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COMMON FIXED POINT THEOREM FOR WEAKLY COMMUTING USING IMPLICIT RELATION ON INTUITIONISTIC FUZZY METRIC SPACE

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Abstract. In this paper, we define the weakly commuting mapping and prove the fixed point theorem for weakly commuting mappings under some conditions on intuitionistic fuzzy metric spaces.

1. Introduction

Several authors([2],[6],[7]) have introduced the basic concepts on fuzzy metric spaces and fuzzy topological spaces induced by fuzzy metrics with different ways. Grabiec[3] obtained the Banach contraction principle in setting of fuzzy metric spaces. Also, Park et al.[9],[10] defined the intuitionistic fuzzy metric space and studied the some properties on this spaces. Recently, Lee[8] proved common fixed theorems for weakly commuting mappings in fuzzy metric spaces.

In this paper, we define the weakly commuting mapping and prove the common fixed point theorem for weakly commuting mappings under some conditions on intuitionistic fuzzy metric space.

2. Preliminaries

We recall some definitions, properties and known results in the intuitionistic fuzzy metric space as following :

Let us recall(see [11]) that a continuous $t-$norm is a operation $* : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions: (a)$*$ is commutative and associative, (b)$*$ is continuous, (c)$a * 1 = a$ for all $a \in [0,1]$, (d)$a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a,b,c,d \in [0,1])$. Also, a continuous $t-$conorm is a operation $\diamond : [0,1]\times[0,1] \to [0,1]$ which

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satisfies the following conditions: (a)$\diamond$ is commutative and associative, (b)$\diamond$ is continuous, (c)$a \diamond 0 = a$ for all $a \in [0,1]$, (d)$a \diamond b \geq c \diamond d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0,1])$.

Definition 2.1. ([9])The 5−tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t-$norm, $\diamond$ is a continuous $t-$conorm and M, N are fuzzy sets on $X^2 \times (0,\infty)$ satisfying the following conditions; for all $x, y, z \in X$, such that

(a)$M(x, y, t) > 0$,
(b)$M(x, y, t) = 1 \Longleftrightarrow x = y$,
(c)$M(x, y, t) = M(y, x, t)$,
(d)$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(e)$M(x, y, \cdot) : (0,\infty) \to (0, 1]$ is continuous,
(f)$N(x, y, t) > 0$,
(g)$N(x, y, t) = 0 \Longleftrightarrow x = y$,
(h)$N(x, y, t) = N(y, x, t)$,
(i)$N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$,
(j)$N(x, y, \cdot) : (0,\infty) \to (0, 1]$ is continuous.

Note that (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Definition 2.2. ([10]) Let X be an intuitionistic fuzzy metric space.

(a) $\{x_n\}$ is said to be convergent to a point $x \in X$ by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$, $\lim_{n\to\infty} N(x_n, x, t) = 0$ for all $t > 0$.

(b) $\{x_n\}$ is called a Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1, \quad \lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$$

for all $t > 0$ and $p > 0$.

(c) X is complete if every Cauchy sequence converges in X.

In this paper, X is considered to be the intuitionistic fuzzy metric space with the following condition:

$$\lim_{t\to\infty} M(x, y, t) = 1, \quad \lim_{t\to\infty} N(x, y, t) = 0 \tag{2.1}$$

for all $x, y \in X$ and $t > 0$.

Lemma 2.3. ([10])*Let $\{x_n\}$ be a sequence in an intuitionistic fuzzy metric space X with the condition* (2.1). *If there exist a number $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,*

$$\begin{aligned} &M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t), \\ &N(x_{n+2}, x_{n+1}, kt) \leq N(x_{n+1}, x_n, t \end{aligned} \tag{2.2}$$

for all $t > 0$ and $n = 1, 2, \cdots$, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.4. ([10])*Let X be an intuitionistic fuzzy metric space. If there exists a number $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,*

$$M(x, y, kt) \geq M(x, y, t),\ N(x, y, kt) \leq N(x, y, t),$$

then $x = y$.

Definition 2.5. Let A, B be mappings from intuitionistic fuzzy metric space X into itself. The mappings are said to be a commuting mappings if

$$M(ABx, BAx, t) = 1, \quad N(ABx, BAx, t) = 0$$

for all $x \in X$ and $t > 0$.

Definition 2.6. Let A, B be mappings from intuitionistic fuzzy metric space X into itself. The mappings are said to be a pair of weakly commuting mappings if

$$M(ABx, BAx, t) \geq M(Ax, Bx, t), \quad N(ABx, BAx, t) \leq N(Ax, Bx, t)$$

for all $x \in X$ and $t > 0$.

We know that commuting mappings are weakly commuting, but not conversely as shown in the following example:

Example 2.7. Let X be an intuitionistic fuzzy metric space, where $X = [0, 1]$ with $a * b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$ for all $a, b \in X$ and

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

for all $x, y \in X$ and $t > 0$. Define $A, B : X \to X$ by $Ax = \frac{x}{2}$, $Bx = \frac{x}{x+2}$. Then we have

$$M(BAx, ABx, t) \geq M(Bx, Ax, t), \quad N(ABx, BAx, t) \leq N(Bx, Ax, t).$$

Hence A, B are weakly commuting mappings, but we have for any $x(\neq 0) \in X$, $BAx = \frac{x}{x+4} > \frac{x}{2x+4} = ABx$. Therefore A and B are not commuting mappings.

Lemma 2.8. *Let X be an intuitionistic fuzzy metric space and let $A, B : X \to X$ be weakly commuting mappings and $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Ax_n = x$ for some $x \in X$. If A is continuous on X, then $\lim_{n\to\infty} BAx_n = Ax$.*

Proof. Suppose that A is continuous at $x \in X$, $\lim_{n\to\infty} ABx_n = Ax$. Since A and B are weakly commoting mappings, we have

$$\lim_{n\to\infty} M(BAx_n, ABx_n, t) \geq \lim_{n\to\infty} M(Bx_n, Ax_n, t) = M(x, x, t) = 1,$$
$$\lim_{n\to\infty} N(BAx_n, ABx_n, t) \leq \lim_{n\to\infty} N(Bx_n, Ax_n, t) = N(x, x, t) = 0.$$

Also, since

$$M(BAx_n, Ax, t) \geq M(BAx_n, ABx_n, \frac{t}{2}) * M(ABx_n, Ax, \frac{t}{2}),$$
$$N(BAx_n, Ax, t) \leq N(BAx_n, ABx_n, \frac{t}{2}) \diamond N(ABx_n, Ax, \frac{t}{2}),$$

hence $\lim_{n\to\infty} M(BAx_n, Ax, t) \geq 1 * 1 = 1$, $\lim_{n\to\infty} N(BAx_n, Ax, t) \leq 0 \diamond 0 = 0$. Therefore $\lim_{n\to\infty} BAx_n = Ax = \lim_{n\to\infty} ABx_n$. □

Now, we will describe the implicit relation. Implicit relation on metric space or fuzzy metric space have been used in many authors(see [1], [4] etc).

Let $\Phi = \{\phi_M, \psi_N\}$, ϕ_M denote a set of all continuous, decreasing functions $\phi_M; [0,1]^6 \to [0,1]$ in each coordinate such that if

$$\phi_M(x, 1, y, 1, y, 1) \geq 0, \quad \phi_M(x, y, 1, 1, y, y) \geq 0,$$
$$\phi_M(x, 1, 1, y, 1, y) \geq 0, \quad \phi_M(x, y, y, y, 1, y) \geq 0,$$

then $x \geq y$ and also, ψ_N denote a set of all continuous, increasing functions $\psi_N; [0,1]^6 \to [0,1]$ in each coordinate such that if

$$\psi_N(x, 0, y, 0, y, 0) \leq 1, \quad \psi_N(x, y, 0, 0, y, y) \leq 1,$$
$$\psi_N(x, 0, 0, y, 0, y) \leq 1, \quad \phi_N(x, y, y, y, 0, y) \leq 1,$$

then $x \leq y$.

3. Common fixed point satisfying weak commuting using implicit relation

In this part, we prove some common fixed point theorem for four mappings satisfying weak commuting using implicit relation conditions.

Theorem 3.1. *Let X be a complete intuitionistic fuzzy metric space with $a * b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$ for all $a, b \in X$. Let A, B, C, D be maps from X into itself satisfying the following conditions:*

(a)$AD(X) \subset DC(X)$, $BC(X) \subset CD(X)$,

(b)the pairs A, C and B, D are weakly commuting mappings,

(c)C and D are continuous,

(d)there exist $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$, where $\phi_M, \psi_N \in \Phi$,

$$\phi_M \left(\begin{array}{c} M(Ax, By, kt), M(Cx, Dy, t), M(Cx, Ax, t), \\ M(Dy, By, t), M(Ax, Dy, t), M(By, Cx, 2t) \end{array} \right) \geq 0,$$

$$\psi_N \left(\begin{array}{c} N(Ax, By, kt), N(Cx, Dy, t), N(Cx, Ax, t), \\ N(Dy, By, t), N(Ax, Dy, t), N(By, Cx, 2t) \end{array} \right) \leq 1.$$

Then A, B, C and D have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Then $ADx_0 \in X$. From (a), since $AD(X) \subset DC(X)$, there exists a point $x_1 \in X$ such that $ADx_0 = DCx_1 = y_0$.Also, since $BC(X) \subset CD(X)$, we can choose a point $x_2 \in X$ for this point x_1 such that $BCx_1 = CDx_2 = y_1$. Inductively, we can define the subsequences $\{x_n\}, \{y_n\} \subset X$ such that $y_{2n} = DCx_{2n+1} = ADx_{2n}$, $y_{2n+1} = CDx_{2n+2} = BCx_{2n+1}$ for $n = 0, 1, 2, \cdots$. We put $e_n(t) = M(y_n, y_{n+1}, t)$, $f_n(t) = N(y_n, y_{n+1}, t)$ for all $n \geq 0$.

Now, replacing $x = Dx_{2n}$ and $y = Cx_{2n+1}$ in (d), we have

$$\phi_M \left(\begin{array}{c} M(y_{2n}, y_{2n+1}, kt), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t), \\ M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n}, t), M(y_{2n+1}, y_{2n-1}, 2t) \end{array} \right) \geq 0,$$

$$\psi_N \left(\begin{array}{c} N(y_{2n}, y_{2n+1}, kt), N(y_{2n-1}, y_{2n}, t), N(y_{2n}, y_{2n-1}, t), \\ N(y_{2n+1}, y_{2n}, t), N(y_{2n}, y_{2n}, t), N(y_{2n+1}, y_{2n-1}, 2t) \end{array} \right) \leq 1.$$

Since we have

$$\begin{aligned} M(y_{2n+1}, y_{2n-1}, 2t) &\geq M(y_{2n+1}, y_{2n}, t) * M(y_{2n}, y_{2n-1}, t) \\ &= \min\{M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t)\}, \\ N(y_{2n+1}, y_{2n-1}, 2t) &\leq N(y_{2n+1}, y_{2n}, t) \diamond N(y_{2n}, y_{2n-1}, t) \\ &= \max\{N(y_{2n+1}, y_{2n}, t), N(y_{2n}, y_{2n-1}, t)\}, \end{aligned}$$

it follows from above inequality and $\phi_M, \psi_N \in \Phi$ that

$$\phi_M(e_{2n}(kt), e_{2n-1}(t), e_{2n-1}(t), e_{2n}(t), 1, \min\{e_{2n}(t), e_{2n-1}(t)\}) \geq 0,$$
$$\psi_N(f_{2n}(kt), f_{2n-1}(t), f_{2n-1}(t), f_{2n}(t), 0, \max\{f_{2n}(t), f_{2n-1}(t)\}) \leq 1.$$

If $e_{2n}(t) \leq e_{2n-1}(t)$ and $f_{2n}(t) \geq f_{2n-1}(t)$, then we have

$$\phi_M(e_{2n}(kt), e_{2n-1}(t), e_{2n-1}(t), e_{2n}(t), 1, e_{2n}(t)) \geq 0,$$
$$\psi_N(f_{2n}(kt), f_{2n-1}(t), f_{2n-1}(t), f_{2n}(t), 0, f_{2n}(t)\}) \leq 1.$$

And so, since $\phi_M, \psi_N \in \Phi$,

$$\phi_M(e_{2n}(kt), e_{2n}(t), e_{2n}(t), e_{2n}(t), 1, e_{2n}(t)) \geq 0,$$
$$\psi_N(f_{2n}(kt), f_{2n}(t), f_{2n}(t), f_{2n}(t), 0, f_{2n}(t)\}) \leq 1.$$

Also, since $e_{2n}(kt) \geq e_{2n}(t)$ and $f_{2n}(kt) \leq f_{2n}(t)$, from Lemma 2.4, $y_{2n} = y_{2n+1}$. i.e., $\{y_n\}$ is a Cauchy sequence in X. Otherwise, we can obtain $e_{2n}(kt) \geq e_{2n-1}(t)$ and $f_{2n}(kt) \leq f_{2n-1}(t)$. Therefore, it follows that $e_{n+1}(kt) \geq e_n(t)$ and $f_{n+1}(kt) \leq f_n(t)$ for all $n \geq 1$. That is,

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t), \quad N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t),$$

which implies that

$$\begin{aligned} M(y_n, y_{n+1}, kt) &\geq M(y_{n-1}, y_n, \frac{t}{k}) \geq \cdots \geq M(y_0, y_1, \frac{t}{k^n}), \\ N(y_n, y_{n+1}, kt) &\leq N(y_{n-1}, y_n, \frac{t}{k}) \leq \cdots \leq N(y_0, y_1, \frac{t}{k^n}). \end{aligned}$$

Hence $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, $\{y_n\}$ converges to a point $x \in X$.

$$\begin{aligned} &\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} ADx_{2n} \\ &= \lim_{n\to\infty} BCx_{2n+1} = \lim_{n\to\infty} CDx_{2n+2} = \lim_{n\to\infty} DCx_{2n+1} = x. \end{aligned}$$

Since D is continuous on X, we have
$\lim_{n\to\infty} DDCx_{2n+1} = \lim_{n\to\infty} DBCx_{2n+1} = Dx$, and so, since B and D are weakly commuting, $\lim_{n\to\infty} DBCx_{2n+1} = Dx$. Also, since C is continuous on X, we have $\lim_{n\to\infty} CCDx_{2n} = \lim_{n\to\infty} CADx_{2n} = Cx$ and so, since A and C are weakly commuting, $\lim_{n\to\infty} ACDx_{2n} = Cx$.

Now, replacing $x = Dx_{2n}$ and $y = DCx_{2n+1}$ in (d), we have, as $n \to \infty$,

$$\phi_M \left(\begin{array}{l} M(x, Dx, kt), M(x, Dx, t), M(x, x, t), \\ M(Dx, Dx, t), M(x, Dx, t), M(Dx, x, t) \end{array} \right) \geq 0,$$

$$\psi_N \left(\begin{array}{l} N(x, Dx, kt), N(x, Dx, t), N(x, x, t), \\ N(Dx, Dx, t), N(x, Dx, t), N(Dx, x, t) \end{array} \right) \leq 1$$

and so, since $\phi_M, \psi_N \in \Phi$, it follows that

$$M(x, Dx, kt) \geq M(x, Dx, t), \quad N(x, Dx, kt) \leq N(x, Dx, t).$$

Hence $Dx = x$.

Now, replacing $x = CDx_{2n}$ and $y = Cx_{2n+1}$ in (d), we have, as $n \to \infty$,

$$\phi_M \left(\begin{array}{l} M(Cx, x, kt), M(Cx, x, t), M(Cx, Cx, t), \\ M(x, x, t), M(Cx, x, t), M(Cx, x, t) \end{array} \right) \geq 0,$$

$$\psi_N \left(\begin{array}{l} N(Cx, x, kt), N(Cx, x, t), N(Cx, Cx, t), \\ N(x, x, t), N(Cx, x, t), N(Cx, x, t) \end{array} \right) \leq 1$$

and so, since $\phi_M, \psi_N \in \Phi$, it follows that

$$M(Cx, x, kt) \geq M(Cx, x, t), \quad N(Cx, x, kt) \leq N(Cx, x, t).$$

Hence $Cx = x$.

By similar method, replacing $x = x$ and $y = Cx_{2n+1}$ in (d), consequently, we have $Ax = x$. Also, replacing $x = Dx_{2n}$ and $y = x$ in (d), we have $Bx = x$. Therefore $Ax = Bx = Cx = Dx = x$. Hence x is a common fixed point of A, B, C and D.

If y is another common fixed point of A, B, C and D, then we have

$$\phi_M \begin{pmatrix} M(Ax, By, kt), M(Cx, Dy, t), M(Cx, Ax, t), \\ M(Dy, By, t), M(Ax, Dy, t), M(By, Cx, 2t) \end{pmatrix} \geq 0,$$

$$\psi_N \begin{pmatrix} N(Ax, By, kt), N(Cx, Dy, t), N(Cx, Ax, t), \\ N(Dy, By, t), N(Ax, Dy, t), N(By, Cx, 2t) \end{pmatrix} \leq 1$$

and so,

$$\phi_M \begin{pmatrix} M(x, y, kt), M(x, y, t), M(x, x, t), \\ M(y, y, t), M(x, y, t), M(y, x, 2t) \end{pmatrix} \geq 0,$$

$$\psi_N \begin{pmatrix} N(x, y, kt), N(x, y, t), N(x, x, t), \\ N(y, y, t), N(x, y, t), N(y, x, 2t) \end{pmatrix} \leq 1$$

Since $\phi_M, \psi_N \in \Phi$, we get

$$M(x, y, kt) \geq M(x, y, t), \quad N(x, y, kt) \leq N(x, y, t).$$

Hence $x = y$. This completes the proof. □

Corollary 3.2. *Let X be a complete intuitionistic fuzzy metric space with $a * b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$ for all $a, b \in X$. Let A, C be maps from X into itself satisfying the following conditions:*

(a)$A(X) \subset C(X)$,

(b)A, C is weakly commuting mappings,

(c)C is continuous on X,

(d)there exist $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$, where $\phi_M, \psi_N \in \Phi$,

$$\phi_M \begin{pmatrix} M(Ax, Ay, kt), M(Cx, Cy, t), M(Cx, Ax, t), \\ M(Cy, Ay, t), M(Ax, Cy, t), M(Ay, Cx, 2t) \end{pmatrix} \geq 0,$$

$$\psi_N \begin{pmatrix} N(Ax, Ay, kt), N(Cx, Cy, t), N(Cx, Ax, t), \\ N(Cy, Ay, t), N(Ax, Cy, t), N(Ay, Cx, 2t) \end{pmatrix} \leq 1.$$

Then A and C have a unique common fixed point in X.

Proof. By above Theorem 3.1, let $A = B$ and $C = D$. Then this completes the proof. □

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