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LOWER BOUND OF LENGTH OF TRIANGLE INSCRIBED IN A CIRCLE ON NON-EUCLIDEAN SPACES

Y. D. Chai and Young Soo Lee

Abstract. Wetzel[5] proved if Γ is a closed curve of length L in E^n, then Γ lies in some ball of radius $[L/4]$. In this paper, we generalize Wetzel's result to the non-Euclidean plane with much stronger version. That is to develop a lower bound of length of a triangle inscribed in a circle in non-Euclidean plane in terms of a chord of the circle.

1. Introduction

The problem concerning the smallest circle containing copies of all closed curves of given length, has been studied from time to time. Among them Wetzel[5] proved the following results:

If Γ is an arc [closed curve] of length L in E^n, then Γ lies in some ball of radius $L/2[L/4]$. If no smaller ball covers Γ, then Γ is a segment of length L [$L/2$ traversed twice].

The circumscribed circle of a plane curve has to meet the curve in a set of points of which convex hull contains the center of the circumcircle. So there must be a triangle with vertices on the intersection of the curve with the circle and having the center of the circle in its interior.

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So above Wetzel's result is equivalent to the following fact:

A triangle inscribed in a circle with the center in its interior has length at least twice the diameter.

In this paper, we prove much stronger results on the unit sphere S^2 and the hyperbolic plane, respectively:

A triangle inscribed in a circle on S^2 and having a given point P in its interior has length at least twice the minimum chord through P
and
A triangle inscribed in a circle on the hyperbolic plane and having a given point P in its interior has length at least twice the minimum chord through P .

2. Lower Bound of Length of Triangle inscribed in a Circle on the Unit Sphere S^2

A study of spherical geometry is very important. Astronomers, surveyors, navigators on ships and airplanes apply the formulas of spherical trigonometry and tables obtained by its use to parts of these spherical triangles and obtain such values at the time of day, directions and sailing and flying, position of ships, airplanes, and reference points. Thus spherical geometry, especially spherical trigonometry is basic in astronomy, in certain kinds of surveying, and in navigation[3].

A plane section of a sphere is a *circle*. When the plane passes through the center of the sphere, the section is a *great circle*; otherwise a *small circle*. From now on, if without any mention, a circle will mean to a small circle.

Two points A and B on a sphere determine a unique great circle provide that A and B are not end points of diameter of the sphere, for A, B and center O of the sphere determine a plane intersecting the sphere in a great circle.

The arc length of a great circle is generally stated as the angle subtended by the arc at the center of the sphere. Generally the length less than π is understood.

A line through the center O of a sphere perpendicular to the plane of a circle on the sphere cuts the sphere in two points called *poles* of the circle. We will call this pole the center of the circle.

The *polar distance* of a circle is the least distance on a sphere from a point on the circle to its pole. We will call this polar distance of a circle the radius of the circle.

A *spherical triangle* consists of three arcs of great circles that form the boundaries of a portion of a spherical surface. As plane geometric notation, in the case of spherical triangle, the vertices of the spherical triangle will be denoted by the capital letters A, B, and C and the sides opposite by a, b, and c, respectively.

The magnitude of an angle of a spherical triangle is that of the plane angle formed by tangents to the sides of the angle at its vertex. *In general, we shall consider only spherical triangles that the length of each side and the degree of each angle is less than* $180°$.

Theorem 1. *A triangle inscribed in a circle on S^2 and having the center in its interior has length at least twice the diameter.*

Proof. Let $\mathcal{C}$ be a circle on S^2 and A, B, and C the vertices of the inscribed spherical triangle $\triangle ABC$ and denote the side opposite by a, b, and c, respectively.

Now consider a plane H containing the circle $\mathcal{C}$, rename the three points A, B, and C in $\triangle ABC \cap H$ by A', B', and C', respectively, and construct a Euclidean triangle on H consisting of three Euclidean segments $\overline{A'B'}$, $\overline{B'C'}$, and $\overline{C'A'}$ and denote sides opposite by c', a', and b', respectively.

Then it is easily verified that

$$a' = 2\sin\frac{a}{2}, \quad b' = 2\sin\frac{b}{2}, \quad \text{and} \quad c' = 2\sin\frac{c}{2}.$$

Let x be the angle $\angle A'B'C'$ on H and $\overline{A'C'} = b'(x)$ and $\overline{A'B'} = c'(x)$. Then, since $\triangle A'B'C'$ contains the center of the circle in its interior, $\pi/2 - A' < x < \pi/2$ and $\angle A'$ on H is constant not depending on x.

By the law of sines on the plane H, it follows that

$$\frac{a'}{\sin A'} = \frac{b'(x)}{\sin x} = \frac{c'(x)}{\sin(\pi - A' - x)},$$

and so

$$b'(x) = \frac{a'\sin x}{\sin A'} \quad \text{and} \quad c'(x) = \frac{a'\sin(A' + x)}{\sin A'}.$$

Now we will find the condition that $b + c$ has the greatest low bound under a fixed side a.

Let's define $f(x)$ by

$$\begin{aligned} f(x) : &= b + c = 2\sin^{-1}\left(\frac{b'(x)}{2}\right) + 2\sin^{-1}\left(\frac{c'(x)}{2}\right) \\ &= 2\sin^{-1}\left(\frac{a'\sin x}{2\sin A'}\right) + 2\sin^{-1}\left(\frac{a'\sin(A'+x)}{2\sin A'}\right). \end{aligned}$$

Then

$$f'(x) = \frac{2a'\cos x}{\sqrt{4\sin^2 A' - a'^2\sin^2 x}} + \frac{2a'\cos(x+A')}{\sqrt{4\sin^2 A' - a'^2\sin^2(x+A')}}$$

and it follows from $f'(x) = 0$ that $x = (\pi - A')/2$.

Using $\cos^2 x = 1 - \sin^2 x$, we obtain

$$f''(x) = \frac{2a'(a'^2\sin x - 4\sin x\sin^2 A' + a'^2\sin(x+A') - 4\sin A'\sin(x+A'))}{(4a'^2\sin^2 A'\sin^2(x+A'))^{3/2}}.$$

Now use $\sin(\frac{\pi-A'}{2}) = \cos\frac{A'}{2}$, $\cos(\frac{\pi-A'}{2}) = \sin\frac{A'}{2}$, $\sin(\frac{\pi+A'}{2}) = \cos\frac{A'}{2}$, and $\cos(\frac{\pi+A'}{2}) = -\sin\frac{A'}{2}$ to obtain

$$f''(\frac{\pi - A'}{2}) = \frac{4a'\cos\frac{A'}{2}}{(4\sin^2 A' - a'^2\cos^2\frac{A'}{2})^{3/2}} \times 4\sin^2 A'(R'^2 - 1)$$

where we use also the law of sines, that is, $a' = 2R'\sin A'$ with the radius R' of the circle derived on H.

Since $R' < 1$,

$$f''(\frac{\pi - A'}{2}) < 0.$$

This tells that $f(x)$ attains the maximum at $x = \frac{\pi-A'}{2}$.

So $f(x)$ has the greatest low bound at $x = \pi/2 - A', \pi/2$, that is, when the side $\overline{A'B'}$ pass through the center, and so side $\overline{AB} = c$ pass through the center of the circle $\mathcal{C}$.

At this case, the sum of three sides is obviously greater than twice the diameter. So any triangle having the center in its interior has length at least twice the diameter. Hence the proof is complete.

Naturally, we can establish the following stronger result:

Theorem 2. *A triangle inscribed in a circle on S^2 and having a given point P in its interior has length at least twice the minimum chord through P.*

Proof. We shall verify this as dividing it into two.

If the triangle have the center of a circle in its interior, then it is obvious, by Theorem 1.

If the triangle does not have the center of a circle in its interior, consider any chord passing through P and not intersecting the side of triangle that has the maximum length. Then, the length of this chord is obviously less than the maximum length of the side of triangle. And the sum of the lengths of other sides is greater than the maximum length of the sides. Hence the proof is complete. The following corollary strengthens Wetzel's result in R^2.

Corollary 1. *A triangle inscribed in a circle on R^2 and having a given point P in its interior has length at least twice the minimum chord through P.*

Proof. Note that Theorem 2 is true for a circle on sphere of arbitrary radius. Considering R^2 as sphere of infinity radius, we complete the proof for circle on R^2.

3. Lower Bound of Length of Triangle inscribed in a Circle on the hyperbolic plane H^2

We will use the Poincaré's upper half plane as the model for a hyperbolic plane. Before introducing the Poincaré's upper half plane, we need to explain the cross ratio which is the fundamental concept in the projective geometry.

Let z_1, z_2, z_3, and z_4 be complex numbers(or real numbers). Then the *cross ratio* $[z_1, z_2, z_3, z_4]$ for four numbers z_1, z_2, z_3, and z_4 is defined by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

If four numbers are all distinct, then $[z_1, z_2, z_3, z_4]$ is a non-zero complex number(or real number). Though not all distinct, if three numbers are distinct, then $[z_1, z_2, z_3, z_4]$ is well-defined under the condition that $[z_1, z_2, z_3, z_4]$ may be 0 or ∞.

Now assume that at least three numbers are distinct when we write $[z_1, z_2, z_3, z_4]$.

Let a, b, c, and d be complex numbers(or real numbers) and $ad - bc \neq 0$. Then the linear fractional transformation

$$w = \frac{az + b}{cz + d}$$

has a inverse transformation

$$z = \frac{dw - b}{-cw + a} \ .$$

For the linear fractional transformation the followings are well-known facts:

(1) The cross ratio does not change under a linear fractional transformation.

(2) A linear fractional transformation preserves the angle.

Now we will discribe the model of hyperbolic geometry. In the Poinca ré's model, a plane means the upper half plane $H^2 = \{\ (x, y) \mid y > 0\}$ and a straight line is considered a vertical line or the circle that its center lies on the x-axis. From now on we only consider a point in H^2 as a point and also a line in H^2 as a line.

For the Poincaré's model, the following fact holds: *Given two points A, B, there is unique straight line connecting these two points.*

Now define a metric between two points. Given two points z and z' on a straight line ℓ in H^2, using the cross ratio, the length of line segment from z to z' is defined by

$$d(z, z') = |\log[z, z', x_0, x_1]| \tag{1}$$

It holds from the definition of the cross ratio that $d(z, z') = d(z', z)$. In H^2, the metric $d(z, z')$ is well-defined, since there is a unique straight line connecting given two points. But (1) can be used only when ℓ is a line induced by a Euclidean circle with a center at x-axis. So if ℓ is a line perpendicular to x-axis, then we need to define another form.

$$d(z, z') = \left| \log \frac{1 + |\frac{z - z'}{z - \bar{z}'}|}{1 - |\frac{z - z'}{z - \bar{z}'}|} \right|$$

always holds together (1).

There is a well-known facts: *Let ℓ and m be two straight lines on H^2. Suppose $z, z' \in \ell$ and $w, w' \in m$ are points such that $d(z, z') = d(w, w')$. Then there is a linear fractional transformation mapping ℓ, z, z' to m, w, w', respectively.*

Let A and B be two points in H^2. Then there is a unique circle with a center A and passing through B. Let's construct a hyperbolic circle on H^2. Let a segment AB be given. Since it can be transfered on a vertical line to the x-axis by some linear fractional transformation, without loss of generality, we can assume that the segment AB lies on the vertical line. Let B' be a point on the line containing the segment AB such that $d(A, B) = d(A, B')$ and $BO \cdot B'O = AO^2$, and draw the Euclidean circle

$\mathcal{C}$ with diameter BB'. Then this circle $\mathcal{C}$ is a hyperbolic circle with a center A in the Poincaré's model.

Now consider the hyperbolic trigonometry in the Poincaré' model.

For the hyperbolic triangle $\triangle ABC$ in H^2, the opposite sides of A, B, and C is denoted by a, b, and c, respectively. Then we have *the law of cosines*:

$$\begin{aligned}\cosh a &= \cosh b \cosh c - \sinh b \cdot \sinh c \cos A \\ \cosh b &= \cosh a \cosh c - \sinh a \cdot \sinh c \cos B \\ \cosh c &= \cosh a \cosh b - \sinh a \cdot \sinh b \cos C\end{aligned}$$

Theorem 3. *A triangle inscribed in a circle on H^2 and having the center in its interior has length at least twice the distance.*

Proof. Let $\mathcal{C}$ be a circle of radius r on H^2 and A, B, and C the vertices of the inscribed hyperbolic triangle $\triangle ABC$ and a, b, and c lengths of opposite sides A, B, and C, respectively. Let A', B', and C' be the reflecting points of A, B, and C about the center O (of hyperbolic sense) of $\mathcal{C}$. That is, A' is a point that the line passing through O and A intersects the circle $\mathcal{C}$. If we denote the length of segment $B'C$ by b' and BB' by c', then it is easily verified that $a + b' > c'$. So if $b + c > b' + c'$, then

$$a + b + c > a + b' + c' > 2c' = 4r.$$

So it suffices to show that $b + c > b' + c'$.

Let it denote lengths of segments $A'B'$ and $A'C'$ by $\bar{c}$ and $\bar{b}$, respectively. By the law of cosines it is easily verified that $b = \bar{b}, c = \bar{c}$, and $b' = BC'$. Therefore to show $b + c > b' + c'$, it suffices to show that $\bar{b} + \bar{c} > b' + c'$. So when A' moves along the arc BC not containing A, we will investigate the changes of length of $\bar{b} + \bar{c}$.

Let $x = \angle BOA'$ $(0 < x < \pi - \alpha)$. Using the hyperbolic trigonometry, from $\triangle OA'B', \triangle OC'A'$, we have

$$\begin{aligned}\cosh \bar{b}(x) &= \cosh^2 r - \sinh^2 r \cos(\pi - x), \\ \cosh \bar{c}(x) &= \cosh^2 r - \sinh^2 r \cos(x + \alpha),\end{aligned}$$

respectively.

Let $f(x) := \bar{b}(x) + \bar{c}(x)$. Then

$$f'(x) = \frac{\sinh^2 r \sin(x + \alpha)}{\sqrt{(\cosh^2 r - \sinh^2 r \cos(x + \alpha))^2 - 1}} - \frac{\sinh^2 r \sin x}{\sqrt{(\cosh^2 r + \sinh^2 r \cos x)^2 - 1}}.$$

From $f'(x) = 0$, if we set $e := \cosh r$ and $f := \sinh r$, then we have

$$\sin^2 x((e^2 - f^2 \cos(x + \alpha))^2 - 1) = \sin^2(x + \alpha)((e^2 + f^2 \cos x)^2 - 1)$$

where we use $e^2 - f^2 = 1$ and $e^4 + f^4 - 1 = 2e^2f^2$.

Using $\sin^2 x = 1 - \cos^2 x$, it follows that

$$2e^2f^2(\cos x + \cos(x+\alpha))(1 - \cos(x+\alpha))(1 + \cos x) = 0.$$

Since $0 < x < \pi - \alpha$, it reduces to that $\cos x + \cos(x+\alpha) = 0$. Therefore, $x = \frac{\pi-\alpha}{2}$ is a unique solution for $f'(x) = 0$. And it is easily verified that $f''(x) < 0$ at $x = \frac{\pi-\alpha}{2}$. So $f(\frac{\pi-\alpha}{2})$ has a maximum value for $f(x)$ on $[0, \pi - \alpha]$ and f attains a minimum value at the boundaries of a domain. So $f(x)$ has the greatest low bound at $x = 0, \pi - \alpha$, that is, $\bar{b} + \bar{c}$ attains a minimum value $b' + c'$. So the proof is complete.

Naturally, we can establish the following stronger result:

Theorem 4. *A triangle inscribed in a circle on H^2 and having a given point P in its interior has length at least twice the minimum chord through P.*

Proof. We shall verify this as dividing it into two.

First, if the triangle have the center of a circumscribed circle in its interior, then, by Theorem 3, it is obvious. So, we will consider the case that the triangle does not have the center of a circle in its interior.

Second, let σ be a chord in a circle induced by any line passing through P and not intersecting the longest side of the triangle. Then length of σ is obviously less than that of longest side. And the sum of lengths of other sides is greater than length of the longest side. Hence the proof is complete.

References

[1] G. D. Chakerian and M. S. Klamkin, Inequalities for sums of distances, *Amer. Math. Monthly*, **80** (1973), 1009-1017.

[2] M. Ghandehari, Sums of distances in normed spaces, *Acta Math. Hungar.*, **67**(1995), 123-129.

[3] L. M. Kells, W. F. Kern, and T. R. Bland,*Plane and Spherical Trigonometry*, McGraw-Hill Inc., New York, 1951.

[4] D. Laugwitz, Konvexe Mittelpunktsbereiche und normierte Raume, Math. Z., 61(1954) 235-244.

[5] J. E Wetzel, Covering balls for curves of constant length, L'Enseignement Math., 17(1971) 275-277.

Y. D. Chai
Department of Mathematics, SungKyunKwan University,

Suwon 440-746, Korea.
E-mail: ydchai@skku.edu

Young Soo Lee
Department of Mathematics, SungKyunKwan University,
Suwon 440-746, Korea.
E-mail: yslee9897@skku.edu