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CONVOLUTION SUM $\sum_{m<n/8}\sigma_1(2m)\sigma_1(n-8m)$

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Abstract. In this paper, we present the convolution sum $\sum_{m<n/8}\sigma_1(2m)\sigma_1$ $(n-8m)$ evaluated for all $n\in\mathbb{N}$.

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and positive integers, respectively. For $N,m,r,s,d\in\mathbb{Z}$ with $d,s>0$ and $r\geq 0$, we defined some necessary divisor functions and infinite products for later use, which appear in many areas of number theory:

$$\sigma_{s,r}(N;m)=\sum_{\substack{d|N\\ d\equiv r \bmod m}} d^s,\qquad \sigma(N):=\sigma_1(N)=\sum_{d|N}d,$$

$$S_1:=\sum_{N\text{ odd}}\sigma_{1,1}(N;2)q^N,\qquad S_2:=\sum_{N\geq 2\text{ even}}\sigma_{1,1}(N;2)q^N,$$

$$(a;q)_\infty:=(a)_\infty:=\prod_{n\geq 0}(1-aq^n),$$

$$A_k(q):=\prod_{n=1}^{\infty}(1+q^n)^{24-4k}(1-q^n)^8(1-q^{4n-2})^{16-2k},$$

$$\sum_{n=1}^{\infty}c_8(n)q^n:=q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4,$$

$$\sum_{n=1}^{\infty}c_{16}(n)q^n:=\frac{1}{32}A_1(q)+\frac{3}{112}A_2(q)+\frac{1}{224}A_3(q)-\frac{1}{32}A_5(q)-\frac{3}{112}A_6(q)-\frac{1}{224}A_7(q),$$

where $q\in\mathbb{C}$ and $|q|<1$. The fundamental knowledge about divisor function is explained well in [8]. In Section 2, using the above notations and the Weierstrass function $\wp(z)$, we briefly comment on the summation $\sum_{k=1}^{N-1}\sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2)$, referred to as $\overline{F}_{1,1}(N):=$

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$\sum_{k=1}^{N-1}\sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2)$, for any integer $N(\geq 2)$, which is important for deducing $\sum_{k<N/4}\sigma_{1,1}(2k;2)\sigma_{1,1}(2N-8k;2)$. The reason why we state the Section 3 is that we are inspired by

$$\sum_{m<n/8}\sigma_1(m)\sigma_1(n-8m)=\frac{1}{192}\sigma_3(n)+\frac{1}{64}\sigma_3(\frac{n}{2})+\frac{1}{16}\sigma_3(\frac{n}{4})+\frac{1}{3}\sigma_3(\frac{n}{8})$$
$$+(\frac{1}{24}-\frac{n}{32})\sigma_1(n)+(\frac{1}{24}-\frac{n}{4})\sigma_1(\frac{n}{8})-\frac{1}{64}c_8(n)$$

in [1, (1.1)] written by A. Alaca, S. Alaca and K. S. Williams. Therefore, we derive the formulae for $\sum_{k<M/8}\sigma_{1,0}(2k;2)\sigma_{1,1}(M-8k;2)$, $\sum_{k<M/8}\sigma_{1,1}(2k;2)\sigma_{1,1}(M-8k;2)$, etc. for odd values of M. This allows us to make the following;
if M (≥ 9) is odd, then

$$\sum_{k<M/8}\sigma_1(2k)\sigma_1(M-8k)=\frac{5}{384}\sigma_3(M)+(\frac{1}{24}-\frac{M}{16})\sigma_1(M)-\frac{3}{64}c_8(M)+\frac{7}{128}c_{16}(M)$$

(see Corollary 3.2). Similarly, for any integer N (≥ 5) we deduce $\sum_{k<N/4}\sigma_{1,0}(2k;2)\sigma_{1,1}(2N-8k;2)$, $\sum_{k<N/4}\sigma_{1,1}(2k;2)\sigma_{1,1}(2N-8k;2)$, etc. Therefore, we can evaluate the summation $\sum_{m<n/8}\sigma_1(2m)\sigma_1(n-8m)$ for any positive integer $n(\geq 9)$. Finally, in Remark 3.7 we show that

$$\sum_{k<N/4}\sigma_{1,0}(2k;2)\sigma_{1,1}(2N-8k;2)=\sum_{k=1}^{n}(k^2-k)=\frac{1}{3}n(n-1)(n+1)$$

for prime $N(=2n+1)$(FIGURE 1).

2. Formulae for $\overline{F}_{1,1}(N)$

Let $\Lambda_\tau=\mathbb{Z}+\tau\mathbb{Z}$ ($\tau\in\mathfrak{H}$ is the complex upper half plane) be a lattice and $z\in\mathbb{C}$. The Weierstrass $\wp$ function relative to Λ_τ is defined by the series

$$\wp(z;\Lambda_\tau)=\frac{1}{z^2}+\sum_{\substack{\omega\in\Lambda_\tau\\ \omega\neq 0}}\left\{\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right\}.$$

Further, the algebraic relation between $\wp(z)$ and $\wp'(z)$ becomes

$$\wp'(z)^2=4\wp(z)^3-g_2(\tau)\wp(z)-g_3(\tau).$$

Proposition 2.1. ([6, p.251]) *Let* $e_1=\wp(\frac{\tau}{2})$, $e_2=\wp(\frac{1}{2})$, *and* $e_3=\wp(\frac{\tau+1}{2})$, *where* $P_0=\prod_{n=1}^{\infty}(1-q^{2n})$, $P_1=\prod_{n=1}^{\infty}(1-q^{2n-1})$, $P_2=\prod_{n=1}^{\infty}(1+q^{2n})$, *and* $P_3=\prod_{n=1}^{\infty}(1+q^{2n-1})$. *Then,*

(a) $e_2 - e_1 = \pi^2 {P_0}^4 {P_3}^8$.
(b) $e_2 - e_3 = \pi^2 {P_0}^4 {P_1}^8$.
(c) $e_3 - e_1 = 2^4\pi^2 q {P_0}^4 {P_2}^8$.

Next, we obtain the identities for $\wp(z)$:(see [4])

(1)
$$\begin{aligned}
\wp(\frac{\tau}{2}) &= -\frac{\pi^2}{3}(\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} + 16\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}) \\
&= -\frac{\pi^2}{3}(1 + 8\sum_{N=1}^{\infty} q^N(2+(-1)^N)\sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 16\sum_{N \text{ odd}}\sigma(N)q^N) \\
&= -\frac{\pi^2}{3}(1 + 8\sum_{N \text{ odd}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 24\sum_{N \text{ even}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 16\sum_{N \text{ odd}}\sigma(N)q^N) \\
&= -\frac{\pi^2}{3}(1 + 24\sum_{N=1}^{\infty} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega) \\
&= -\frac{\pi^2}{3}(1 + 24\sum_{N=1}^{\infty}\sigma_{1,1}(N;2)q^N) \\
&= -\frac{\pi^2}{3}(1 + 24S_1 + 24S_2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\wp(\frac{\tau+1}{2}) &= -\frac{\pi^2}{3}(\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} - 32\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}) \\
&= -\frac{\pi^2}{3}(1 - 24S_1 + 24S_2),
\end{aligned} \tag{2}$$

$$\begin{aligned}
\wp(\frac{1}{2}) &= \frac{2\pi^2}{3}(\frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} - 8\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}) \\
&= \frac{2\pi^2}{3}(1 + 24S_2).
\end{aligned} \tag{3}$$

Consequently, the following can be deduced [7, p. 59]:

$$E_4(\tau) = \frac{2^2 \cdot 3}{(2\pi)^4} g_2(\tau) = \frac{2^2 \cdot 3}{(2\pi)^4}(-4(e_1e_2 + e_2e_3 + e_3e_1))$$
$$= \frac{2^2 \cdot 3}{(2\pi)^4} \cdot \frac{4\pi^4}{9}[3(1 + 24S_2)^2 + 24^2 S_1^2].$$

Further, we can change the degree of q^2 to obtain q in $E_4(\tau')$ with $\tau' = \frac{\tau}{2}$;

(4)

$$E_4(\tau') = 1 + 240q + \sum_{M=2}^{\infty}[48\sigma_{1,1}(2M;2) + 576 \sum_{\substack{k=1 \\ k+l=M}}^{M-1} \sigma_{1,1}(2k;2)\sigma_{1,1}(2l;2)$$
$$+ 192 \sum_{\substack{k=1 \\ k+l-1=M}}^{M} \sigma_{1,1}(2k-1;2)\sigma_{1,1}(2l-1;2)]q^M.$$

From [7, p. 59], we already know that

$$E_4(\tau') = 1 + 240 \sum_{M \geq 1} \sigma_3(M) q^M. \tag{5}$$

In [2, p. 300], Glaisher proved that

(6)

$$\sigma(1)\sigma(2N-1)+\sigma(3)\sigma(2N-3)+\cdots+\sigma(2N-1)\sigma(1) = \frac{1}{8}[\sigma_3(2N)-\sigma_3(N)].$$

From (4), (5), and (6), we deduce following (see [5]):

(7)

$$\overline{F}_{1,1}(N) := \sum_{k=1}^{N-1} \sigma_{1,1}(k;2)\sigma_{1,1}(N-k;2) = \frac{1}{24}[11\sigma_3(N) - \sigma_3(2N) - 2\sigma_{1,1}(N;2)]$$

with $N \geq 2$.

3. Convolution Sums for $\sum_{k<M/8}\sigma_{1,0}(2k;2)\sigma_{1,1}(M-8k;2)$

The purpose of this study is to obtain the convolution sum $\sum_{m<n/8}\sigma_1(2m)\sigma_1(n-8m)$. Hence, we deduce the formulae for both odd n and even n. Let us consider Theorem 3.1 here for odd n.

Theorem 3.1. *Let M be odd positive integers. Then,*

(a)

$$\sum_{k<M/8}\sigma_{1,0}(2k;2)\sigma_{1,1}(M-8k;2)=\frac{1}{96}\sigma_3(M)+(\frac{1}{12}-\frac{M}{16})\sigma_1(M)-\frac{1}{32}c_8(M).$$

(b)

$$\sum_{k<M/8}\sigma_{1,1}(2k;2)\sigma_{1,1}(M-8k;2)=\frac{1}{384}\sigma_3(M)-\frac{1}{24}\sigma_1(M)-\frac{1}{64}c_8(M)+\frac{7}{128}c_{16}(M).$$

(c)

$$\sum_{k<M/8}\sigma_{1,1}(2k;2)\sigma_{1,0}(M-8k;2)=0.$$

(d)

$$\sum_{k<M/8}\sigma_{1,0}(2k;2)\sigma_{1,0}(M-8k;2)=0.$$

Proof. (a) Because $\sigma_{1,0}(2k;2)=2\sigma_1(k)$ and $\sigma_{1,1}(M-8k;2)=\sigma_{1,1}(odd;2)$, we have

$$\sum_{k<M/8}\sigma_{1,0}(2k;2)\sigma_{1,1}(M-8k;2)=2\sum_{k<M/8}\sigma_1(k)\sigma_1(M-8k).$$

Then, we can refer to

$$\sum_{m<n/8}\sigma_1(m)\sigma_1(n-8m)=\frac{1}{192}\sigma_3(n)+\frac{1}{64}\sigma_3(\frac{n}{2})+\frac{1}{16}\sigma_3(\frac{n}{4})+\frac{1}{3}\sigma_3(\frac{n}{8})+(\frac{1}{24}-\frac{n}{32})\sigma_1(n)+(\frac{1}{24}-\frac{n}{4})\sigma_1(\frac{n}{8})-\frac{1}{64}c_8(n)$$

in [1, (1.1)].

(b) It is sufficient to determine $\sum_{k<M/8}\sigma_{1,1}(k;2)\sigma_{1,1}(M-8k;2)$ using $\sigma_{1,1}(2k;2)=\sigma_{1,1}(k;2)$. Therefore, we note that

$$\sum_{k<M/8}\sigma_1(k)\sigma_1(M-8k)=\sum_{k<M/8}\sigma_{1,1}(k;2)\sigma_{1,1}(M-8k;2)+\sum_{k<M/8}\sigma_{1,0}(k;2)\sigma_{1,1}(M-8k;2). \tag{8}$$

The last term in equation (8) becomes

$$\begin{aligned}\sum_{k<M/8}\sigma_{1,0}(k;2)\sigma_{1,1}(M-8k;2)&=\sum_{k<M/16}\sigma_{1,0}(2k;2)\sigma_{1,1}(M-16k;2)\\&=2\sum_{k<M/16}\sigma_1(k)\sigma_1(M-16k).\end{aligned}$$

For this reason, we require that the following condition:

$$\begin{aligned}\sum_{m<n/16}\sigma_1(m)\sigma_1(n-16m)=&\frac{1}{768}\sigma_3(n)+\frac{1}{256}\sigma_3(\frac{n}{2})+\frac{1}{64}\sigma_3(\frac{n}{4})\\&+\frac{1}{16}\sigma_3(\frac{n}{8})+\frac{1}{3}\sigma_3(\frac{n}{16})+(\frac{1}{24}-\frac{n}{64})\sigma_1(n)\\&+(\frac{1}{24}-\frac{n}{4})\sigma_1(\frac{n}{16})-\frac{7}{256}c_{16}(n)\end{aligned}$$

in [1, Theorem 1.1]. Thus, (b) is obtained by the fact that

$$\begin{aligned}\sum_{k<M/8}\sigma_{1,1}(k;2)\sigma_{1,1}(M-8k;2)=&\sum_{k<M/8}\sigma_1(k)\sigma_1(M-8k)\\&-2\sum_{k<M/16}\sigma_1(k)\sigma_1(M-16k).\end{aligned}$$

(c), (d) We use $\sigma_{1,0}(M-8k;2)=\sigma_{1,0}(\text{odd};2)=0$.

□

Let us introduce some values of $c_8(n)$ and $c_{16}(n)$ for $n\in\mathbb{N}$.

n	$c_8(n)$	$c_{16}(n)$	n	$c_8(n)$	$c_{16}(n)$	n	$c_8(n)$	$c_{16}(n)$
1	1	1	26	0	$\frac{264}{7}$	51	-200	$-\frac{13739}{28}$
2	0	$\frac{12}{7}$	27	152	$-\frac{152}{7}$	52	0	$\frac{248757}{28}$
3	-4	$\frac{4}{7}$	28	0	0	53	-242	$\frac{11933}{28}$
4	0	0	29	198	198	54	0	$-\frac{297867}{28}$
5	-2	-2	30	0	$\frac{96}{7}$	55	88	$\frac{8771}{4}$
6	0	$-\frac{48}{7}$	31	-160	$\frac{613}{28}$	56	0	$\frac{487489}{28}$
7	24	$-\frac{24}{7}$	32	0	$-\frac{135}{28}$	57	-176	$\frac{47697}{28}$
8	0	0	33	176	$\frac{4017}{28}$	58	0	$-\frac{448943}{28}$
9	-11	-11	34	0	$\frac{225}{28}$	59	-668	$\frac{56097}{28}$
10	0	$-\frac{24}{7}$	35	-48	$-\frac{495}{28}$	60	0	$\frac{661033}{28}$
11	-44	$\frac{44}{7}$	36	0	$\frac{2857}{28}$	61	550	$-\frac{1404}{7}$
12	0	0	37	-162	$-\frac{4719}{28}$	62	0	$-\frac{235613}{8}$
13	22	22	38	0	$-\frac{3799}{28}$	63	-260	$-\frac{17069}{14}$
14	0	$\frac{288}{7}$	39	-88	$\frac{7545}{28}$	64	0	$\frac{261819}{8}$
15	8	$-\frac{8}{7}$	40	0	$\frac{28349}{28}$	65	-56	$-\frac{23745}{14}$
16	0	0	41	-198	$-\frac{219}{28}$	66	0	$-\frac{2322723}{56}$
17	50	50	42	0	$-\frac{39811}{28}$	67	168	$-\frac{3633}{2}$
18	0	$-\frac{132}{7}$	43	52	$\frac{6253}{28}$	68	0	$\frac{2539949}{56}$
19	44	$-\frac{44}{7}$	44	0	$\frac{60661}{28}$	69	300	$-\frac{29803}{14}$
20	0	0	45	22	$-\frac{18787}{28}$	70	0	$-\frac{483261}{8}$
21	-96	-96	46	0	$-\frac{112747}{28}$	71	760	-5206
22	0	$-\frac{528}{7}$	47	528	$-\frac{13819}{28}$	72	0	$\frac{3170453}{56}$
23	-56	8	48	0	$\frac{109357}{28}$	73	10	$-\frac{34004}{7}$
24	0	0	49	233	$-\frac{23039}{28}$	74	0	$-\frac{4294811}{56}$
25	-121	-121	50	0	$-\frac{207835}{28}$	75	428	-10107

TABLE 1. Examples for odd n $(1 \leq n \leq 75)$

Using Theorem 3.1, the following is deduced.

Corollary 3.2. *If $M(\geq 9)$ is odd, then*

$$\sum_{k<M/8} \sigma_1(2k)\sigma_1(M-8k) = \frac{5}{384}\sigma_3(M) + (\frac{1}{24} - \frac{M}{16})\sigma_1(M) - \frac{3}{64}c_8(M) + \frac{7}{128}c_{16}(M).$$

Proof. We consider the equation

$$\begin{aligned}\sum_{k<M/8} \sigma_1(2k)\sigma_1(M-8k) = &\sum_{k<M/8} \sigma_{1,1}(2k;2)\sigma_{1,1}(M-8k;2)\\ &+ \sum_{k<M/8} \sigma_{1,1}(2k;2)\sigma_{1,0}(M-8k;2)\\ &+ \sum_{k<M/8} \sigma_{1,0}(2k;2)\sigma_{1,1}(M-8k;2)\\ &+ \sum_{k<M/8} \sigma_{1,0}(2k;2)\sigma_{1,0}(M-8k;2)\end{aligned}$$

and use Theorem 3.1.

□

Example 3.3. *The first eleven values of $\sum_{k<M/8} \sigma_1(2k)\sigma_1(M-8k)$ are listed in the following table.*

M	9	11	13	15	17	19	21	23	25	27	29
$\sum_{k<M/8} \sigma_1(2k)\sigma_1(M-8k)$	3	12	18	24	46	64	84	128	157	192	266

TABLE 2. Examples for odd M $(9 \leq M \leq 29)$

To generalize the convolution sum $\sum_{m<n/8} \sigma_1(2m)\sigma_1(n-8m)$, we consider even n in Theorem 3.4.

Theorem 3.4. *Let $N(\geq 5)$ be any set of positive integers. Then,*

(a)

$$\sum_{k<N/4} \sigma_{1,0}(2k;2)\sigma_{1,1}(2N-8k;2)$$
$$= \begin{cases} \frac{1}{24}\{\sigma_3(N) + (2-3N)\sigma_1(N)\}, & \text{for odd } N,\\ \frac{1}{24}\{\sigma_3(N) - 5\sigma_3(\frac{N}{2}) + (2-3N)\sigma_1(N) - 2(2-3N)\sigma_1(\frac{N}{2}) & \\ -16\sigma_3(\frac{N}{4}) - 2\sigma_1(\frac{N}{4})\}, & \text{otherwise.}\end{cases}$$

(b)

$$\sum_{k<N/4}\sigma_{1,1}(2k;2)\sigma_{1,1}(2N-8k;2)$$
$$=\begin{cases}\frac{1}{96}\sigma_3(N)-\frac{1}{24}\sigma_1(N)+\frac{1}{32}c_8(N), & \textit{for odd } N,\\ \frac{1}{24}\{\sigma_3(\frac{N}{2})-\sigma_1(\frac{N}{2})\}, & N=4l+2, l\in\mathbb{N}\ ,\\ \frac{1}{24}\{11\sigma_3(\frac{N}{4})-\sigma_3(\frac{N}{2})-2\sigma_{1,1}(\frac{N}{2};2)\}, & N=4l, l\in\mathbb{N}.\end{cases}$$

(c)

$$\begin{aligned}\sum_{k<N/4}\sigma_{1,1}(2k;2)\sigma_{1,0}(2N-8k;2)=&\frac{1}{48}\sigma_3(N)+\frac{1}{16}\sigma_3(\frac{N}{2})+\frac{5}{12}\sigma_3(\frac{N}{4})\\&-\frac{4}{3}\sigma_3(\frac{N}{8})-\frac{1}{12}\sigma_1(N)+(\frac{1}{12}-\frac{N}{2})\sigma_1(\frac{N}{4})\\&-(\frac{1}{6}-N)\sigma_1(\frac{N}{8})+\frac{1}{16}c_8(N).\end{aligned}$$

(d)

$$\begin{aligned}\sum_{k<N/4}\sigma_{1,0}(2k;2)\sigma_{1,0}(2N-8k;2)=&\frac{1}{12}\{\sigma_3(N)+(2-3N)\sigma_1(N)+3\sigma_3(\frac{N}{2})\\&+16\sigma_3(\frac{N}{4})+2(1-6N)\sigma_1(\frac{N}{4})\}.\end{aligned}$$

Proof. (a)
We have

$$\sum_{k<N/4}\sigma_{1,0}(2k;2)\sigma_{1,1}(2N-8k;2)=2\sum_{k<N/4}\sigma_1(k)\sigma_{1,1}(N-4k;2). \tag{9}$$

First, because $\sigma_{1,1}(N-4k;2)=\sigma_1(N-4k)$ for odd N, (9) becomes

$$\sum_{k<N/4}\sigma_{1,0}(2k;2)\sigma_{1,1}(2N-8k;2)=2\sum_{k<N/4}\sigma_1(k)\sigma_1(N-4k).$$

Then, we can refer to

$$\begin{aligned}\sum_{m<n/4}\sigma_1(m)\sigma_1(n-4m)=&\frac{1}{48}\sigma_3(n)+(\frac{1}{24}-\frac{n}{16})\sigma_1(n)+\frac{1}{16}\sigma_3(\frac{n}{2})+\frac{1}{3}\sigma_3(\frac{n}{4})\\&+(\frac{1}{24}-\frac{n}{4})\sigma_1(\frac{n}{4})\end{aligned}$$

in [1, (2.39)]. Second, for even N, we can consider $N = 2m$ and change (9) as

$$2\sum_{k<N/4}\sigma_1(k)\sigma_{1,1}(N-4k;2) = 2\sum_{k<N/4}\sigma_1(k)(\sigma_1(N-4k)-\sigma_{1,0}(N-4k;2))$$
$$= 2\sum_{k<m/2}\sigma_1(k)\sigma_1(2m-4k) - 4\sum_{k<m/2}\sigma_1(k) \cdot \sigma_1(m-2k)$$

by using $\sigma_{1,0}(2k;2) = 2\sigma_1(k)$. Similarly, we note that

$$\sum_{m<n/2}\sigma_1(m)\sigma_1(n-2m) = \frac{1}{24}\{2\sigma_3(n)+(1-3n)\sigma_1(n)+8\sigma_3(\frac{n}{2}) + (1-6n)\sigma_1(\frac{n}{2})\}$$

in [3, (4.4)].
(b) In

$$\sum_{k<N/4}\sigma_{1,1}(2k;2)\sigma_{1,1}(2N-8k;2) = \sum_{k<N/4}\sigma_{1,1}(k;2)\sigma_{1,1}(N-4k;2), \tag{10}$$

we can track the method of above (a), and hence, we can separate the cases for odd N and even N. For odd N, (10) can be written as

$$\sum_{k<N/4}\sigma_{1,1}(k;2)\sigma_{1,1}(N-4k;2) = \sum_{k<N/4}\sigma_{1,1}(k;2)\sigma_1(N-4k)$$
$$= \sum_{k<N/4}(\sigma_1(k)-\sigma_{1,0}(k;2))\sigma_1(N-4k)$$
$$= \sum_{k<N/4}\sigma_1(k)\sigma_1(N-4k) - 2\sum_{k<N/8}\sigma_1(k) \cdot \sigma_1(N-8k).$$

Likewise, if N is even, i.e., $N = 2m$, then (10) becomes

$$\sum_{k<N/4}\sigma_{1,1}(k;2)\sigma_{1,1}(N-4k;2) = \sum_{k<m/2}\sigma_{1,1}(k;2)\sigma_{1,1}(m-2k;2). \tag{11}$$

Again, let us separately consider the cases for which m is odd and those for which it is even. To obtain the formulae for odd $m(= 2l+1)$, by numbering the index k in (7), we get

$$\sum_{k=1}^{2l}\sigma_{1,1}(k;2)\sigma_{1,1}(2l+1-k;2) = 2\sum_{k=1}^{l}\sigma_{1,1}(2k;2)\sigma_{1,1}(2l+1-2k;2).$$

Hence, we obtain

$$\sum_{k=1}^{l} \sigma_{1,1}(2k;2)\sigma_{1,1}(2l+1-2k;2) = \frac{1}{48}[11\sigma_3(2l+1) - \sigma_3(4l+2) - 2\sigma_1(2l+1)].$$

Finally, we use $\sigma_{1,1}(2k;2) = \sigma_{1,1}(k;2)$ and $\sigma_3(2l) = \sigma_3(2)\sigma_3(l)$. On the other hand, for even $m(=2l)$, (11) becomes

$$\sum_{k<m/2} \sigma_{1,1}(k;2)\sigma_{1,1}(m-2k;2) = \sum_{k<l} \sigma_{1,1}(k;2)\sigma_{1,1}(l-k;2).$$

Thus, we quote (7).
(c) We use

$$\begin{aligned}\sum_{k<N/4} \sigma_{1,1}(2k;2)\sigma_{1,0}(2N-8k;2) =& 2\sum_{k<N/4} \sigma_{1,1}(k;2)\sigma_1(N-4k)\\ =& 2\sum_{k<N/4} (\sigma_1(k) - \sigma_{1,0}(k;2))\sigma_1(N-4k)\\ =& 2\sum_{k<N/4} \sigma_1(k)\sigma_1(N-4k) - 4\sum_{k<N/8} \sigma_1(k)\\ & \cdot \sigma_1(N-8k).\end{aligned}$$

(d)

$$\sum_{k<N/4} \sigma_{1,0}(2k;2)\sigma_{1,0}(2N-8k;2) = 4\sum_{k<N/4} \sigma_1(k)\sigma_1(N-4k)$$

is obtained using $\sigma_{1,0}(2k;2) = 2\sigma_1(k)$. □

Corollary 3.5. *Let M and N be odd integers.*

(a) *In Theorem 3.1, let us consider $\sum_{k<M/8} \sigma_{1,i}(2k;2)\sigma_{1,j}(M-8k;2)$ for $i, j \in \{0,1\}$. If we put*

$$\begin{aligned}\sum_{k<M/8} \sigma_{1,i}(2k;2)\sigma_{1,j}(M-8k;2) = \frac{1}{u}[& a\sigma_3(M) + (b+cM)\sigma_1(M)\\ & + dc_8(M) \quad + ec_{16}(M)]\end{aligned}$$

with $a, b, c, d, e, u \in \mathbb{Z}$, then $a+b+c+d+e = 0$ and $(a, b, c, d, e) = 1$.

(b) *Similarly, for Theorem 3.4, let us consider $\sum_{k<N/4} \sigma_{1,i}(2k;2)\ \sigma_{1,j}(2N-8k;2)$ for $i, j \in \{0,1\}$. If we put the expression*

$$\sum_{k<N/4} \sigma_{1,i}(2k;2)\sigma_{1,j}(2N-8k;2) = \frac{1}{u}[a\sigma_3(N) + (b+cN)\sigma_1(N) + dc_8(N)]$$

with $a, b, c, d, u \in \mathbb{Z}$, *then* $a + b + c + d = 0$.

Let us introduce a familiar formula with prime $N(= 2n+1)$ appearing in Theorem 3.4(a).

Corollary 3.6. *If* N *is prime, i.e., for instance, it is* $2n + 1$, *then*

$$\sum_{k<N/4} \sigma_{1,0}(2k; 2)\sigma_{1,1}(2N - 8k; 2) = \sum_{k=1}^{n}(k^2 - k) = \frac{1}{3}n(n - 1)(n + 1).$$

Proof. Because N is prime, we have $\sigma_3(N) = N^3 + 1$ and $\sigma_1(N) = N + 1$. Therefore,

$$\begin{aligned}\sum_{k<N/4} \sigma_{1,0}(2k; 2)\sigma_{1,1}(2N - 8k; 2) &= \frac{1}{24}\{\sigma_3(N) + (2 - 3N)\sigma_1(N)\}\\ &= \frac{1}{24}\{N^3 + 1 + (2 - 3N)(N + 1)\}\\ &= \frac{1}{24}(N + 1)(N - 1)(N - 3).\end{aligned}$$

Then, we can replace N with the expression $2n+1$ in the above equation. □

Remark 3.7. We make a Table 3 and a Figure 1 about the Corollary 3.6 in the following, so let $f(n) = \sum_{k<(2n+1)/4} \sigma_{1,0}(2k; 2)\sigma_{1,1}(2(2n+1) - 8k; 2)$ and $g(n) = \sum_{k=1}^{n}(k^2 - k) = \frac{1}{3}n(n-1)(n+1)$ with $N = 2n+1$. The values of $f(n)$ and $g(n)$ for $n = 2, 3, \cdots, 15$ are given in the following table.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(n)$	2	8	18	40	70	104	168	240	320	440	562	720	910	1120
$g(n)$	2	8	20	40	70	112	168	240	330	440	572	728	910	1120

TABLE 3. $f(n)$ and $g(n)$ for $2 \leq n \leq 15$

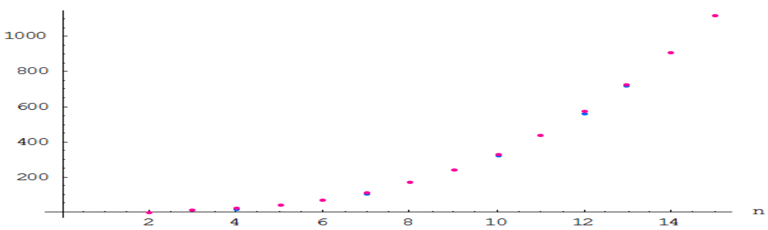

FIGURE 1. $f(n)$ and $g(n)$ for $2 \leq n \leq 15$

In the above picture the blue dot and purple dot represent values of $f(n)$ and $g(n)$, respectively. The purple dot appears only when $f(n)$ and $g(n)$ have the same value for the corresponding n $(2n+1$: prime$)$ while they are dislocated when the values are different.

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