Honam Mathematical J. **34** (2012), No. 1, pp. 001–009
http://dx.doi.org/10.5831/HMJ.2012.34.1.1

ON THE EXTENDED q-EULER NUMBERS AND POLYNOMIALS OF HIGHER-ORDER WITH WEIGHT

Hyun-Mee Kim, Jongsung Choi and Taekyun Kim

Abstract. The purpose of this paper is to give a new construction of the extended q-Euler numbers and polynomials of higher-order with weight by using p-adic q-integral on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, the symbol $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of p-adic integers, the field of rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm of $\mathbb{C}_p$ is defined by $|p|_p = 1/p$. We assume that $\alpha \in \mathbb{Q}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

As an indeterminate, we consider that $q \in \mathbb{C}$ or $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, the we assume thst $|1-q|_p < 1$.

The q-number of x is defined by $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

Recently, the q-Euler numbers with weight α are defined by

$$(1) \qquad \tilde{\mathcal{E}}_{0,q}^{(\alpha)} = 1, \quad \text{and} \quad q\left(q^{\alpha}\tilde{\mathcal{E}}_q^{(\alpha)} + 1\right)^n + \tilde{\mathcal{E}}_{n,q}^{(\alpha)} = 0 \;\text{ if }\; n > 0,$$

with the usual convention about replacing $\left(\tilde{\mathcal{E}}_q^{(\alpha)}\right)^n$ by $\tilde{\mathcal{E}}_{n,q}^{(\alpha)}$ (see[5]).

The q-Euler polynomials with weight α also defined by

$$(2) \qquad \begin{aligned} \tilde{\mathcal{E}}_{n,q}^{(\alpha)}(x) &= \sum_{l=0}^{n} \binom{n}{l} [x]_{q^{\alpha}}^{n} q^{\alpha l x} \tilde{\mathcal{E}}_{l,q}^{(\alpha)} \\ &= \left([x]_{q^{\alpha}} + q^{\alpha x}\tilde{\mathcal{E}}_q^{(\alpha)}\right)^n, \quad \text{for } n \geq 0. \end{aligned}$$

Received September 26, 2011. Accepted October 11, 2011.
2000 Mathematics Subject Classification. 11S80, 11B68.
Key words and phrases. Bernoulli numbers and polynomials, Euler numbers and polynomials, fermionic p-adic integrlal, bosonic p-adic integral.

Let $f \in C(\mathbb{Z}_p)$ = the space of continuous functions on $\mathbb{Z}_p$. Then the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows ([1-16]):

$$\begin{aligned} I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) \\ &= \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \end{aligned} \tag{3}$$

From (3), we have

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} f(l) q^l (-1)^{n-1-l}, \tag{4}$$

where $f_n(x) = f(x+n)$ (see$[1-16]$).

From (2) and (3), we note that

$$\tilde{\mathcal{E}}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [y+x]_{q^\alpha}^n d\mu_{-q}(y). \tag{5}$$

Thus, by (5), we have

$$\tilde{\mathcal{E}}_{n,q}^{(\alpha)}(x) = \frac{[2]_q}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{1}{1+q^{\alpha l+1}}. \tag{6}$$

Note that $\lim_{q\to 1} \tilde{\mathcal{E}}_{n,q}^{(\alpha)}(x) = E_n(x)$ where $E_n(x)$ are the n-th ordinary Euler polynomial which are defined by $\frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$ (see[1-16]).

By using the fermionic multivariate p-adic q-integral on $\mathbb{Z}_p$, we give a new construction of the extended q-Euler numbers and polynomials of higher-order with weight α.

From the extended q-Euler numbers and polynomials of higher-order with weight α, we derive a new explicit formulae by those numbers and polynomials.

2. On the extended q-Euler numbers of higher-order with weight α

In this section, we assume that $h_1, h_2, \cdots, h_k \in \mathbb{Z}_+$ and $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Now we consider a sequence of p-adic rational numbers as expansion of the q-Euler numbers and polynomials of order k with weight α as follows:

$$\tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1, h_2, \cdots, h_k)$$

$$(7) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(h_j-1)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k),$$

and

$$\tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1, h_2, \cdots, h_k|x)$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{\sum_{j=1}^k x_j(h_j-1)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k), \tag{8}$$

By (7) and (8), we get

$$\tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1, h_2, \cdots, h_k) = \frac{[2]_q^k}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{\prod_{j=1}^k (1 + q^{\alpha l + h_j})}, \tag{9}$$

and

$$\begin{aligned}\tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1, h_2, \cdots, h_k|x) &= \frac{[2]_q^k}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{\prod_{j=1}^k (1 + q^{\alpha l + h_j})} \\ &= \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{\mathcal{E}}_{l,q}^{(k,\alpha)}(h_1, h_2, \cdots, h_k).\end{aligned}$$

From (9) we note that

$$\sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{\prod_{j=1}^k (1 + q^{\alpha l + h_j})}$$

$$= \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} (1-q)^{n-l} [\alpha]_q^{n-l} \sum_{s=0}^l \binom{l}{s} (-1)^s \left(\prod_{j=1}^k (1 + q^{\alpha s + h_j}) \right)^{-1}. \tag{10}$$

Therefore, by (10), we obtain the following theorem.

Theorem 1. *Let $h_1, h_2, \cdots h_k \in \mathbb{Z}_+$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \left(\prod_{j=1}^{k} (1 + q^{\alpha l + h_j}) \right)^{-1}$$

$$= \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} (1-q)^{n-l} [\alpha]_q^{n-l} \sum_{s=0}^{l} \binom{l}{s} (-1)^s \left(\prod_{j=1}^{k} (1 + q^{\alpha s + h_j}) \right)^{-1}.$$

By (4), we get

$$q^{h_1} \int_{\mathbb{Z}_p} [x_1 + x + 1]_{q^\alpha}^n q^{(h_1-1)x_1} d\mu_{-q}(x_1)$$

(11) $$= - \int_{\mathbb{Z}_p} [x_1 + x]_{q^\alpha}^n q^{(h_1-1)x_1} d\mu_{-q}(x_1) + [2]_q [x]_{q^\alpha}^n.$$

Therefore, by (11) we obtain the following theorem.

Theorem 2. *For $n \in \mathbb{Z}_+$, we have*

$$q^{h_1} \tilde{\mathcal{E}}_{n,q}^{(1,\alpha)}(h_1 | x+1) + \tilde{\mathcal{E}}_{n,q}^{(1,\alpha)}(h_1 | x) = [2]_q [x]_{q^\alpha}^n.$$

By (8) we get

$$\begin{aligned}
& q^{\alpha x} \tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1 + \alpha, h_2 + \alpha, \cdots, h_k + \alpha | x) \\
= \ & q^{\alpha x} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^\alpha}^n q^{\sum_{j=1}^k x_j (h_j + \alpha - 1)} \\
& d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\
= \ & (q^\alpha - 1) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^\alpha}^{n+1} q^{\sum_{j=1}^k x_j (h_j - 1)} \\
& d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \\
& + \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k + x]_{q^\alpha}^n q^{\sum_{j=1}^k x_j (h_j - 1)} \\
& d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k).
\end{aligned}$$

Thus, we obtain the following theorem.

Theorem 3. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, we have*

$$q^{\alpha x} \tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1 + \alpha, \cdots, h_k + \alpha | x)$$

$$= (q^{\alpha}-1)\tilde{\mathcal{E}}_{n+1,q}^{(k,\alpha)}(h_1,\cdots,h_k|x)+\tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1,\cdots,h_k|x).$$

Let $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$. Then we have

$$\begin{aligned}
&\tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1,\cdots,h_k|x)\\
=\ &\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_k]_{q^{\alpha}}^{n}q^{\sum_{j=1}^{k}x_j(h_j-1)}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k)\\
=\ &\frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q}^{k}}\sum_{a_1,\cdots,a_k=0}^{d-1}q^{\sum_{j=1}^{k}h_ja_j}(-1)^{\sum_{j=1}^{k}a_j}\\
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[\frac{\sum_{j=1}^{k}a_j+x}{d}+\sum_{j=1}^{k}x_j\right]_{q^{\alpha d}}^{n}\times q^{d\sum_{j=1}^{k}x_j(h_j-1)}d\mu_{-q^d}(x_1)\\
&\cdots d\mu_{-q^d}(x_k)\\
=\ &\frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q}^{k}}\sum_{a_1,\cdots,a_k=0}^{d-1}q^{\sum_{j=1}^{k}h_ja_j}(-1)^{\sum_{j=1}^{k}a_j}\tilde{\mathcal{E}}_{n,q^d}^{(k,\alpha)}\\
&\left(h_1,\cdots,h_k|\frac{\sum_{j=1}^{k}a_j+x}{d}\right).
\end{aligned}\tag{12}$$

Therefore, by (12), we obtain the following theroem.

Theorem 4. *For $n\in\mathbb{Z}_+$ and $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$, we have*

$$\begin{aligned}
&\tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1,\cdots,h_k|x)\\
=\ &\frac{[d]_{q^{\alpha}}^{n}}{[d]_{-q}^{k}}\sum_{a_1,\cdots,a_k=0}^{d-1}q^{\sum_{j=1}^{k}h_ja_j}(-1)^{\sum_{j=1}^{k}a_j}\tilde{\mathcal{E}}_{n,q^d}^{(k,\alpha)}\left(h_1,\cdots,h_k|\frac{\sum_{j=1}^{k}a_j+x}{d}\right).
\end{aligned}$$

Let $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$. For $N\in\mathbb{N}$, we get

$$X=X_d=\varprojlim_{N}\mathbb{Z}/dp^N\mathbb{Z},\quad X_1=\mathbb{Z}_p,\quad X^*=\bigcup_{0<a<dp,(a,p)=1}(a+dp\mathbb{Z}_p),$$

$$a+dp^N\mathbb{Z}_p=\{x\in X|x\equiv a \pmod{dp^N}\},$$

where $a\in\mathbb{Z}$ lies in $0\le a<dp^N$.

Let χ be a primitive Dirichlet character with conductor $d\in\mathbb{N}$. Then we consider the generalized q-Bernoulli numbers of order k with weight

α as follows:

$$\begin{aligned}
(13)\qquad & \tilde{\mathcal{E}}_{n,\chi,q}^{(k,\alpha)}(h_1,\cdots,h_k) \\
= & \int_X \cdots \int_X \left(\prod_{i=1}^{k}\chi(x_i)\right)[x_1+\cdots+x_k]_{q^\alpha}^n q^{\sum_{i=1}^k x_i(h_i-1)} \\
& d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_k).
\end{aligned}$$

By (13), we note that

$$\begin{aligned}
& \tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1,\cdots,h_k) \\
= & \frac{[d]_{q^\alpha}^n}{[d]_{-q}^k}\sum_{a_1,\cdots,a_k=0}^{d-1} q^{\sum_{i=1}^k h_i a_i}(-1)^{\sum_{j=1}^k a_j}\left(\prod_{i=1}^k \chi(a_i)\right)\tilde{\mathcal{E}}_{n,q}^{(k,\alpha)} \\
& \left(h_1,\cdots,h_k|\frac{\sum_{j=1}^k a_j}{d}\right).
\end{aligned}$$

Let $F_q^{(k,\alpha|h_1,\cdots,h_k)}(t,x)=\sum_{n=0}^{\infty}\tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1,\cdots,h_k|x)\frac{t^n}{n!}$. Then, by (9), we get

$$\begin{aligned}
& F_q^{(k,\alpha|h_1,\cdots,h_k)}(t,x) \\
= & [2]_q^k \sum_{m_1,\cdots,m_k=0}^{\infty} q^{\sum_{j=1}^k h_j m_j}(-1)^{\sum_{j=1}^k m_j}\sum_{n=0}^{\infty}[m_1+\cdots+m_k+x]_{q^\alpha}^n\frac{t^n}{n!} \\
(14)\quad = & [2]_q^k \sum_{m_1,\cdots,m_k=0}^{\infty} q^{\sum_{j=1}^k h_j m_j}(-1)^{\sum_{j=1}^k m_j}e^{[m_1+\cdots+m_k+x]_{q^\alpha}t}.
\end{aligned}$$

Therefore, by (14), we obtain the following theroem.

Theorem 5. *Let* $F_q^{(k,\alpha|h_1,\cdots,h_k)}(t,x) = \sum_{n=0}^{\infty}\tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1,\cdots,h_k|x)\frac{t^n}{n!}$. *Then we have*

$$F_q^{(k,\alpha|h_1,\cdots,h_k)}(t,x)=[2]_q^k\sum_{m_1,\cdots,m_k=0}^{\infty}q^{\sum_{j=1}^k h_j m_j}(-1)^{\sum_{j=1}^k m_j}e^{[m_1+\cdots+m_k+x]_{q^\alpha}t}.$$

3. Further Remark

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. As well known definition, the gamma function is defined by

$$(15)\ \Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt, \quad \text{where} \quad s \in \mathbb{C} \quad \text{with} \quad Re(s) > 0.$$

From (15), we have

$$\Gamma(s+1) = s\Gamma(s), \quad \text{and} \quad \Gamma(n+1) = n! \quad (n \in \mathbb{N}).$$

In $\mathbb{C}$, the extended q-Euler polynomials of order k with weight α are given by

$$\begin{aligned} F_q^{(k,\alpha|h_1,\cdots,h_k)}(t,x) &= [2]_q^k \sum_{m_1,\cdots,m_k=0}^{\infty} q^{\sum_{j=1}^k h_j m_j} (-1)^{\sum_{j=1}^k m_j} e^{[m_1+\cdots+m_k+x]_{q^\alpha} t} \\ (16) \qquad &= \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1,\cdots,h_k|x) \frac{t^n}{n!}. \end{aligned}$$

For $s \in \mathbb{C}$, it is easy to show that

$$\begin{aligned} &\frac{1}{\Gamma(s)} \int_0^1 F_q^{(k,\alpha|h_1,\cdots,h_k)}(-t,x) t^{s-1} dt \\ (17) \qquad &= [2]_q^k \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{q^{\sum_{j=1}^k h_j m_j} (-1)^{\sum_{j=1}^k m_j}}{[m_1+\cdots+m_k+x]_{q^\alpha}^s}, \end{aligned}$$

where $x \neq 0, -1, -2, \cdots$.

From (17), we can define the multiple q-Euler Zeta functio with weight α as follows: For $s \in \mathbb{C}$, define

$$(18)\ \zeta_q^{(k,\alpha)}(h_1,\cdots,h_k|s,x) = [2]_q^k \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{q^{\sum_{j=1}^k h_j m_j} (-1)^{\sum_{j=1}^k m_j}}{[m_1+\cdots+m_k+x]_{q^\alpha}^s},$$

where $x \neq 0, -1, -2, \cdots$.

Note that $\zeta_q^{(k,\alpha)}(h_1,\cdots,h_k|s,x)$ is analytic function in whole complex s-plane. By using (16), (17), (18), and Laurent series, we obtain the following theroem.

Theorem 6. *For $n \in \mathbb{Z}_+$, we have*

$$\zeta_q^{(k,\alpha)}(h_1,\cdots,h_k|-n,x) = \tilde{\mathcal{E}}_{n,q}^{(k,\alpha)}(h_1,\cdots,h_k|x).$$

References

[1] M. Can, M. Cenkci, V. Kurt, Y. Simsek, *Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-functions*, Adv. Stud. Contemp. Math. **18** (2009), 135-160.

[2] D. Ding, J. Yang, *Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials*, Adv. Stud. Contemp. Math. **20** (2010), 7-21.

[3] C. Chao-Ping, L. Lin, *An inequality for the generalized-Euler-constant function*, Adv. Stud. Contemp. Math. **17** (2008), 105-107.

[4] M. Cenkci, *The p-adic generalized twisted (h, q)-Euler-l-function and its applications*, Adv. Stud. Contemp. Math. **15** (2007), 37-47.

[5] K-W. Hwang, D.V. Dology, T. Kim, S.H. Lee, *On the higher-order q-Euler numbers and polynomials with weight α*, Discrete Dynamics in Nature and Society **2011** (2011), Article ID 354329, 12pp.

[6] T. Kim, *New approach to q-Euler polynomials of higher order*, Russ. J. Math. Phys. **17** (2010), 218-225.

[7] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$* , Russ. J. Math. Phys. **16** (2009), 484-491.

[8] I. N. Cangul, V. Kurt,H. Ozden, Y. Simsek, *On the higher-order $w-q$-Genocchi numbers*, Adv. Stud. Contemp. Math. **19** (2009), 39-57.

[9] H. Ozden,I.N. Cangul, Y. Simsek, *Remarks on q-Bernoulli numbers associated with their interpolation functions*, Adv. Stud. Contemp. Math. **18** (2009), 41-48.

[10] Y. Simsek, *Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions*,Adv. Stud. Contemp. Math. **16** (2008), 251-278.

[11] C.S. Ryoo, Y.H. Kim, *A numerical investigation on the structure of the roots of the twisted q-Euler polynomials*, Adv. Stud. Contemp. Math. **19** (2009), 131-141.

[12] C.S. Ryoo, *Calculating zeros of the twisted Genocchi polynomials*,Adv. Stud. Contemp. Math. **17** (2008), 147-159.

[13] C.S. Ryoo, H. Song, R.P. Agarwal, *On the roots of the q-analogue of Euler-Barnes' polynomials*, Adv. Stud. Contemp. Math. (2004), 153-163.

[14] S.H. Rim, S.J. Lee, E.J. Moon *On the q-Genocchi numbers and polynomials associated with q-Zeta function*, Proc. Jangjeon Math. Soc. **12** (2009), 261-267.

[15] C. S. Ryoo, *On the generalized Barnes type multiple q-Euler polynomials twisted by ramified roots of unity*, Proc. Jangjeon Math. Soc. **13** (2010), 255-263.

[16] A Bayad, *Modular properties of elliptic Bernoulli and Euler functions*, Adv. Stud. Contemp. Math. **20** (2010), 389-401.

Hyun-Mee Kim
Division of General Education-Mathematics, Kwangwoon University,
Seoul 139-701, Republic of Korea.
E-mail: kagness@khu.ac.kr

Jongsung Choi
Division of General Education-Mathematics, Kwangwoon University,
Seoul 139-701, Republic of Korea.
E-mail: jeschoi@kw.ac.kr

Taekyun Kim
Division of General Education-Mathematics, Kwangwoon University,
Seoul 139-701, Republic of Korea.
E-mail: tkkim@kw.ac.kr