Bull. Korean Math. Soc. **49** (2012), No. 1, pp. 145–154
http://dx.doi.org/10.4134/BKMS.2012.49.1.145

THE LINEAR 2-ARBORICITY OF PLANAR GRAPHS WITHOUT ADJACENT SHORT CYCLES

Hong-Yu Chen, Xiang Tan, and Jian-Liang Wu

Abstract. Let G be a planar graph with maximum degree Δ. The linear 2-arboricity $la_2(G)$ of G is the least integer k such that G can be partitioned into k edge-disjoint forests, whose component trees are paths of length at most 2. In this paper, we prove that (1) $la_2(G) \leq \lceil \frac{\Delta}{2} \rceil + 8$ if G has no adjacent 3-cycles; (2) $la_2(G) \leq \lceil \frac{\Delta}{2} \rceil + 10$ if G has no adjacent 4-cycles; (3) $la_2(G) \leq \lceil \frac{\Delta}{2} \rceil + 6$ if any 3-cycle is not adjacent to a 4-cycle of G.

1. Introduction

In this paper, all graphs are finite, simple and undirected. For a real number x, $\lceil x \rceil$ is the least integer not less than x and $\lfloor x \rfloor$ is the largest integer not larger than x. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set, respectively. If $uv \in E(G)$, then u is said to be the neighbor of v, and $N(v)$ is the set of neighbors of v. The degree of a vertex v $d(v) = |N(v)|$, $\delta(G)$ is the minimum degree and $\Delta(G)$ is the maximum degree of G. A k-, k^+- or k^-- vertex is a vertex of degree k, at least k, or at most k, respectively. A k- cycle is a cycle of length k. Two cycles are said to be adjacent if they are incident with a common edge. For $s \geq 2$, an even cycle $C = v_1v_2 \cdots v_{2s}v_1$ is called a 2-alternating cycle if $d(v_1) = d(v_3) = \cdots = d(v_{2s-1}) = 2$.

An edge-partition of a graph G is a decomposition of G into subgraphs $G_1, G_2, \ldots, G_m$ such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_m)$ and $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$. A linear k-forest is a graph in which each component is a path of length at most k. The linear k-arboricity $la_k(G)$ of a graph G is the least integer m such that G can be edge-partitioned into m linear k-forests. Clearly, $la_k(G) \geq la_{k+1}(G)$ for any $k \geq 1$. For extremities, $la_1(G)$ is the edge chromatic number $\chi^{'}(G)$ of G; $la_\infty(G)$ representing the case when component paths have unlimited lengths is the ordinary linear arboricity $la(G)$ of G.

Received September 13, 2010; Revised May 21, 2011.
2010 *Mathematics Subject Classification.* 05C15.
Key words and phrases. planar graph, linear 2-arboricity, cycle.
This work was partially supported by grants 10971121 and 60873207 from NSFC and by grants Y52012-20 from Scientific research Foundation for Talent Introduction of Shanghai Institute of Technology.

The linear k-arboricity of a graph was first introduced by Habib and Péroche [7]. They posed the following conjecture.

Conjecture A. For a graph G of order n and a positive integer i,

$$la_i(G) \leq \begin{cases} \lceil (\Delta n+1)/2\lfloor \frac{in}{i+1} \rfloor \rceil \text{ if } \Delta \neq n-1, \\ \lceil (\Delta n)/2\lfloor \frac{in}{i+1} \rfloor \rceil \text{ if } \Delta = n-1. \end{cases}$$

The linear k-arboricity of cycles, trees, complete graphs and complete bipartite graphs has been determined in [5], [6]. Thomassen [12] proved that $la_k \leq 2$ for a cubic graph G, where $k \geq 5$, and this result is best possible. Chang [3] and Chang et al. [4] investigated the algorithmic aspects of the linear k-arboricity. It was further studied by Bermond et al. [2], Jackson and Wormald [8], and Aldred and Wormald [1]. Lih, Tong and Wang [9] proved that for a planar graph G, $la_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 12$; Moreover, $la_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 6$ if G does not contain 3-cycles. Qian and Wang [11] proved that for a planar graph G without 4-cycles, $la_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 3$. Ma, Wu and Yu [10] proved that for a planar graph G without 5- or 6-cycles, $la_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 6$. For a planar graph G, we will prove that (1) $la_2(G) \leq \lceil \frac{\Delta}{2} \rceil + 8$ if G has no adjacent 3-cycles; (2) $la_2(G) \leq \lceil \frac{\Delta}{2} \rceil + 10$ if G has no adjacent 4-cycles; (3) $la_2(G) \leq \lceil \frac{\Delta}{2} \rceil + 6$ if any 3-cycle is not adjacent to a 4-cycle of G.

2. Main results and their proofs

In the section, we always assume that a planar graph G has always been embedded in the plane. Let G be a planar graph and $F(G)$ be the face set of G. For $f \in F(G)$, the degree of f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k-, k^+- or k^-- face is a face of degree k, at least k, or at most k, respectively. Let $n_i(v)$ denote the number of i-vertices of G adjacent to the vertex v, $q_i(v)$ the number of i-faces of G incident with v. A k-face with consecutive vertices $v_1, v_2, \ldots, v_k$ along its boundary in some direction is often said to be $(d(v_1), d(v_2), \ldots, d(v_k))$-face.

Lemma 1. *Let G be a connected planar graph with $\delta(G) \geq 2$. If G has no adjacent 3-cycles, then G contains an edge xy such that $d(x) + d(y) \leq 11$, or G contains a 2-alternating cycle.*

Proof. Suppose, to the contrary, that G is such a connected planar graph not satisfying the lemma. Then we have

(a) For any vertex v, $q_3(v) \leq \lfloor \frac{d(v)}{2} \rfloor$;

(b) For any vertex v, $n_2(v) + n_3(v) + q_3(v) \leq d(v)$;

(c) Let G_2 be the subgraph induced by the edges incident with the 2-vertices of G, then G_2 is a forest and there exists a matching M such that all 2-vertices in G_2 are saturated.

(a) is obvious. For (b), suppose f is a 3-face incident with v. Since $d(x) + d(y) \geq 12$ for any edge $xy \in E(G)$, f is incident with at most one 5^--vertex.

So v is adjacent to at least $q_3(v)$ 6^+-vertices. Hence, $d(v) - n_2(v) - n_3(v) \geq d(v) - \sum_{i=2}^{5} n_i(v) \geq q_3(v)$.

For (c), since $d(x) + d(y) \geq 12$ for every edge $xy \in E(G)$, every pair of 2-vertices is nonadjacent. Hence, G_2 does not contain any odd cycle. Since G does not contain any 2-alternating cycle, G_2 does not contain any even cycle. So every component of G_2 is a tree and there exists a matching M such that all 2-vertices in G_2 are saturated.

If $uv \in M$ and $d(u) = 2$, we call v the 2-master of u.

By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\text{(1)} \qquad \sum_{v \in V}(d(v) - 4) + \sum_{f \in F}(d(f) - 4) = -4(|V| - |E| + |F|) = -8 < 0.$$

We define ch to be the initial charge. Let $ch(x) = d(x) - 4$ for each $x \in V(G) \cup F(G)$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\text{(2)} \qquad \sum_{x \in V(G) \cup F(G)} ch'(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -8.$$

In the following, we will show that $ch'(x) \geq 0$ for each $x \in V(G) \cup F(G)$, a contradiction to (2), completing the proof.

The discharging rules are defined as follows.

R1-1. Each 2-vertex receives 2 from its 2-master.

R1-2. Each 3-vertex receives $\frac{1}{3}$ from each of its neighbors.

R1-3. If a 3-face f is incident with a 4^--vertex, then f receives $\frac{1}{2}$ from each of another two incident vertices; Otherwise, f receives $\frac{1}{3}$ from each of its incident vertices.

Let f be a face of G. If $d(f) \geq 4$, then $ch'(f) = ch(f) \geq 0$. If $d(f) = 3$, then it is incident with at most one 4^--vertex. It follows that $ch'(f) \geq ch(f) + \min\{2 \times \frac{1}{2}, 3 \times \frac{1}{3}\} = 0$ by R1-3.

Let v be a vertex of G. If $d(v) = 2$, then $ch'(v) = ch(v) + 2 = 0$ by R1-1. If $d(v) = 3$, then $ch'(v) = ch(v) + 3 \times \frac{1}{3} = 0$ by R1-2. If $d(v) = 4$, then $ch'(v) = ch(v) = d(v) - 4 = 0$. If $5 \leq d(v) \leq 8$, then $q_3(v) \leq \lfloor \frac{d(v)}{2} \rfloor$ by (a), it follows that $ch'(v) \geq ch(v) - \frac{1}{2}q_3(v) \geq 0$ by R1-3. If $d(v) = 9$, then $q_3(v) \leq 4$ by (a), and $n_3(v) \leq d(v) - q_3(v)$ by (b). It follows that $ch'(v) \geq ch(v) - \frac{1}{2}q_3(v) - \frac{1}{3}n_3(v) \geq 0$ by R1-2 and R1-3. If $d(v) \geq 10$, then $q_3(v) \leq \lfloor \frac{d(v)}{2} \rfloor$ by (a), and $n_3(v) \leq d(v) - q_3(v) - n_2(v)$ by (b). It follows that $ch'(v) \geq ch(v) - \max\{2 + \frac{1}{2}q_3(v) + \frac{1}{3}(d(v) - q_3(v) - n_2(v)),\ \frac{1}{2}q_3(v) + \frac{1}{3}(d(v) - q_3(v))\} \geq$

$\max\{2+\frac{1}{2}q_3(v)+\frac{1}{3}(d(v)-q_3(v)-1),\ \frac{1}{2}q_3(v)+\frac{1}{3}(d(v)-q_3(v))\}\geq 0$ by R1-1, R1-2 and R1-3.

Hence we complete the proof of the lemma. □

Lemma 2. *Every planar graph G without adjacent 3-cycles has an edge-partition into two forests T_1, T_2 and a subgraph H such that for every $v\in V(G)$, $d_{T_1}(v)\leq\lceil\frac{d_G(v)}{2}\rceil$, $d_{T_2}(v)\leq\lceil\frac{d_G(v)}{2}\rceil$ and $d_H(v)\leq 6$.*

Proof. The proof of the lemma is by induction on the number $|V(G)|+|E(G)|$. For a planar graph G with $|V(G)|+|E(G)|\leq 5$, the lemma holds obviously. For a planar graph G with $|V(G)|+|E(G)|\geq 6$, if $\Delta(G)\leq 6$, then let $H=G$ and $T_1=T_2=\emptyset$, the result holds.

Suppose that $\Delta(G)\geq 7$. We may assume that G is a connected planar graph. By the induction, if G' is a proper subgraph of G, the lemma is true for the graph G', that is, G' has an edge-partition into two forests T_1', T_2' and a subgraph H' such that for every $v\in V(G')$, $d_{H'}(v)\leq 6$ and $d_{T_i'}(v)\leq\lceil\frac{d_{G'}(v)}{2}\rceil$ for $i=1,2$. We will choose an appropriate subgraph G' to extend $T_1'\cup T_2'\cup H'$ to an edge-partition $T_1\cup T_2\cup H$ of G satisfying the lemma.

We now consider two cases according to the minimum degree of G.

Case 1. $\delta(G)=1$. Let $uv\in E(G)$ and $d_G(u)=1$. Define the graph $G'=G-uv$.

If $d_{H'}(v)\leq 5$, then let $H=H'+uv$ and $T_i=T_i'$ for $i=1,2$. It is easy to see that the result holds.

If $d_{H'}(v)=6$, suppose that $d_{T_1'}(v)\leq d_{T_2'}(v)$. Since $d_{G'}(v)=d_{T_1'}(v)+d_{T_2'}(v)+d_{H'}(v)=d_{T_1'}(v)+d_{T_2'}(v)+6$ and $d_{G'}(v)=d_G(v)-1$, we have $d_{T_1'}(v)\leq\frac{d_G(v)-7}{2}$. Let $T_1=T_1'+uv$, $T_2=T_2'$ and $H=H'$. Thus $d_{T_2}(x)=d_{T_2'}(x)$ and $d_H(x)=d_{H'}(x)$ for all $x\in V(G')$. Moreover, $d_{T_1}(u)=1=\lceil\frac{d_G(u)}{2}\rceil$, $d_{T_1}(v)=1+d_{T_1'}(v)\leq 1+\frac{d_G(v)-7}{2}<\lceil\frac{d_G(v)}{2}\rceil$, and $d_{T_1}(x)=d_{T_1'}(x)$ for all $x\in V(G)\backslash\{u,v\}$.

Case 2. $\delta(G)\geq 2$. By Lemma 1, we only need to consider two subcases.

Subcase 1. G contains an edge $xy\in E(G)$ such that $d_G(x)+d_G(y)\leq 11$.

Define the graph $G'=G-xy$ and assume that $d_{H'}(x)\leq d_{H'}(y)$. If $d_{H'}(y)\leq 5$, let $H=H'+xy$, $T_1=T_1'$ and $T_2=T_2'$, then the lemma holds obviously.

Suppose that $d_{H'}(y)=6$. Then $1\leq d_{G'}(x)\leq 3$ and $d_{T_1'}(y)+d_{T_2'}(y)+d_{G'}(x)\leq 3$. We may assume $d_{T_1'}(x)\leq d_{T_2'}(x)$.

If $d_{G'}(x)=3$, then $y\notin V(T_1')$ and $y\notin V(T_2')$. Let $T_1=T_1'+xy$, $T_2=T_2'$ and $H=H'$. If $d_{G'}(x)=2$, then $x\in V(T_1')$ and $x\in V(T_2')$ since $d_{T_i'}(x)\leq\lceil\frac{d_{G'}(x)}{2}\rceil$ for $i=1,2$. Also note that $y\notin V(T_1')$ or $y\notin V(T_2')$, Assume that $y\notin V(T_1')$. Again let $T_1=T_1'+xy$, $T_2=T_2'$ and $H=H'$. We see that T_1 is a forest and $d_{T_1}(x)=2=\lceil\frac{3}{2}\rceil=\lceil\frac{d_G(x)}{2}\rceil$. If $d_{G'}(x)=1$, then $x\notin V(T_1')$. Let $T_1=T_1'+xy$, $T_2=T_2'$ and $H=H'$. We see that T_1 is a forest and

$d_{T_1}(x) = 1 = \lceil \frac{d_G(x)}{2} \rceil$. Furthermore, $d_{T_1}(y) = d_{T_1'}(y) + 1 \leq 3 < \lceil \frac{d_G(y)}{2} \rceil$, the result holds.

Subcase 2. G contains a 2-alternating cycle $C = v_1 v_2 \cdots v_{2s} v_1$, $s \geq 2$, such that $d_G(v_1) = d_G(v_3) = \cdots = d_G(v_{2s-1}) = 2$.

Define the graph $G' = G - E(C)$. Let $T_1 = T_1' + \{v_1v_2, v_3v_4, \ldots, v_{2s-1}v_{2s}\}$, $T_2 = T_2' + \{v_2v_3, v_4v_5, \ldots, v_{2s}v_1\}$ and $H = H'$. Note that both T_1 and T_2 are forests. Since $d_G(x) = d_{G'}(x) + 2$ for vertices x of the cycle C, we see that $d_{T_1}(v_j) = d_{T_2}(v_j) = 1 = \frac{d_G(v_j)}{2}$ for $j = 1, 3, \ldots, 2s-1$, and $d_{T_i}(v_j) = d_{T_i'}(v_j) + 1 \leq \lceil \frac{d_{G'}(v_j)}{2} \rceil + 1 = \lceil \frac{d_G(v_j)}{2} \rceil$ for $i = 1, 2$ and $j = 2, 4, \ldots, 2s$, the lemma holds. □

The following is a direct consequence of Lemma 2.

Corollary 3. *Every planar graph G without adjacent 3-cycles can be edge-partitioned into two forests T_1, T_2 and a subgraph H such that $\Delta(T_1) \leq \lceil \frac{\Delta(G)}{2} \rceil$, $\Delta(T_2) \leq \lceil \frac{\Delta(G)}{2} \rceil$ and $\Delta(H) \leq 6$.*

Lemma 4. *If a graph G can be edge-partitioned into m subgraphs $G_1, G_2, \ldots, G_m$, then $la_2(G) \leq \sum_{i=1}^{m} la_2(G_i)$.*

The above lemma is obvious since we just need to use disjoint color sets on the G_i's.

Lemma 5 ([5])**.** *For a forest T, we have $la_2(T) \leq \lceil \frac{\Delta(T)+1}{2} \rceil$.*

Lemma 6 ([2])**.** *For a graph G, we have $la_2(G) \leq \Delta(G)$.*

Now we are ready to prove our first main result.

Theorem 7. *If G is a planar graph without adjacent 3-cycles, then $la_2(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 8$.*

Proof. By Corollary 3, G has an edge-partition into two forests T_1, T_2 and a subgraph H such that $\Delta(T_1) \leq \lceil \frac{\Delta(G)}{2} \rceil$, $\Delta(T_2) \leq \lceil \frac{\Delta(G)}{2} \rceil$ and $\Delta(H) \leq 6$. Combining Lemmas 4, 5, 6, we obtain the following sequence of inequalities.

$$\begin{aligned} la_2(G) &\leq la_2(T_1) + la_2(T_2) + la_2(H) \\ &\leq \lceil \frac{\Delta(T_1)+1}{2} \rceil + \lceil \frac{\Delta(T_2)+1}{2} \rceil + \Delta(H) \\ &\leq 2\lceil \frac{\lceil \frac{\Delta(G)}{2} \rceil + 1}{2} \rceil + 6 \\ &\leq (\lceil \frac{\Delta(G)}{2} \rceil + 2) + 6 \\ &= \lceil \frac{\Delta(G)}{2} \rceil + 8. \end{aligned}$$

□

Lemma 8. *Let G be a connected planar graph with $\delta(G) \geq 2$. If G has no adjacent 4-cycles, then G contains an edge xy such that $d(x) + d(y) \leq 13$, or G contains a 2-alternating cycle.*

Proof. Suppose, to the contrary, that G is such a connected planar graph not satisfying the lemma. Then we have

(a) For any vertex v, $q_3(v) \leq \lfloor \frac{2d(v)}{3} \rfloor$;

(b) For any vertex v, $n_2(v) + n_3(v) + \lceil \frac{q_3(v)}{2} \rceil \leq d(v)$;

(c) Let G_2 be the subgraph induced by the edges incident with the 2-vertices of G, then G_2 is a forest and there exists a matching M such that all 2-vertices in G_2 are saturated.

(a) is obvious. For (b), suppose f is a 3-face incident with v. Since $d(x) + d(y) \geq 14$ for any edge $xy \in E(G)$, f is incident with at most one 6^--vertex. So v is adjacent to at least $\lceil \frac{q_3(v)}{2} \rceil$ 7^+-vertices. Hence, $d(v) - n_2(v) - n_3(v) \geq d(v) - \sum_{i=2}^{6} n_i(v) \geq \lceil \frac{q_3(v)}{2} \rceil$.

For (c), it is similar to that of Lemma 1(c).

If $uv \in M$ and $d(u) = 2$, we call v the 2-master of u.

By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v\in V}(d(v)-4) + \sum_{f\in F}(d(f)-4) = -4(|V|-|E|+|F|) = -8 < 0. \tag{3}$$

We define ch to be the initial charge. Let $ch(v) = d(v) - 4$ for each $v \in V(G)$ and $ch(f) = d(f) - 4$ for each $f \in F(G)$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x\in V(G)\cup F(G)} ch'(x) = \sum_{x\in V(G)\cup F(G)} ch(x) = -8. \tag{4}$$

In the following, we will show that $ch'(x) \geq 0$ for each $x \in V(G) \cup F(G)$, a contradiction to (4), completing the proof.

Now, let us introduce the needed discharging rules as follows:

R2-1. Each 2-vertex receives 2 from its 2-master.
R2-2. Each 3-vertex receives $\frac{4}{15}$ from each of its neighbors.
R2-3. If a vertex v is incident with a 5^+-face f, then v receives $\frac{1}{5}$ from f.
R2-4. Each 3-face receives $\frac{1}{2}$ from each of its incident 7^+-vertices.

Let f be a face of G. If $d(f) \geq 5$, then $ch'(f) \geq ch(f) - d(f) \times \frac{1}{5} \geq 0$ by R2-3. If $d(f) = 4$, then $ch'(f) = ch(f) = d(f) - 4 = 0$. If $d(f) = 3$, then it is incident with at least two 7^+-vertices. It follows that $ch'(f) \geq ch(f) + 2 \times \frac{1}{2} = 0$ by R2-4.

Let v be a vertex of G. If $d(v) = 2$, then $ch^{'}(v) = ch(v) + 2 = 0$ by R2-1. If $d(v) = 3$, then v is incident with at least one 5^{+}-face and $ch^{'}(v) \geq ch(v) + \frac{1}{5} + 3 \times \frac{4}{15} = 0$ by R2-2 and R2-3. If $4 \leq d(v) \leq 6$, then $ch^{'}(v) = ch(v) = d(v) - 4 \geq 0$. If $7 \leq d(v) \leq 10$, then v is incident with at most $\lfloor \frac{2d(v)}{3} \rfloor$ 3-faces by (a), it follows that $ch^{'}(v) \geq ch(v) - \frac{1}{2}\lfloor \frac{2d(v)}{3} \rfloor > 0$ by R2-4. If $d(v) = 11$, then $q_3(v) \leq 7$ by (a), and $n_3(v) \leq d(v) - \lceil \frac{q_3(v)}{2} \rceil$ by (b). It follows that $ch^{'}(v) \geq ch(v) - \frac{1}{2}q_3(v) - \frac{4}{15}n_3(v) > 0$ by R2-2 and R2-4. If $d(v) \geq 12$, then $q_3(v) \leq \lfloor \frac{2d(v)}{3} \rfloor$ by (a), and $n_3(v) \leq d(v) - n_2(v) - \lceil \frac{q_3(v)}{2} \rceil$ by (b). It follows that $ch^{'}(v) \geq ch(v) - \max\{2 + \frac{1}{2}q_3(v) + \frac{4}{15}(d(v) - n_2(v) - \lceil \frac{q_3(v)}{2} \rceil),\ \frac{1}{2}q_3(v) + \frac{4}{15}(d(v) - \lceil \frac{q_3(v)}{2} \rceil) \geq ch(v) - \max\{2 + \frac{1}{2}q_3(v) + \frac{4}{15}(d(v) - 1 - \lceil \frac{q_3(v)}{2} \rceil),\ \frac{1}{2}q_3(v) + \frac{4}{15}(d(v) - \lceil \frac{q_3(v)}{2} \rceil) \geq 0$ by R2-1, R2-2 and R2-4.

Hence, we complete the proof of the lemma. □

Using Lemma 8, the next result can be proved analogously to Lemma 2.

Lemma 9. *Every planar graph G without adjacent 4-cycles can be edge-partitioned into two forests T_1, T_2 and a subgraph H such that $\Delta(T_1) \leq \lceil \frac{\Delta(G)}{2} \rceil$, $\Delta(T_2) \leq \lceil \frac{\Delta(G)}{2} \rceil$ and $\Delta(H) \leq 8$.*

Our second main result is the following theorem.

Theorem 10. *If G is a planar graph without adjacent 4-cycles, then $la_2(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 10$.*

Proof. We can prove it using an argument similar to the proof of Theorem 7. □

Lemma 11. *Let G be a connected planar graph with $\delta(G) \geq 2$. If any 3-cycle is not adjacent to a 4-cycle of G, then G contains an edge xy such that $d(x) + d(y) \leq 9$, or G contains a 2-alternating cycle.*

Proof. Suppose, to the contrary, that G is such a connected planar graph not satisfying the lemma. Then we have

(a) Any 3-face is not adjacent to a 3-face;

(b) For any vertex v, $q_3(v) \leq \lfloor \frac{d(v)}{2} \rfloor$;

(c) Let G_2 be the subgraph induced by the edges incident with the 2-vertices of G, then G_2 is a forest and there exists a matching M such that all 2-vertices in G_2 are saturated.

By Euler's formula $|V| - |E| + |F| = 2$, we have

$$(5) \qquad \sum_{v \in V}(2d(v) - 6) + \sum_{f \in F}(d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0.$$

We define ch to be the initial charge. Let $ch(v) = 2d(v) - 6$ for each $v \in V(G)$ and $ch(f) = d(f) - 6$ for each $f \in F(G)$. In the following, we will reassign a new charge denoted by $ch^{'}(x)$ to each $x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x\in V(G)\cup F(G)} ch^{'}(x) = \sum_{x\in V(G)\cup F(G)} ch(x) = -12. \tag{6}$$

In the following, we will show that $ch^{'}(x) \geq 0$ for each $x \in V(G) \cup F(G)$, a contradiction to (6), completing the proof.

The discharging rules are defined as follows.

R3-1. Each 2-vertex receives 2 from its 2-master.

R3-2. Each 5-vertex sends 1 to each of its incident 3-faces, $\frac{1}{2}$ to each of its incident other faces.

R3-3. Each 6^+-vertex sends $\frac{3}{2}$ to each of its incident 3-faces, 1 to each of its incident 4-faces, $\frac{1}{3}$ to each of its incident 5-faces.

In particular, we have

Remark 1. Let $d(v) \geq 6$, $f_1, f_2, \dots, f_d$ be the faces incident with v in a clockwise order. If $d(f_i) = 3$, then $d(f_{i+1}) \geq 5$. v sends at most $\frac{3}{2} + \frac{1}{3} = \frac{11}{6}$ to f_i and f_{i+1}; If $d(f_i) = d(f_{i+1}) = 4$, then v sends 2 to f_i and f_{i+1}.

Let f be a face of G. If $d(f) \geq 6$, then $ch^{'}(f) = ch(f) \geq 0$. If $d(f) = 5$, then it is incident with at most two 4^--vertices. If f is incident with two 4^--vertices, then the other three vertices must be 6^+-vertices. It follows that $ch^{'}(f) \geq ch(f) + 3 \times \frac{1}{3} = 0$ by R3-3. If f is incident with one 4^--vertices, then $ch^{'}(f) \geq ch(f) + 4 \times \frac{1}{3} > 0$ by R3-2 and R3-3. If f is not incident with any 4^--vertices, then $ch^{'}(f) \geq ch(f) + 5 \times \frac{1}{3} > 0$ by R3-2 and R3-3. If $d(f) = 4$, then it is incident with at most two 4^--vertices. If f is incident with at least one 4^--vertex, then f is incident with at least two 6^+-vertices. Hence, $ch^{'}(f) \geq ch(f) + 2 \times 1 = 0$ by R3-3. If f is not incident with 4^--vertices, then f receives at least $\frac{1}{2}$ from each of its incident vertices by R3-2 and R3-3. Hence, $ch^{'}(f) \geq ch(f) + 4 \times \frac{1}{2} = 0$. If $d(f) = 3$, then it is incident with at most one 4^--vertex. If f is incident with one 4^--vertex, then the other two vertices must be 6^+-vertices. Hence, $ch^{'}(f) \geq ch(f) + 2 \times \frac{3}{2} = 0$ by R3-3. Otherwise, f receives at least 1 from each of its incident vertices by R3-2 and R3-3. It follows that $ch^{'}(f) \geq ch(f) + 3 \times 1 = 0$.

Let v be a vertex of G. If $d(v) = 2$, then $ch^{'}(v) = ch(v) + 2 = 0$ by R3-1. If $3 \leq d(v) \leq 4$, then $ch^{'}(v) = ch(v) \geq 0$. If $d(v) = 5$, then v is incident with at most two 3-faces. It follows that $ch^{'}(v) \geq ch(v) - 2 \times 1 - 3 \times \frac{1}{2} > 0$ by R3-2.

By Remark 1, for $d(v) \geq 6$, we only need to consider the case that v is incident with $d(v)$ 4-faces.

If $6 \leq d(v) \leq 7$, then $ch^{'}(v) \geq ch(v) - d(v) \times 1 \geq 0$ by R3-3. If $d(v) \geq 8$, then $ch^{'}(v) \geq ch(v) - 2 - d(v) \times 1 \geq 0$ by R3-1 and R3-3.

Hence we complete the proof of the lemma. □

Using Lemma 11, the next result can be proved analogously to Lemma 2.

Lemma 12. *Let G be a planar graph. If any 3-cycle is not adjacent to a 4-cycle, then G has an edge-partition into two forests T_1, T_2 and a subgraph H such that $\Delta(T_1) \leq \lceil \frac{\Delta(G)}{2} \rceil$, $\Delta(T_2) \leq \lceil \frac{\Delta(G)}{2} \rceil$ and $\Delta(H) \leq 4$.*

Our third main result is the following theorem.

Theorem 13. *If G is a planar graph that any 3-cycle is not adjacent to a 4-cycle, then $la_2(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 6$.*

The proof of Theorem 13 is similar to that of Theorem 7, we omit here.

References

[1] R. E. L. Aldred and N. C. Wormald, *More on the linear k-arboricity of regular graphs*, Australas. J. Combin. **18** (1998), 97–104.

[2] J. C. Bermond, J. L. Fouquet, M. Habib, and B. Péroche, *On linear k-arboricity*, Discrete Math. **52** (1984), no. 2-3, 123–132.

[3] G. J. Chang, *Algorithmic aspects of linear k-arboricity*, Taiwanese J. Math. **3** (1999), no. 1, 73–81.

[4] G. J. Chang, B. L. Chen, H. L. Fu, and K. C. Huang, *Linear k-arboricity on trees*, Discrete Appl. Math. **103** (2000), no. 1-3, 281–287.

[5] B. L. Chen, H. L. Fu, and K. C. Huang, *Decomposing graphs into forests of paths with size less than three*, Australas. J. Combin. **3** (1991), 55–73.

[6] H. L. Fu and K. C. Huang, *The linear 2-arboricity of complete bipartite graphs*, Ars Combin. **38** (1994), 309–318.

[7] M. Habib and B. Péroche, *Some problems about linear arboricity*, Discrete Math. **41** (1982), no. 2, 219–220.

[8] B. Jackson and N. C. Wormald, *On linear k-arboricity of cubic graphs*, Discrete Math. **162** (1996), no. 1-3, 293–297.

[9] K. W. Lih, L. D. Tong, and W. F. Wang, *The linear 2-arboricity of planar graphs*, Graphs Combin. **19** (2003), no. 2, 241–248.

[10] Q. Ma and J. L. Wu, *Planar graphs without 5-cycles or without 6-cycles*, Discrete Math. (2008), doi:10.1016/j.disc.2008.07.033.

[11] J. Qian and W. F. Wang, *The linear 2-arboricity of planar graphs without 4-cycles*, J. Zhejiang Norm. Univ. **29** (2006), no. 2, 121–125.

[12] C. Thomassen, *Two-coloring the edges of a cubic graph such that each monochromatic component is a path of length at most 5*, J. Combin. Theory Ser. B **75** (1999), no. 1, 100–109.

Hong-Yu Chen
School of Sciences
Shanghai Institute of Technology
Shanghai, 201418, P. R. China
E-mail address: hongyuchen86@163.com

Xiang Tan
School of Statistics and Mathematics
Shandong University of Finance
Jinan, Shandong, 250014, P. R. China
E-mail address: xtan@ruyi.com

Jian-Liang Wu
School of Mathematics
Shandong University
Jinan, Shandong, 250100, P. R. China
E-mail address: jlwu@sdu.edu.cn