Honam Mathematical J. **34** (2012), No. 1, pp. 085–092
http://dx.doi.org/10.5831/HMJ.2012.34.1.85

GENERALIZED PROPERTIES OF STRONGLY FRÉCHET

Myung Hyun cho*, Junhui Kim†, and Mi Ae Moon

Abstract. Our purpose of this paper is to introduce and study some properties related to approximations by points. More precisely, we introduce strongly AP, strongly AFP, strongly ACP, and strongly WAP properties which are stronger than AP, AFP, ACP, and WAP respectively. Also they are weaker than strongly Fréchet property. And we study general properties and topological operations on such spaces and give some examples.

1. Introduction

In this paper, all spaces under consideration are assumed to be Hausdorff, ω is the first countable ordinal, and $|X|$ is the cardinality of any set X.

We first introduce some properties which we are interested in. A space X is said to be *AP* (resp. *ACP*) if for every non-closed subset A of X and for every $p \in \overline{A} \setminus A$, there exists a (resp. countable) subset F of A such that $\overline{F} = F \cup \{p\}$. A space X is said to be *WAP* if for every non-closed subset A of X, there exist a point $p \in \overline{A} \setminus A$ and a subset F of A such that $\overline{F} = F \cup \{p\}$.

Approximation by points (AP) and weak approximation by points (WAP) were introduced by A. Pultr and A. Tozzi in 1993 ([7]) and studied by A. Bella ([2]) and P. Simon ([8]) etc. And W. Hong ([5]) introduced many properties by generalizing AP and WAP. Specially, he defined approximation by countable points (ACP).

Received December 14, 2011. Accepted February 13, 2012.
2000 Mathematics Subject Classification. 54A20, 54C05, 54D55.
Key words and phrases. Fréchet, AP, ACP, WAP, strongly Fréchet, strongly AP, strongly AFP, strongly ACP, strongly WAP.
* The first author's research was supported by Wonkwang University Research Grant, 2010.
† corresponding author

A space X is said to be *Fréchet* if for every subset A of X and for every $p \in \overline{A}$, there exists a sequence in A which converges to p.

A space X is said to be *strongly Fréchet* if for every decreasing family $\{A_n : n \in \omega\}$ of subsets of X and for every $x \in \cap\{\overline{A_n} : n \in \omega\}$, there exists $x_n \in A_n$ for each $n \in \omega$ such that $\{x_n : n \in \omega\}$ converges to x. This property was introduced by E. Michael in [6].

By definitions, any strongly Fréchet space is Fréchet, any Fréchet space is ACP, any ACP space is AP, and any AP space is WAP.

2. New Properties and Examples

We define some properties (strongly AP, strongly AFP, strongly ACP, strongly WAP) related to approximations by points as follows:

Definition 1. A space X is said to be *strongly AP* if for every decreasing family $\{A_n : n \in \omega\}$ of non-closed subsets of X and for every $p \in \cap\{\overline{A_n} : n \in \omega\} \setminus \cup\{A_n : n \in \omega\}$, there exists a subset F_n of A_n for each $n \in \omega$ such that $\overline{F} = F \cup \{p\}$ where $F = \cup\{F_n : n \in \omega\}$.

Definition 2. A space X is said to be *strongly AFP* (resp. *strongly ACP*) if for every decreasing family $\{A_n : n \in \omega\}$ of non-closed subsets of X and for every $p \in \cap\{\overline{A_n} : n \in \omega\} \setminus \cup\{A_n : n \in \omega\}$, there exists a subset F_n of A_n for each $n \in \omega$ such that $\overline{F} = F \cup \{p\}$ where $F = \cup\{F_n : n \in \omega\}$ and $|F_n| < \omega$ (resp. $|F_n| \leq \omega$).

Definition 3. A space X is said to be *strongly WAP* if for every decreasing family $\{A_n : n \in \omega\}$ of non-closed subsets of X, there exist a point $p \in \cap\{\overline{A_n} : n \in \omega\} \setminus \cup\{A_n : n \in \omega\}$ and a subset F_n of A_n for each $n \in \omega$ such that $\overline{F} = F \cup \{p\}$ where $F = \cup\{F_n : n \in \omega\}$

By definitions, we have the following properties:

- Every strongly Fréchet space is Fréchet;
- Every strongly Fréchet space is strongly AFP;
- Every strongly AFP space is strongly ACP;
- Every strongly ACP space is strongly AP;
- Every strongly AP space is AP;
- Every strongly WAP space is WAP.

So we have the following diagram:

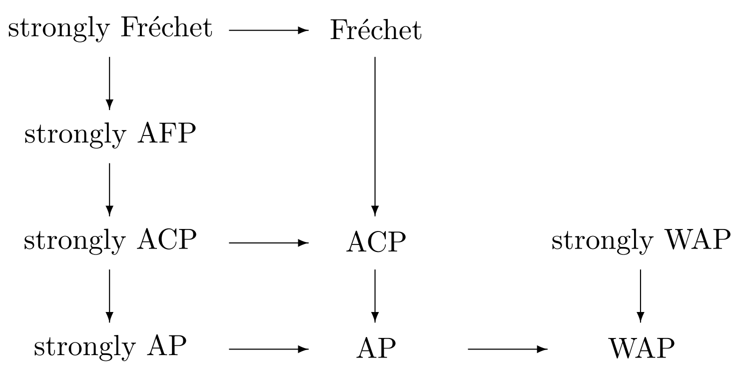

Diagram 1

Example 2.1. The sequential fan S_ω is the set $(\omega\times\omega)\cup\{p\}$ equipped with a topology defined as follows:

- each point in $\omega \times \omega$ is isolated;
- a basic open neighborhood of p is the form of
$$O_f(p) = \{p\} \cup \{(n, k) \in \omega \times \omega : k \geq f(n)\}$$
for each function $f : \omega \to \omega$.

The sequential fan S_ω is regular, but not countably compact. Since S_ω is Fréchet, it is AP. Also it is well known that it not strongly Fréchet. Furthermore, S_ω is strongly ACP since S_ω is countable and S_ω has the only one limit point p.

Claim : S_ω is not strongly AFP.

Let $A_n = [n, \omega) \times \omega$. Then $A_{n+1} \subset A_n$ for all $n \in \omega$ and $p \in \cap\{\overline{A_n} : n \in \omega\}$. Choose any finite subset F_n of A_n for each $n \in \omega$ and denote $F = \cup\{F_n : n \in \omega\}$. Then $|F \cap B_n| < \omega$ for each $n \in \omega$ where $B_n = \{n\} \times \omega$. Define a function $f : \omega \to \omega$ by $f(n) = 1 + \max\{k : (n, k) \in F\}$ if $F \cap B_n \neq \emptyset$ and $f(n) = 0$ otherwise. Then $O_f(p) \cap F = \emptyset$, that is, $p \notin \overline{F}$. Therefore S_ω is not strongly AFP. $\square$

Example 2.2. Let $X = (\omega \times \omega) \cup \{p\}$. We topologize X as follows:

- each point in $\omega \times \omega$ is isolated;
- U is a basic open neighborhood of p if and only if for all but a finite number of $n \in \omega$, each set $\{k \in \omega : (n, k) \notin U\}$ is finite.

Then the space X is regular, but not countably compact. Also $p \in \overline{\omega \times \omega} \setminus (\omega \times \omega)$, but there is no sequence of points in $\omega \times \omega$ which converges to p. So X is not Fréchet. Since X is countable and p is the unique non-isolated point in X, X is strongly ACP. Furthermore, one can prove that X is not strongly AFP with the same argument in Example 2.1. Therefore we can not say that all spaces satisfying ACP and strongly AP are Fréchet or strongly AFP. □

In general, every AP space is a WAP space, but we have a strongly AP space which is not strongly WAP as follows:

Example 2.3. Let $X = [0, 1]$ be the Euclidean space. Since X is a compact Hausdorff space which is Fréchet, X is a strongly AP space which is WAP.

To prove that X is not strongly WAP, we take an interval $A_n = [0, \frac{1}{n+1})$ for each $n \in \omega$. Then $\{A_n : n \in \omega\}$ is a decreasing family of non-closed subsets of X. Since $\cap\{\overline{A_n} : n \in \omega\} = \{0\}$ and $0 \in \cup\{A_n : n \in \omega\}$, $\cap\{\overline{A_n} : n \in \omega\} \setminus \cup\{A_n : n \in \omega\} = \emptyset$. Therefore X is not strongly WAP. □

Example 2.4. Let D be a uncountable discrete space and $p \notin D$. Assume that $X = D \cup \{p\}$ is a one-point Lindelöf extension, that is, X is topologized as follows:

- each point in D is isolated in X;
- if $p \in U$ and $X \setminus U$ is countable.

Since X has the only one non-isolated point p, one can easily prove that X is strongly AP and strongly WAP. But every countable subset is closed. Therefore X is not strongly ACP. □

3. Relevant Results

Theorem 3.1.

(1) *Every subspace of a strongly AP space is strongly AP;*
(2) *Every subspace of a strongly ACP space is strongly ACP;*
(3) *Every subspace of a strongly AFP space is strongly AFP.*

Proof. (1) Assume that Y is a subspace of a strongly AP space X. Let $\{A_n : n \in \omega\}$ be a decreasing family of non-closed subsets of Y and let $p \in \cap\{\overline{A_n}^Y : n \in \omega\} \setminus \cup\{A_n : n \in \omega\}$. Then $p \in \overline{A_n}^X \cap Y$ for each $n \in \omega$. Since X is strongly AP, there exists a subset $F_n \subset A_n$ such that $\overline{F}^X = F \cup \{p\}$ where $F = \cup\{F_n : n \in \omega\}$. Since $\overline{F}^Y = \overline{F}^X \cap Y$ and $p \in Y$, $p \in \overline{F}^Y$. Also $\overline{F}^Y = \overline{F}^X \cap Y = (F \cup \{p\}) \cap Y = F \cup \{p\}$. Therefore X is strongly AP.

(2) and (3) can be proved by the same argument with (1). □

Theorem 3.2.

(1) *Every closed continuous image of a strongly AP space is strongly AP;*
(2) *Every closed continuous image of a strongly ACP space is strongly ACP;*
(3) *Every closed continuous image of a strongly AFP space is strongly AFP;*

Proof. (1) Assume that X is a strongly AP space and $f : X \to Y$ is a closed continuous onto map. Let $\{A_n : n \in \omega\}$ be a decreasing family of non-closed subsets of Y and let $p \in \cap\{\overline{A_n} : n \in \omega\} \setminus \cup\{A_n : n \in \omega\}$. Denote $B_n = f^{-1}(A_n)$ for each $n \in \omega$. Notice that $f(\overline{B_n}) = \overline{f(B_n)} = \overline{A_n}$ for each $n \in \omega$ (because f is a closed continuous onto map). Hence $\overline{B_n} = f^{-1}(\overline{A_n})$ for each $n \in \omega$. So we obtain that

$$f^{-1}(p) \subset f^{-1}(\cap\{\overline{A_n} : n \in \omega\}) = \cap\{\overline{B_n} : n \in \omega\}$$

and

$$f^{-1}(p) \cap (\cup\{B_n : n \in \omega\}) = \emptyset.$$

Pick a point $x \in f^{-1}(p)$. Since X is strongly AP, there exists a subset $C_n \subset B_n$ for each $n \in \omega$ such that $\overline{C} = C \cup \{x\}$ where $C = \cup\{C_n : n \in \omega\}$. Let $F_n = f(C_n)$ for each $n \in \omega$ and $F = \cup\{F_n : n \in \omega\}$. Then one can prove that $\overline{F} = F \cup \{p\}$. Therefore Y is strongly AP.

It follows from the same argument that (2) and (3) can be proved. □

A continuous map $f : X \to Y$ is called *pseudo-open* if for each $y \in Y$ and every neighborhood U of $f^{-1}(y)$ we have $y \in f(U)^\circ$ where A° is the interior of A in Y.

The following lemma is a characterization of a pseudo-open map.

Lemma 3.3. [1] *Let $f : X \to Y$ be a continuous onto map. Then the following conditions are equivalent:*

(1) *for each $Y' \subset Y$ the restriction f' of f to $X' = f^{-1}(Y')$, the inverse image of Y', is a quotient map of X' onto Y';*
(2) *for each $y \in Y$ and every open set U in X containing $f^{-1}(y)$, $y \in f(U)^\circ$;*

(3) *whenever* $B \subset Y$ *and* $y \in Y$ *satisfies* $y \in \overline{B}$, *we have* $f^{-1}(y) \cap \overline{f^{-1}(B)} \neq \emptyset$.

It is well known that every continuous closed (or open) map is pseudo-open. And it was proved that any pesudo-open image of an AP space (resp. WAP space) is an AP space (resp. WAP space) in [4]. As corollaries of this result, we have the following ([4]): any closed continuous image of an AP (resp. WAP) space is an AP (resp. WAP) space; any open continuous image of an AP (resp. WAP) space is an AP (resp. WAP) space.

One may wonder whether such a situation occurs on strongly AP (strongly ACP, or strongly AFP) space as Theorem 3.2, But to get the similar result, we have to give an additional condition that a map is injective.

Theorem 3.4.

(1) *Any pseudo-open injective image of a strongly AP space is strongly AP;*
(2) *Any pseudo-open injective image of a strongly ACP space is strongly ACP;*
(3) *Any pseudo-open injective image of a strongly AFP space is strongly AFP.*

Proof. (1) : Assume that X is a strongly AP space and $f : X \to Y$ is a pseudo-open bijection. Let $\{A_n : n \in \omega\}$ be a decreasing family of non-closed subsets of Y and let $p \in \cap\{\overline{A_n} : n \in \omega\} \setminus \cup\{A_n : n \in \omega\}$. Denote $B_n = f^{-1}(A_n)$ for each $n \in \omega$. Then, by Lemma 3.3, $f^{-1}(p) \cap \overline{B_n} \neq \emptyset$. Since f is injective, $f^{-1}(p)$ is a singleton (denote $f^{-1}(p) = \{x\}$). Hence $x \in \overline{B_n}$ for each $n \in \omega$. Namely, $x \in \cap\{\overline{B_n} : n \in \omega\} \setminus \cup\{B_n : n \in\}$ where $\{B_n : n \in \omega\}$ is a decreasing family of non-closed subsets of X. Since X is strongly AP, there exists a subset $C_n \subset B_n$ such that $\overline{C} = C \cup \{x\}$ where $C = \cup\{C_n : n \in \omega\}$. Let $F_n = f(C_n)$ for each $n \in \omega$ and $F = \cup\{F_n : n \in \omega\}$. Then $\overline{F} = F \cup \{p\}$. Therefore Y is strongly AP.

(2)-(3) : The proofs follow the same argument with (1). □

Theorem 3.5. *Every strongly AP space X with $|X| \leq \omega$ is strongly ACP.*

Proof. Let $\{A_n : n \in \omega\}$ be a decreasing family of non-closed subsets of X and let $p \in \cap\{\overline{A_n} : n \in \omega\} \setminus \cup\{A_n : n \in \omega\}$. Since X is strongly AP, there exists a subset F_n of A_n for each $n \in \omega$ such that $\overline{F} = F \cup \{p\}$

where $F = \cup\{F_n : n \in \omega\}$. And each F_n is countable because X is countable. Therefore X is strongly ACP. □

Theorem 3.6. [10] *Every sequential and AP space is Fréchet.*

We obtain a similar statement with Theorem 3.6 was proved by V.V. Tkachuk and I.V. Yaschenko as follows:

Theorem 3.7. *Every sequential and strongly AP space is strongly Fréchet.*

Proof. Assume that a space X is sequential and strongly AP. Let $\{A_n : n \in \omega\}$ be a decreasing family of non-closed subsets of X and let $p \in \cap\{\overline{A_n} : n \in \omega\} \setminus \cup\{A_n : n \in \omega\}$. Since X is strongly AP, there is a subset $F_n \subset A_n$ such that $\overline{F} = F \cup \{p\}$ where $F = \cup\{F_n : n \in \omega\}$. Since F is not closed in X and since X is sequential, there is a sequence $S = \{x_k : k \in \omega\} \subset F$ which converges to p. Note that each x_k belongs to at least one A_n and that $\{A_n : n \in \omega\}$ is a decreasing family.

Inductively, we shall construct a sequence $\{y_i : y_i \in A_i, i \in \omega\}$ which converges to p. Let $n_0 \in \omega$ be the smallest ordinal number such that $x_0 \in A_{n_0}$. Then we take $y_i = x_0$ for all $i \leq n_0$. Suppose that, for all $i \leq k$, we have chosen y_i such that $y_i \in A_i$ and $y_i \in S$.

Step 1. If $n_{k+1} \in \omega$ is the smallest ordinal number such that $n_k < n_{k+1}$ and $x_{k+1} \in A_{n_{k+1}}$, then we take $y_i = x_{k+1}$ for all i with $n_k < i \leq n_{k+1}$.

Step 2. If there is no such an $n_{k+1} \in \omega$, then we take $n_{k+1} = n_k$, drop the point x_{k+1}, and go to Step 1.

Then $y_i \in A_i$ for all $i \in \omega$. Since the sequence $\{y_i : i \in \omega\}$ is a natural expansion of $\{x_k : k \in \omega\}$, the sequence $\{y_i : i \in \omega\}$ converges to p. Therefore X is strongly Fréchet. □

Theorem 3.8. [10] *Every countably compact AP space is Fréchet.*

Corollary 3.9. *All properties except two properties "strongly WAP" and "WAP" mentioned in* Diagram 1 *are equivalent for a countably compact regular space.*

Proof. It is well known that a countably compact regular space is Fréchet if and only if it is strongly Fréchet. Therefore, by Theorem 3.8, every countably compact regular AP space is strongly Fréchet. Example 2.3 is a compact Hausdorff space satisfies all properties except "strongly WAP". □

References

[1] A. V. Arhangel'skii and V. I. Ponomarev, *Fundamentals of General Topology*, D. Reidel Publishing Co., Dordrecht/Boston/Lancaster, 1984.

[2] A. Bella, *On spaces with the property of weak approximation by points*, Comment. Math. Univ. Carolin. **35** (1994), no. 2, 357-360.

[3] A. Bella, C. Costantini and P. Simon, *Fréchet versus strongly Fréchet*, *Topology Appl.* **153** (2006) 1651-1657.

[4] M.H. Cho, J. Kim and M. A. Moon, *Examples and function theorems around AP and WAP spaces*, Commun. Korean Math. Soc. **23** (2008), no.3, 447-452.

[5] W. C. Hong, *Generalized Fréchet-Urysohn Spaces*, J. Korean Math. Soc. **44** (2007), no. 2, 261-273.

[6] E. Michael, A quintuple quotient quest, *Gen. Topology Appl.* **2** (1972) 91-138.

[7] A. Pultr and A. Tozzi, *Equationally closed subframes and representation of quotient spaces*, Cahiers Topologie Géom. Différentielle Catég. **34** (1993), no. 3, 167-183.

[8] P. Simon, *On accumulation points*, Cahiers Topologie Géom. Différentielle Catég. **35** (1994), no. 4, 321-327.

[9] F. Siwiec, Generalizations of the first axiom of countability, *Rocky Mountain J. Math.* **5** (1975) 1-60.

[10] V.V. Tkachuk and I.V. Yaschenko, Almost closed sets and topologies they determine, *Comment. Math. Univ. Carolinae*, **42(2)** (2001), 393-403.

Myung Hyun Cho
Department of Mathematics Education, Wonkwang University,
Iksan 570-749, Korea.
E-mail: mhcho@wonkwang.ac.kr

Junhui Kim
Department of Mathematics Education, Wonkwang University,
Iksan 570-749, Korea.
E-mail: junhikim@wonkwang.ac.kr

Mi Ae Moon
Division of Mathematics & Informational Statistics, Wonkwang University,
Iksan 570-749, Korea.
E-mail: moonmae@wonkwang.ac.kr