mathbf{x}, \mathbf{u}), ..., f_n(t, \mathbf{x}, \mathbf{u})]^\top \in R^n$ is continuously differentiable with respect to its respective arguments. Vectors $\mathbf{x}^0 = [x_1^0, ..., x_n^0]^\top \in R^n$ and $\mathbf{x}^T = [x_1^T, ..., x_n^T]^\top \in R^n$ are given constants. Let

$$A_{lu} = \{\mathbf{y}(.) = [y_1(.), ..., y_n(.)]^\top \in R^n;\ {y_i}^l \leq y_i(.) \leq {y_i}^u,\ i = 1, ..., n\}, \tag{4}$$

$$U_{lu} = \{\mathbf{v}(.) = [v_1(.), ..., v_m(.)]^\top \in R^m;\ {v_j}^l \leq v_j(.) \leq {v_j}^u,\ j = 1, ..., m\}, \tag{5}$$

where lower and upper bounds are given real numbers. It is obvious U_{lu} is a compact subset of R^m. A Boral measurable function $\mathbf{u} : [0, T] \to R^m$ is called an *admissible control* if $\mathbf{u} \in U_{lu}$. Let U denote the class of all such admissible controls. For each admissible control $\mathbf{u} \in U_{lu}$, let $\mathbf{x}(., \mathbf{u})$ denote the corresponding solution of the system (1) and satisfy the initial and final conditions (2)-(3). Such this state vector is referred to as an *admissible solution* of system (1) and (2)-(3) corresponding to $\mathbf{u} \in U_{lu}$, if $\mathbf{x} \in A_{lu}$. Let A denote the class of all such admissible states.

The canonical optimal control problem is now formulated as the following. (See [16])

<u>**Problem C:**</u> Subject to the dynamical system (1) and (2)-(3), find a control $\mathbf{u} \in \mathsf{U}$ such that the cost functional

$$G_0(\mathbf{u}) = \Phi_0(\mathbf{x}(T|\mathbf{u})) + \int_0^T \mathsf{L}_0(t, \mathbf{x}(t|\mathbf{u}(t)), \mathbf{u}(t))\ dt, \tag{6}$$

is minimized over U and subject to

$$G_i(\mathbf{u}) = \Phi_i(\mathbf{x}(T|\mathbf{u})) + \int_0^T \mathsf{L}_i(t, \mathbf{x}(t|\mathbf{u}(t)), \mathbf{u}(t))\ dt = 0,\ \ i = 1, ..., N_c, \tag{7}$$

$$G_i(\mathbf{u}) = \Phi_i(\mathbf{x}(T|\mathbf{u})) + \int_0^T \mathsf{L}_i(t, \mathbf{x}(t|\mathbf{u}(t)), \mathbf{u}(t))\ dt \leq 0,\ \ i = N_c + 1, ..., N, \tag{8}$$

where Φ_i and L_i for $i = 0, 1, ..., N$, are given real-valued functions.

Remark 1. Here, we assume that $\mathbf{f}$, L_i, $i = 0, 1, ..., N$, and their partial derivatives are piecewise continuous on $[0, T]$ for each $(\mathbf{x}, \mathbf{u}) \in R^n \times R^m$ and continuous on $R^n \times R^m$ for each $t \in [0, T]$, and Φ_i, $i = 0, 1, ..., N$, is continuously differentiable with respect to $\mathbf{x}$.

3. Modified measure theoretical (MMT) approach

In this section, we present only the principle steps of MMT approach and further remarks in connection with MMT algorithm are presented in [1, 2, 3, 12, 13].

Firstly, without lose of the generality, we suppose that

$$\Phi_i = 0, \qquad i = 0, 1, ..., N.$$

A pair $\mathsf{p} = [\mathbf{x}, \mathbf{u}]$ is said to be an *admissible pair* if $\mathbf{x} \in \mathsf{A}$ and $\mathbf{u} \in \mathsf{U}$. Let P_{ad} denote the class of all such admissible pairs.
Since measure theoretical approach deals with integral equations, we should change Problem C to a problem with integral equations. For this purpose, let B be an open ball in R^{n+1} containing $\mathsf{J} \times \mathsf{A}$ where $\mathsf{J} = [0, T]$ and let $\mathsf{C}^1(\mathsf{B})$ be the space of all real-valued continuously differentiable functions on B. Suppose that the function $\varphi^{\mathbf{f}}$ is defined as follows:

$$\varphi^{\mathbf{f}}(t, \mathbf{x}(t), \mathbf{u}(t)) = \varphi_{\mathbf{x}}(t, \mathbf{x}(t))\mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) + \varphi_t(t, \mathbf{x}(t)), \quad \forall \varphi \in \mathsf{C}^1(\mathsf{B}), \tag{9}$$

for all $\mathsf{p} = [\mathbf{x}, \mathbf{u}] \in \mathsf{P}_{ad}$ and $t \in \mathsf{J}$. Here, φ_* denotes $\frac{\partial \varphi}{\partial *}$.
Since p is an admissible pair, it implies that

$$\begin{aligned}
\int_0^T \varphi^{\mathbf{f}}(t, \mathbf{x}(t), \mathbf{u}(t))\ dt &= \int_0^T \big(\varphi_{\mathbf{x}}(t, \mathbf{x}(t))\mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) + \varphi_t(t, \mathbf{x}(t))\big)\ dt \\
&= \int_0^T \frac{d}{dt}\varphi(t, \mathbf{x}(t))\ dt \\
&= \varphi(T, \mathbf{x}(T)) - \varphi(0, \mathbf{x}(0)) \\
&= \triangle\varphi. \qquad (10)
\end{aligned}$$

One can easily verify that the integral equation (10) for any $(t, \mathbf{x}(t), \mathbf{u}(t)) \in \Omega$ is equivalent to the differential equation (1).

We know that a larger optimization problem with more constraints has a better optimal solution if there exists. Hence, we attempt to provide more constraints for the optimization problem.
Let $\mathsf{D}(\mathsf{J}^\circ)$ be the space of all infinitely differentiable real-valued functions with compact support in an open interval J°.
In view of dynamical system (1) and $\mathsf{p} \in \mathsf{P}_{ad}$, the extra functions ϕ_i, $i = 1, ..., n$, are constructed such that:

$$\phi_i^\psi(t, \mathbf{x}(t), \mathbf{u}(t)) = x_i(t)\frac{d\psi(t)}{dt} + f_i(t, \mathbf{x}(t), \mathbf{u}(t))\psi(t), \quad \forall \psi \in \mathsf{D}(\mathsf{J}^\circ). \tag{11}$$

Following the integrating of the above relation, we have

$$\begin{aligned} \int_0^T \phi_i^\psi(t, \mathbf{x}(t), \mathbf{u}(t))\ dt &= \int_0^T x_i(t)\frac{d\psi(t)}{dt}\ dt + \int_0^T f_i(t, \mathbf{x}(t), \mathbf{u}(t))\psi(t)\ dt \\ &= x_i(t)\psi(t)\big|_0^T - \int_0^T (\frac{dx_i(t)}{dt} - f_i(t, \mathbf{x}(t), \mathbf{u}(t)))\psi(t)\ dt. \end{aligned} \tag{12}$$

The fact that $\psi \in \mathsf{D}(\mathsf{J}^\circ)$ has a compact support in J° gives rise to $\psi(0) = \psi(T) = 0$. As for the dynamical system (1), the right-hand side of (12) becomes zero. Thus, the following constraint can be added to Problem C

$$\int_0^T \Phi_\psi(t, \mathbf{x}(t), \mathbf{u}(t))\ dt = \mathbf{0}, \tag{13}$$

where $\Phi_\psi = [\phi_1^\psi, ..., \phi_n^\psi]^\top$.
By choosing the functions which are dependent only on time $t \in \mathsf{J}$, one gets

$$\int_0^T h(t, \mathbf{x}(t), \mathbf{u}(t))\ dt = a_h, \quad \forall h \in \mathsf{C}_1(\Omega), \tag{14}$$

where $\mathsf{C}_1(\Omega)$ as a subspace of $\mathsf{C}(\Omega)$, contains all continuous functions on Ω depending only on $t \in \mathsf{J}$, and a_h is the integral of h on $[0, T]$.

From the above assumptions, Problem C is equivalent to the following problem with integral equations.

Problem CI: Find an admissible pair $\mathsf{p} \in \mathsf{P}_{ad}$ such that the cost functional

$$G_0(\mathbf{u}) = \int_0^T \mathsf{L}_0(t, \mathbf{x}(t), \mathbf{u}(t))\ dt, \tag{15}$$

is minimized over P_{ad} and subject to

$$G_i(\mathbf{u}) = \int_0^T \mathsf{L}_i(t, \mathbf{x}(t), \mathbf{u}(t))\ dt \ = \ 0, \quad i = 1, ..., N_c, \tag{16}$$

$$G_i(\mathbf{u}) = \int_0^T \mathsf{L}_i(t, \mathbf{x}(t), \mathbf{u}(t))\ dt \ \leq \ 0, \quad i = N_c + 1, ..., N, \tag{17}$$

$$\int_0^T \varphi^{\mathbf{f}}(t, \mathbf{x}(t), \mathbf{u}(t))\ dt \ = \ \triangle\varphi, \quad \forall \varphi \in \mathsf{C}^1(\mathsf{B}), \tag{18}$$

$$\int_0^T \Phi_\psi(t,\mathbf{x}(t),\mathbf{u}(t))\ dt = \mathbf{0}, \quad \forall \psi \in \mathsf{D}(\mathsf{J}^\circ), \tag{19}$$

$$\int_0^T h(t,\mathbf{x}(t),\mathbf{u}(t))\ dt = a_h, \quad \forall h \in \mathsf{C}_1(\Omega). \tag{20}$$

The key to MMT approach lies in establishing the integral form of constraints and cost functional. Note that the requirement for using MMT approach has been justified so far.
Let $\mathsf{p} \in \mathsf{P}_{ad}$ and define the functional

$$\Lambda_{\mathsf{p}} : F \to \int_{\mathsf{J}} F(t,\mathbf{x}(t),\mathbf{u}(t))\ dt, \quad \forall F \in \mathsf{C}(\Omega), \tag{21}$$

which is a positive linear functional on $\mathsf{C}(\Omega)$, the space of all bounded continuous functions on Ω.

Based on the above positive linear functional, the next proposition shows that Problem C can be considered on the dual space of $\mathsf{C}(\Omega)$, denoted by $\mathsf{C}^*(\Omega)$, instead of P_{ad}.

Proposition 1. *The mapping $\mathsf{p} \mapsto \Lambda_{\mathsf{p}}$ from P_{ad} into $\mathsf{C}^*(\Omega)$ is an injection.*

Proof. Let Ω_1 be a subset of Ω on which the function $F(t,\mathbf{x}(t),\mathbf{u}(t))$ in (21) can be constructed independent of t and $\mathbf{x}(t)$. Consequently, for two distinct admissible pairs $\mathsf{p}_1 \neq \mathsf{p}_2$, we get $\Lambda_{\mathsf{p}_1} \neq \Lambda_{\mathsf{p}_2}$. ☐

With respect to the definition of Λ_{p} (see (21)), the functional representation of Problem CI is expressed as:

Problem CIF: Find a functional $\Lambda_{\mathsf{p}} \in \mathsf{C}^*(\Omega)$ corresponding to admissible pair p such that the cost functional

$$\Lambda_{\mathsf{p}}(\mathsf{L}_0), \tag{22}$$

is minimized over $\mathsf{C}^*(\Omega)$ and subject to

$$\Lambda_{\mathsf{p}}(\mathsf{L}_i) = 0, \quad i = 1,...,N_c, \tag{23}$$

$$\Lambda_{\mathsf{p}}(\mathsf{L}_i) \leq 0, \quad i = N_c+1,...,N, \tag{24}$$

$$\Lambda_{\mathsf{p}}(\varphi^{\mathbf{f}}) = \triangle\varphi, \quad \forall \varphi \in \mathsf{C}^1(\mathsf{B}), \tag{25}$$

$$\Lambda_{\mathsf{p}}(\Phi_\psi) = \mathbf{0}, \quad \forall \psi \in \mathsf{D}(\mathsf{J}^\circ), \tag{26}$$

$$\Lambda_{\mathsf{p}}(h) = a_h, \quad \forall h \in \mathsf{C}_1(\Omega). \tag{27}$$

Linear functional Λ_{p} can be uniquely defined in term of a positive Radon measure such that

$$\Lambda_{\mathsf{p}}(F) = \int_{\mathsf{J}} F\ dt = \int_\Omega F\ d\mu \equiv \mu(F), \quad \forall F \in \mathsf{C}(\Omega). \tag{28}$$

This result is a direct corollary of Riesz' representation theorem.(See Theorem 2.14 of [14])

In conjunction with the positive Radon measure given by (28), Problem CIF is stated in the sense of measure space as the following.

Problem CIM: Find a measure $\mu \in \mathsf{M}^+(\Omega)$ such that the cost functional

$$\mu(\mathsf{L}_0), \tag{29}$$

is minimized over $\mathsf{M}^+(\Omega)$ and subject to

$$\begin{aligned} \mu(\mathsf{L}_i) &= 0, \quad i = 1, ..., N_c, & (30)\\ \mu(\mathsf{L}_i) &\leq 0, \quad i = N_c + 1, ..., N, & (31)\\ \mu(\varphi^{\mathbf{f}}) &= \triangle\varphi, \quad \forall\varphi \in \mathsf{C}^1(\mathsf{B}), & (32)\\ \mu(\Phi_\psi) &= \mathbf{0}, \quad \forall\psi \in \mathsf{D}(\mathsf{J}^\circ), & (33)\\ \mu(h) &= a_h, \quad \forall h \in \mathsf{C}_1(\Omega), & (34) \end{aligned}$$

where $\mathsf{M}^+(\Omega)$ is referred to as the space of all positive Radon measures on Ω.

Let Q be a subset of $\mathsf{M}^+(\Omega)$ whose elements satisfy (30)-(34). If $\mathsf{M}^+(\Omega)$ is equipped by weak*-topology, then by the use of Alaoghlu theorem one can prove that Q is a compact set. In the sense of this topology, the functional $\mathsf{I} : \mathsf{Q} \to R$ defined by $\mathsf{I}(\mu) = \mu(\mathsf{L}_0)$ is a linear continuous functional on the compact set Q. In fact, the functional I has at least a minimum on Q. This fact is summarized in the following proposition.

Proposition 2. *There exists an optimal measure $\mu^* \in \mathsf{Q}$ that minimizes the functional $\mathsf{I}(\mu) = \mu(\mathsf{L}_0)$.*

The main advantage of the above measure representation of Problem C is that the objective functional and the constraint functions in Problem CIM are linear in measure μ, even though the classical problem is nonlinear. Therefore, the whole machinery of linear analysis can be utilize to attack the problem.

It should be mentioned that only the appearance of the control optimization problem has been changed so far and nothing else.
Problem CIM is an infinite-dimensional linear programming(LP) problem because the underlying spaces $\mathsf{C}^1(\mathsf{B})$, $\mathsf{D}(\mathsf{J}^\circ)$ and $\mathsf{C}_1(\Omega)$ have not finite dimension. Hereafter, the aim is to approximate Problem CIM by a finite-dimensional LP problem whose optimal solution converges to minimizer of Problem CIM. For this purpose, a two-phase scheme of approximation is outlined. The first phase is completed when an approximation of Q is obtained.
Let $\{\varphi_k,\ k = 1, 2, ...\}$, $\{\psi_j,\ j = 1, 2, ...\}$ and $\{h_s,\ s = 1, 2, ...\}$ be sets of total functions, that is, their linear combinations are dense in $\mathsf{C}^1(\mathsf{B})$, $\mathsf{D}(\mathsf{J}^\circ)$ and $\mathsf{C}_1(\Omega)$, respectively.
If $\mathsf{Q}(M_1, M_2, L)$ denotes the subset of $\mathsf{M}^+(\Omega)$ containing of all measures which satisfy the following finite number of constraints:

$$\begin{aligned} \mu(\mathsf{L}_i) &= 0, \quad i = 1, ..., N_c, & (35)\\ \mu(\mathsf{L}_i) &\leq 0, \quad i = N_c + 1, ..., N, & (36)\\ \mu(\varphi_k^{\mathbf{f}}) &= \triangle\varphi_k, \quad k = 1, 2, ..., M_1, & (37) \end{aligned}$$

$$\mu(\Phi_{\psi_j}) = \mathbf{0}, \quad j = 1, 2, ..., M_2, \tag{38}$$
$$\mu(h_s) = a_{h_s}, \quad s = 1, 2, ..., L, \tag{39}$$

then, the convergence result is established in the next proposition.

Proposition 3. *If M_1, M_2 and L tend to infinity, then, the sequence of suboptimal solutions of* Problem CIM

$$\{\eta_{(M_1,M_2,L)} = \inf_{\mathsf{Q}(M_1,M_2,L)} \mu(\mathsf{L}_0)\}$$

converges to $\eta = \inf_{\mathsf{Q}} \mu(\mathsf{L}_0)$.

Proof. The proof is much like that of Prop. III.1 in [12], and is therefore omitted. □

Note that the latter proposition guarantees theoretically the existence of optimal solution of Problem CIM for sufficiently large M_1, M_2 and L.
In the second phase of approximation, the aim is to characterize optimal measure, say μ^*, in the space $\mathsf{Q}(M_1, M_2, L)$ at which $\mathsf{I}(\mu) = \mu(\mathsf{L}_0)$ takes minimum value. By Theorem A.5 of [12], measure $\mu^* \in \mathsf{Q}(M_1, M_2, L)$ the minimizer of $\mathsf{I}(\mu) = \mu(\mathsf{L}_0)$ has the form

$$\mu^* = \sum_{r=1}^{M_1+M_2+L} \alpha_r^* \delta_{(z_r^*)}, \tag{40}$$

where $z_r^* \in \Omega$ and α_r^* for $r = 1, 2, ..., M_1 + M_2 + L$, are non-negative coefficients. In the above formula $\delta_{(z_r^*)}$ is a unitary atomic measure defined by

$$\delta_{(z)}(F) = F(z), \quad \forall F \in \mathsf{C}(\Omega). \tag{41}$$

Remark 2. Consider the finite-dimensional optimization problem with objective functional $\mathsf{I}(\mu) = \mu(\mathsf{L}_0)$ and constraints (35)-(39). If μ in the latter optimization problem is substituted by μ^* defined by (40), then, the recent optimization problem is a non-linear programming problem because there exist unknown coefficients α_r^* and unknown supports z_r^* for $r = 1, 2, ..., M_1 + M_2 + L$.

It would be convenient if we could obtain a linear programming problem in which the unknowns are only the coefficients α_r^*. Hence, we take the fixed and determined points z_r, which are the approximation of z_r^*, from a countable and dense subset of Ω.

Proposition 4. *Let ω be a countable and dense subset of Ω. Given $\epsilon > 0$, a measure $\widehat{\mu} \in \mathsf{M}^+(\Omega)$ can be found such that*

$$|(\mu^* - \widehat{\mu})(\zeta_l)| < \epsilon, \quad l = 0, 1, ..., N + 1 + M_1 + M_2 + L, \tag{42}$$

and $\widehat{\mu}$ has the form

$$\widehat{\mu} = \sum_{r=1}^{M_1+M_2+L} \alpha_r^* \delta_{(z_r)}, \tag{43}$$

where coefficients α_r^* *are the same as in* (40), $z_r \in \omega$ *and* $\{\zeta_l,\ l = 0, ..., 1 + N + M_1 + M_2 + L\}$ *are* $\{\mathsf{L}_i,\ i = 0, ..., N\}$, $\{\varphi_k^{\mathbf{f}},\ k = N+1, ..., N+1+M_1\}$, $\{\Phi_{\psi_j},\ j = N+1+M_1, ..., N+1+M_1+M_2\}$ *and* $\{h_s,\ s = N+1+M_1+M_2, ..., N+1+M_1+M_2+L\}$.

Proof. Let ω be a countable dense subset of Ω. Given $\epsilon > 0$, the points $z_r \in \omega,\ r = 1, ..., M_1 + M_2 + L$ can be taken so that (42) holds. For $l = 0, 1, ..., N + 1 + M_1 + M_2 + L$, one gets

$$\begin{aligned} |(\mu^* - \widehat{\mu})(\zeta_l)| &= |\sum_{r=1}^{M_1+M_2+L} \alpha_r^* (\ \zeta_l(z_r^*) - \zeta_l(z_r))| \\ &\le \max_{l,r} |\ \zeta_l(z_r^*) - \zeta_l(z_r)| \sum_{r=1}^{M_1+M_2+L} \alpha_r^*. \end{aligned}$$

Now from (40), we have

$$\sum_{r=1}^{M_1+M_2+L} \alpha_r^* = \int_\Omega 1 d\mu^* < \infty.$$

By the continuity of finite numbers of functions ζ_l and choosing z_r sufficiently close to z_r^*, the $\max_{l,r}$ can be made less than $\frac{\epsilon}{|\int_\Omega 1 d\mu^*|}$. Hence, the inequalities (42) follow. □

Based on the above concepts, the finite-dimensional LP problem may be stated as follows:

<u>Problem CILP:</u> Minimize

$$\sum_{r=1}^{\overline{N}} \alpha_r^* \mathsf{L}_0(z_r), \tag{44}$$

subject to

$$\sum_{r=1}^{\overline{N}} \alpha_r^* \mathsf{L}_i(z_r) = 0, \quad i = 1, ..., N_c, \tag{45}$$

$$\sum_{r=1}^{\overline{N}} \alpha_r^* \mathsf{L}_i(z_r) \le 0, \quad i = N_c + 1, ..., N, \tag{46}$$

$$\sum_{r=1}^{\overline{N}} \alpha_r^* \varphi_k^{\mathbf{f}}(z_r) = \triangle\varphi, \quad k = 1, 2, ..., M_1, \tag{47}$$

$$\sum_{r=1}^{\overline{N}} \alpha_r^* \Phi_{\psi_j}(z_r) = \mathbf{0}, \quad j = 1, 2, ..., M_2, \tag{48}$$

$$\sum_{r=1}^{\overline{N}} \alpha_r^* h_s(z_r) = a_h, \quad s = 1, 2, ..., L, \tag{49}$$

$$\alpha_r^* \geq 0, \quad r = 1, 2, ..., \overline{N}. \tag{50}$$

Suppose that $\{\alpha_r^*\}_{r=1}^{\overline{N}}$ is the optimal solution of problem CILP. In the sequel, we explain the procedure of constructing a piecewise constant control function. To begin with, let $\alpha_{i_1}^*, ..., \alpha_{i_q}^*$ be nonzero optimal values sorted by indices. Obviously, q must not be greater than the number of constraints $N + M_1 + M_2 + L$. Optimal control u^* as a piecewise constant function is then constructed such that $u^* = u_{i_1}$ on $[0, \alpha_{i_1}^*)$, $u^* = u_{i_2}$ on $[\alpha_{i_1}^*, \alpha_{i_1}^* + \alpha_{i_2}^*)$ and in a similar manner $u^* = u_{i_q}$ on $[\ \Sigma_{r=1}^{q-1} \alpha_{i_r}^*, \Sigma_{r=1}^{q} \alpha_{i_r}^*)$ where each constant u_{i_r} is the corresponding value to Ω_{i_r} in dividing Ω, for $r = 1, ..., q$.
The response of system (1)-(3) is determined according to the latter optimal control u^*.

Remark 3. The same conclusions are also valid for the general case of Problem C where $\Phi_i(.)$, $i = 1, ..., N$, are non-zeros. In general case, as shown in (7) and (8)

$$G_i(\mathbf{u}) = \Phi_i(\mathbf{x}(T|\mathbf{u})) + \int_0^T \mathsf{L}_i(t, \mathbf{x}(t|\mathbf{u}(t)), \mathbf{u}(t))\ dt, \quad i = 1, ..., N.$$

It is easily shown that

$$G_i(\mathbf{u}) = \int_0^T (\dot{\Phi}_i(\mathbf{x}(t|\mathbf{u})) + \mathsf{L}_i(t, \mathbf{x}(t|\mathbf{u}(t)), \mathbf{u}(t)))\ dt + \Phi_i(\mathbf{x_0}), \quad i = 1, ..., N,$$

where $\dot{\Phi}_i = \frac{d\Phi_i}{dt}$ and $\Phi_i(\mathbf{x_0})$, $i = 1, ..., N$, are given real-valued functions.

By virtue of Remark 3, the only differences between Problem CILP and that which is obtained with respect to the general case, are the two following constraints

$$\sum_{r=1}^{\overline{N}} \alpha_r^* (\mathsf{L}_i(z_r) + \dot{\Phi}_i(z_r)) \quad = \quad -\Phi_i(\mathbf{x_0}), \quad i = 1, ..., N_c, \tag{51}$$

$$\sum_{r=1}^{\overline{N}} \alpha_r^* (\mathsf{L}_i(z_r) + \dot{\Phi}_i(z_r)) \quad \leq \quad -\Phi_i(\mathbf{x_0}), \quad i = N_c + 1, ..., N. \tag{52}$$

4. The optimal car-deriving strategy

In this section, we consider the following variational problem which is referred to it as a driving strategy(DS) problem (see [7]).

Problem DS: Minimize

$$\Gamma(x) = \int_0^1 \sqrt{1 + \dot{x}^2(t)}\ dt, \tag{53}$$

subject to

$$x^L(t) \leq x(t) \leq x^U(t), \quad \forall t \in [0,1], \tag{54}$$

$$x(0) = x_0, \; x(1) = x_1, \tag{55}$$

where $x \in \mathsf{C}^1([0,1])$, the set of all continuously differentiable functions on $[0,1]$, and $\dot{x}$ denotes $\frac{dx}{dt}$.
For any given constants x_0, x_1 and defining $u(t) = \dot{x}(t)$ for all $t \in [0,1]$, Problem DS can be regarded as an optimal control problem which describes a journey begins from the origin $(0, x_0)$ and terminates to the destination $(1, x_1)$ while the trajectory $x(t)$ is restricted by the two boundaries $(t, x^L(t))$ and $(t, x^U(t))$. Note that the trajectory $x(t)$ conserves smoothness property when we take $|u(t)| < \epsilon$, for all $t \in [0,1]$ and given small constant $\epsilon > 0$.

Furthermore, in the presence of some traffic circles(TC) through the path, the goal is to construct the trajectory $x(t)$ approaches any traffic circle as close as possible. Let us consider an allowable region around a traffic circle characterized by

$$R_{in}^2 \leq (x(t) - O(t_{TC}))^2 + (t - t_{TC})^2 \leq R_{out}^2, \quad \forall t \in [0,1]. \tag{56}$$

Here, R_{in}^2, R_{out}^2 and $(t_{TC}, O(t_{TC}))^\top$ are given constants and the coordinate of center of region, respectively.

Now we are in a position to transform DS problem into the form of Problem C. Consider the inequality constraint

$$H(t, x(t)) \geq 0, \quad \forall t \in [0,1]. \tag{57}$$

The above inequality is equivalent to

$$\int_0^1 \min\{H(t, x(t)), 0\} \; dt = 0. \tag{58}$$

Obviously, equality constraint (58) is nonsmooth. However, the nonsmooth function $g = \min\{H, 0\}$ can be replaced by the following smooth function (see [16])

$$g_\varepsilon(H) = \begin{cases} H & \text{if } H < -\varepsilon, \\ -\frac{(H-\varepsilon)^2}{4\varepsilon} & \text{if } -\varepsilon \leq H \leq \varepsilon, \\ 0 & \text{if } H > \varepsilon, \end{cases} \tag{59}$$

where g_ε is obtained by smoothing out the sharp corner of g.

Let G_0 be the original objective functional. If $G_0^{\varepsilon,\gamma}$ denotes a new penalty objective functional which is defined as

$$G_0^{\varepsilon,\gamma} = G_0 - \gamma \int_0^1 g_\varepsilon(H(t, x(t))) \; dt. \tag{60}$$

Then for each $\varepsilon > 0$, the penalty parameter γ can be made large enough such that the solution of the new problem is feasible point of the original problem. For more details the interested reader is refereed to [16].

From the above discussion, we can add the inequalities (54) and (56) to the objective functional as the penalty terms. Hence, the new penalty objective functional is expressed as

$$\int_0^1 \sqrt{1+u^2(t)}\ dt - \sum_{i=1}^{4} \gamma_i \int_0^1 g_{\varepsilon_i}(H_i(t,x(t)))\ dt, \tag{61}$$

where for all $t \in [0,1]$

$$\begin{aligned} H_1(t,x(t)) &= x^U(t) - x(t) \geq 0, \\ H_2(t,x(t)) &= x(t) - x^L(t) \geq 0, \\ H_3(t,x(t)) &= R_{out}^2 - (x(t) - O(t_{TC}))^2 - (t - t_{TC})^2 \geq 0, \\ H_4(t,x(t)) &= (x(t) - O(t_{TC}))^2 + (t - t_{TC})^2 - R_{in}^2 \geq 0. \end{aligned}$$

Now, we may state the canonical form of the optimal control problem DS as follows.
Problem DSC: Minimize

$$\int_0^1 \sqrt{1+u^2(t)}\ dt - \sum_{i=1}^{4} \gamma_i \int_0^1 g_{\varepsilon_i}(H_i(t,x(t)))\ dt, \tag{62}$$

subject to

$$\dot{x}(t) = u(t), \quad \forall t \in [0,1], \tag{63}$$

$$x(0) = x_0,\ x(1) = x_1, \tag{64}$$

$$-\epsilon < u(t) < \epsilon, \quad \forall t \in [0,1], \tag{65}$$

where $H_i, i = 1,...,4$, are mentioned above. In order to solve Problem DSC by the use of MMT approach, we assume that:

- $x_0 = 0.5,\ x_1 = 0.3$
- $\varepsilon_i = 10^{-2}$ and $\gamma_i = 10^2,\ i = 1,...,4,$.
- $x^L(t) = \frac{1}{2}\sin(2\pi t)$ and $x^U(t) = 1 + \sin(2\pi t)$.
- $R_{in}^2 = 0.05$, $R_{out}^2 = 0.15$ and in
  - *: *Case 1:* $(t_{TC}, O(t_{TC}))^\top = (0.6, 0.2)^\top$.
  - *: *Case 2:* $(t_{TC}, O(t_{TC}))^\top = (0.7, -0.2)^\top$.
- $\mathsf{J} = [0,1]$, $\mathsf{A} = [-1,1]$ and $\epsilon = 2$ so $\mathsf{U} = [-2,2]$.
- Total functions in Problem CILP are chosen as
  - *: $\varphi_k = x^k,\ k = 1,...,4,$ and so $M_1 = 4$.
  - *: $\psi_j = \begin{cases} \sin(2\pi jt) & j = 1,...,4, \\ 1 - \cos(2\pi jt) & j = 5,...,8, \text{ and so } M_2 = 8. \end{cases}$
  - *: $h_s(t) \begin{cases} 1 & t \in J_s, \\ 0 & \text{etc.}, \end{cases}$
    where $J_s = (\frac{s-1}{L}, \frac{s}{L}),\ s = 1,...,L = 10$.

The set $\Omega = \mathsf{J} \times \mathsf{A} \times \mathsf{U}$ is covered with a grid of $20 \times 10 \times 40$ points where these are equidistance along the $t-$, $x-$, and $u-$axis, respectively. Taking all

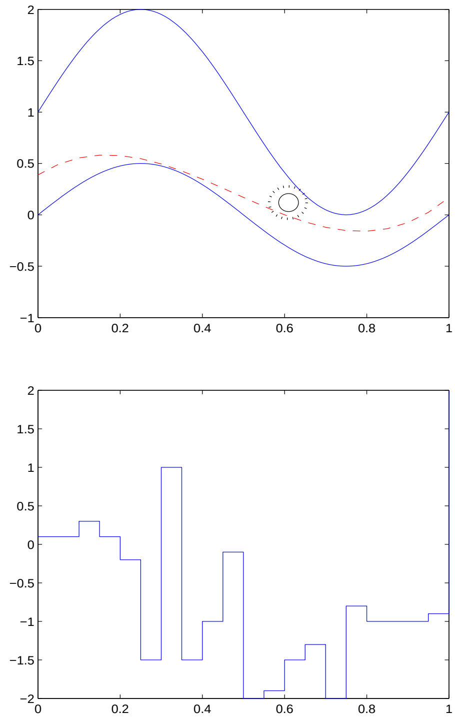

FIGURE 1. Approximate trajectory and optimal control in *Case 1*.

points in Ω as $z_r = (t_r, x_r, u_r)$, $r = 1, ..., \overline{N} = 20 \times 10 \times 40$, characterizes the mentioned grid.

However, Problem CILP derived from Problem DSC is solved by using MMT algorithm written in the MATLAB 7.1 code. The approximate trajectory $x(.)$, fitted by polynomial of degree 5, and the constant piecewise optimal control $u(.)$ are depicted in FIGURE 1 and FIGURE 2, according to *Case 1* and *Case 2*, respectively.

5. Conclusion

The aim of this paper is to devise a computational algorithm using the concept of measure theory for solving a class of optimal control problems constrained by the canonical formulation. In this non-iterative algorithm no initial guess is needed in advance and the computations of the approximate optimal solution can be carried out easily by solving an LP problem which its optimal solution approximates the one of original optimal control problem. In the numerical discussion, an optimal control problem known as deriving strategy problem is

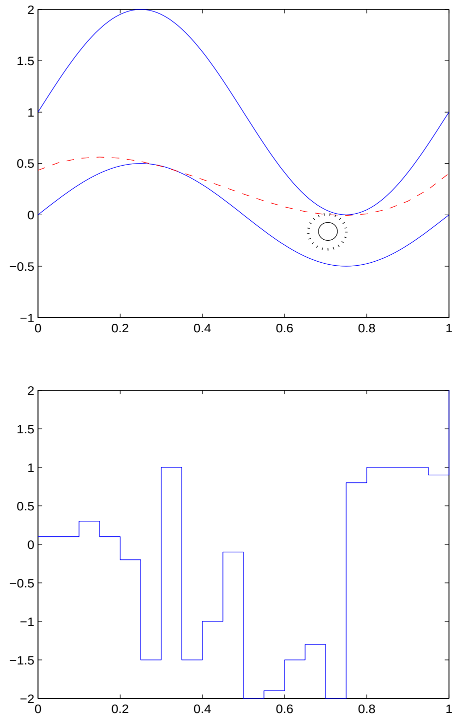

FIGURE 2. Approximate trajectory and optimal control in *Case 2*.

taken and then transformed to the one which is referred to as the canonical optimal control problem. Intuitively, the obtained results show that the proposed solution procedure is effective to find the approximate optimal solution of the deriving strategy problem and even in the more complex optimal control problems in canonical form.

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