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BLOW UP OF SOLUTIONS WITH POSITIVE INITIAL ENERGY FOR THE NONLOCAL SEMILINEAR HEAT EQUATION

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ABSTRACT. In this paper, we investigate a nonlocal semilinear heat equation with homogeneous Dirichlet boundary condition in a bounded domain, and prove that there exist solutions with positive initial energy that blow up in finite time.

1. INTRODUCTION

In this paper, we consider the initial Dirichlet-boundary problem for nonlocal semilinear heat equation

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = f(u), \quad x \in \Omega, t > 0, \tag{1.1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \tag{1.2}$$

with the initial condition

$$u(x,0) = u_0(x) \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega), \quad x \in \Omega. \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with sufficient smooth boundary, relaxation function $g : R^+ \to R^+$ is a bounded C^1 function and nonlinearity $f(u) \in C(R)$.

Equation (1.1) arises naturally from a variety of mathematical models in engineering and physical science. For example, in the study of heat conduction in materials with memory term, the classical Fourier's law of heat flux is replaced by the following form:

$$q = -d\nabla u - \int_{-\infty}^t \nabla[k(x,t)u(x,\tau)]d\tau, \tag{1.4}$$

where u, d and the integral term represent temperature, diffusion coefficient, and the effect of memory term in the material, respectively. The study of this type of equations has drawn a considerable attention, see [1-5]. In mathematical view, one would expect the integral term in the equation to be dominated by the leading term. Therefore, one can apply the theory of parabolic equations to this type of equations.

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When $g=0$ and the nonlinearity is in the form of power function in equation (1.1), problem (1.1)-(1.3) has been studied by various authors and several results concerning global and non-global existence have been established. For instance, in the early 1970s, Levine [6] introduced concavity method and showed that solutions with negative energy blow up in finite time. Later, this method was improved by Kalantarov et al. [7] to more general situations. Other studies about equations with gradient term in bounded or unbounded domains, we refer to [8,9]. In addition, for quasilinear equations, there are some results about the influence of initial energy on blow-up solutions of initial boundary value problem as well. For instance, Zhao [10] established a nonglobal existence result of blow-up solutions with initial energy satisfying

$$E(0)=\frac{1}{p}\int_{\Omega}|\nabla u_0(x)|^p dx-\int_{\Omega}F(u_0(x))dx\leq -\frac{4(p-1)}{pT(p-2)^2}\int_{\Omega}u_0^2(x)dx,$$

which was generalized by Levine et al. [11]. Existence results of blow-up solutions with nonpositive and positive initial energy, one can see [12,13]. Lately, Messaoudi [14] studied problem (1.1)-(1.3) when there is a memory term (i.e.$g\neq 0$) and the nonlinearity is in the form of power function. He established a blow-up result when initial energy is nonpositive. For more results about relations between energy and blow-up solutions to nonlocal hyperbolic equation, we refer to [15] and references therein.

In the works mentioned above, most problems were supposed that the initial energy is negative or non-positive to ensure the occurrence of blow-up. But to our knowledge, there are few works about the influence of positive initial energy on blow-up solutions of parabolic equation. The aim of this paper is to find sufficient condition of existence of blow-up solutions with positive initial energy when the fully nonlinear term $f(u)$ and relaxation function g satisfy some proper conditions.

In order to show our result, we assume that $f\in C(R)$, $F(u)=\int_0^u f(s)ds$, and

$$\inf\{\int_{\Omega}F(u)dx:|u|=1\}>0. \tag{1.5}$$

Denote by C_* the optimal constant of Sobolev embedding inequality

$$(\int_{\Omega}F(u)dx)^{\frac{1}{r}}\leq C_*\|\nabla u\|_2, u\in W_0^{1,2}(\Omega), \tag{1.6}$$

where $r\in(2,\frac{2N}{N-2}]$ is a fixed constant, that is

$$C_*^{-1}=\inf_{u\in W_0^{1,2}(\Omega),u\neq 0}\frac{\|u\|_2}{(\int_{\Omega}F(u)dx)^{\frac{1}{r}}}.$$

For relaxation function g, we assume that

$$g(s)\geq 0,\ g'(s)\leq 0,\ 1-\int_0^{\infty}g(s)ds=l>0. \tag{1.7}$$

We also set

$$\alpha=B^{-\frac{r}{r-2}},\ E_1=(\frac{1}{2}-\frac{1}{r})B^{-\frac{2r}{r-2}}=(\frac{1}{2}-\frac{1}{r})\alpha^2. \tag{1.8}$$

where $B = C_*/l$.

Our main result is as follows:

Theorem 1.1. *Let $f \in C(R)$ satisfy condition (1.5)(1.6) and*

$$sf(s) \geq rF(s) \geq |s|^r,\ r > 2. \tag{1.9}$$

For g, we let it satisfy (1.7) and

$$\int_0^\infty g(s)ds < \frac{1-c_0}{1-\frac{3}{4}c_0}, \tag{1.10}$$

where $c_0 = \frac{1+(r-2)(\alpha_1/\alpha_2)^r}{r} < 1$. If the initial datum is chosen to ensure that

$$E(0) < E_1 \tag{1.11}$$

and

$$\|\nabla u_0\|_2 > \alpha_1, \tag{1.12}$$

then the strong solution u blows up in finite time.

Remark 1.2. *Our result improves the results of Messaoudi [14].*

2. The proof of Theorem 1.1

In order to prove Theorem 1.1, we first introduce the "modified" energy functional

$$E(t) = \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2}(1 - \int_0^\infty g(s)ds)\|\nabla u(t)\|_2^2 - \int_\Omega F(u)dx, \tag{2.1}$$

where

$$(g \circ v)(t) = \int_0^t g(t-s)\|v(t) - v(s)\|_2^2 ds. \tag{2.2}$$

Multiplying (1.1) with $-u_t$ and integrating over Ω, after some manipulations (see [15]), we get

$$-\frac{d}{dt}E(t) = -\int_0^t g'(t-s)\int_\Omega \frac{1}{2}|\nabla u(s) - \nabla u(t)|^2 dxds + g(t)\int_\Omega \frac{1}{2}|\nabla u(t)|^2 dx + \int_\Omega |u_t|^2 dx \geq 0,$$

for regular solutions, from which we can deduce that

$$\frac{d}{dt}E(t) \leq 0.$$

The same result can be established for almost every t by simple density argument.

Similar to [16], we give a definition for a strong solution of (1.1)-(1.3).

Definition A strong solution of (1.1)-(1.3) is a function $u \in C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega))$, satisfying $\frac{d}{dt}E(t) \leq 0$ and

$$\int_0^t \int_\Omega (\nabla u \cdot \nabla \phi - \int_0^s \nabla u(\tau) \cdot \nabla \phi(s) d\tau + u_t \phi - f(u)\phi) dxds = 0$$

for all $t \in [0,T)$ and all $\phi \in C([0,T); H_0^1(\Omega))$.

Remark 2.1. *The condition* $1-\int_0^\infty g(s)ds = l > 0$ *is necessary to guarantee the parabolicity of system (1.1)-(1.3).*

We prove the following two lemmas by using the idea of Vitillaro in [17].

Lemma 2.2. *Suppose that* u *is a strong solution of (1.1)-(1.3), and* $E(0) < E_1$*,* $\|\nabla u_0\|_2 > \alpha_1$*, then there exists a positive constant* $\alpha_2 > \alpha_1$*, so that*

$$[(1-\int_0^\infty g(s)ds)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)]^{\frac{1}{2}} \geq \alpha_2, \tag{2.3}$$

$$(\int_\Omega rF(u)dx)^{\frac{1}{r}} \geq B\alpha_2. \tag{2.4}$$

Proof. First, by (1.6), we can get

$$\begin{aligned}
E(t) &= \frac{1}{2}(g\circ\nabla u)(t) + \frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 - \int_\Omega F(u)dx \\
&\geq \frac{1}{2}(g\circ\nabla u)(t) + \frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 - \frac{1}{r}B^r\|\nabla u\|_2^r \\
&\geq \frac{1}{2}(g\circ\nabla u)(t) + \frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 \\
&\quad -\frac{1}{r}B^r[(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)]^{\frac{r}{2}} \\
&= \frac{1}{2}\zeta^2 - \frac{B^r}{r}\zeta^r \doteq h(\zeta),
\end{aligned} \tag{2.5}$$

where $\zeta = (1-\int_0^t g(s)ds)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)$.

It is easy to see that h is increasing for $0 < \zeta < \alpha_1$ and decreasing for $\zeta > \alpha_1$; $h(\zeta) \to -\infty$ as $\zeta \to +\infty$ and $h(\alpha_1) = E_1$, where α_1 and E_1 are constants defined in (1.8). Since $E(0) < E_1$, $\|\nabla u_0\|_2 > \alpha_1$, we can know that there exists a constant $\alpha_2 > \alpha_1$ such that $E(0) = h(\alpha_2)$. Then by (2.5), we have $h(\|\nabla u_0\|_2) < E(0) = h(\alpha_2)$, which implies that $\|\nabla u_0\|_2 \geq \alpha_2$.

To establish (2.3), we assume that there exists a $t_0 > 0$ such that

$$[(1-\int_0^{t_0} g(s)ds)\|\nabla u\|_2^2 + (g\circ\nabla u)(t_0)]^{\frac{1}{2}} < \alpha_2.$$

Because of the continuity of $(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)$, we can choose t_0 such that

$$[(1-\int_0^{t_0} g(s)ds)\|\nabla u\|_2^2 + (g\circ\nabla u)(t_0)]^{\frac{1}{2}} > \alpha_1.$$

And from (2.5), we get

$$E(t_0) \geq h([(1-\int_0^{t_0} g(s)ds)\|\nabla u\|_2^2 + (g\circ\nabla u)(t_0)]^{\frac{1}{2}}) > h(\alpha_2) = E(0),$$

which is impossible according to Lemma 1, then (2.3) is established.

It follows from (2.1) that

$$\begin{aligned}\int_\Omega F(u)dx &\geq \frac{1}{2}(g\circ\nabla u)(t)+\frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 - E(0)\\ &\geq \frac{1}{2}\alpha_2^2 - h(\alpha_2) = \frac{B^r}{r}\alpha_2^r,\end{aligned}$$

from which we can draw inequality (2.4), then the proof is complete. □

Setting

$$H(t) = E_1 - E(t),\ t\geq 0, \tag{2.6}$$

we have the following Lemma:

Lemma 2.3. *Suppose that $E(0) < E_1$, then for all $t\geq 0$,*

$$0 < H(0) \leq H(t) \leq \int_\Omega F(u)dx. \tag{2.7}$$

Proof. By $\frac{d}{dt}E(t)\leq 0$, we have

$$\frac{d}{dt}H(t)\geq 0,$$

and thus

$$H(t)\geq H(0) = E_1 - E(0) > 0,\ t\geq 0.$$

From (2.1) and (2.6), we have

$$H(t) = E_1 - \frac{1}{2}(g\circ\nabla u)(t) - \frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 + \int_\Omega F(u)dx.$$

From (2.3) and (2.5), we then obtain that

$$\begin{aligned}&E_1 - \frac{1}{2}[(1-\int_0^t g(s)ds)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)]\\ &\leq E_1 - \frac{1}{2}\alpha_2^2 \leq E_1 - \frac{1}{2}\alpha_1^2 = -\frac{1}{r}\alpha_1^2 < 0,\ \forall t\geq 0,\end{aligned}$$

which guarantees $H(t)\leq \int_\Omega F(u)dx$. The proof is complete. □

Proof of Theorem 1.1 We define $L(t) = \frac{1}{2}\int_\Omega u^2(x,t)dx$ and differentiate L to get

$$\begin{aligned}L'(t) &= \int_\Omega uu_t dx\\ &= \int_\Omega u(\Delta u - \int_0^t g(t-s)\Delta u(x,s)ds + f(u)]dx\\ &= -\int_\Omega |\nabla u|^2dx + \int_0^t g(t-s)\int_\Omega \nabla u(x,t)\cdot\nabla u(x,s)dxds + \int_\Omega uf(u)dx\\ &= -\int_\Omega |\nabla u|^2dx + \int_0^t g(t-s)\int_\Omega \nabla u(x,t)\cdot[\nabla u(x,s)-\nabla u(x,t)]\end{aligned}$$

$$\geq -\int_\Omega |\nabla u|^2 dx - \int_0^t g(t-s)\int_\Omega |\nabla u(x,t)\cdot[\nabla u(x,s)-\nabla u(x,t)]|$$

$$+\int_0^t g(t-s)\|\nabla u(x,t)\|_2^2 ds + r\int_\Omega F(u)dx. \tag{2.8}$$

By using Schwartz inequality, we have,

$$L'(t) \geq r\int_\Omega F(u)dx - (1-\int_0^t g(s)ds)\|\nabla u\|_2^2$$

$$-\int_0^t g(t-s)\|\nabla u(x,t)\|_2\|\nabla u(x,s)-\nabla u(x,t)\|_2 ds, \tag{2.9}$$

Then by using Young inequality to the last term, we get

$$L'(t) \geq r\int_\Omega F(u)dx - (1-\frac{3}{4}\int_0^t g(s)ds)\|\nabla u(x,t)\|_2^2 - (g\circ\nabla u)(t). \tag{2.10}$$

Next from (2.9), we deduce that

$$\|\nabla u(x,t)\|_2^2 = \frac{1}{1-\int_0^t g(s)ds}[2E(t)-(g\circ\nabla u)(t)+2\int_\Omega F(u)dx]$$

. Substitute into (2.10), we arrive at

$$L'(t) \geq r\int_\Omega F(u)dx - 2\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds}E(t) + \frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds}(g\circ\nabla u)(t)$$

$$-2\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds}\int_\Omega F(u)dx - (g\circ\nabla u)(t)$$

$$= r\int_\Omega F(u)dx + 2\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds}(H(t)-E_1)$$

$$+[\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds}-1](g\circ\nabla u)(t) - 2\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds}\int_\Omega F(u)dx. \tag{2.11}$$

By combining (2.4) and Lemma 2.2, we get that

$$L'(t) \geq 2\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds}H(t) + [\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds}-1](g\circ\nabla u)(t)$$

$$+[r-2\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds}]\int_\Omega F(u)dx - 2\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds}\frac{r-2}{2r}\frac{\alpha_1^r}{\alpha_2^r}B^r\alpha_2^r$$

$$\geq \gamma H(t) + C_0\int_\Omega F(u)dx, \tag{2.12}$$

where

$$\gamma = 2\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds} > 0,$$

$$\begin{aligned} C_0 &= r - 2\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds} - (r-2)\frac{\alpha_1^r}{\alpha_2^r}\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds} \\ &= r - [2 + (r-2)\frac{\alpha_1^r}{\alpha_2^r}]\frac{1-\frac{3}{4}\int_0^t g(s)ds}{1-\int_0^t g(s)ds} > 0, \end{aligned}$$

because of $r > 2,\ \alpha_2 > \alpha_1$.

Next we use Hölder inequality to estimate $L^{\frac{r}{2}}(t)$:

$$L^{\frac{r}{2}}(t) \le C\|u\|_r^r \le Cr\int_\Omega F(u)dx, \tag{2.13}$$

then from (2.12), (2.13) and Lemma 2.3, we arrive at

$$L'(t) \ge \beta L^{\frac{r}{2}}(t),\ \beta = \frac{C_0}{Cr}. \tag{2.14}$$

A direct integration of (2.14) from 0 to t yields

$$L^{\frac{r}{2}-1}(t) \ge \frac{1}{L^{1-\frac{r}{2}}(0) - (\frac{r}{2}-1)\beta t}, \tag{2.15}$$

Then $L(t)$ blows up at a finite time $t_* \le \frac{L^{1-\frac{r}{2}(0)}}{(\frac{r}{2}-1)\beta}$, and so does $u(x,t)$. The proof of Theorem 1.1 is complete.

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