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선형 음의 사분 종속확률변수에서 가중합에 대한 수렴성 연구

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Convergence of weighted sums of linearly negative quadrant dependent random variables

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Abstract

We in this paper discuss the strong law of large numbers for weighted sums of arrays of rowwise *LNQD* random variables by using a new exponential inequality of *LNQD* r.v.'s under suitable conditions and we obtain one of corollary.

Keywords : Negative associated random variables, Negative quadrant dependent, linearly negative quadrant dependent random variables, AMS 1991 Subject Classification : 60F15

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1. Introduction

Let $\{X_n | n \geq 1\}$ be a sequence of random variables. Hsu and Robbins (1947) introduced the concept of complete convergence of $\{X_n | n \geq 1\}$. A sequence $\{X_n | n \geq 1\}$ of random variable converges to a constant c completely if

$$\sum_{n=1}^{\infty} P(|X_n - c| > \varepsilon) < \infty \text{ for all } \varepsilon > 0.$$

If $X_n \to c$ completely, then the Borel-Cantelli lemma implies that $X_n \to c$ almost surely, but the converse is not true in general.

Hu et al.(1989) had obtained the following result in complete convergence and they had established Theorem A for non identically random variable when no assumption of independence between rows of the array is made.

Theorem A. Let $\{X_{ni} | 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise independent random variables with $EX_{ni} = 0$. Suppose that $\{X_{ni} | 1 \leq i \leq n, n \geq 1\}$ are uniformly bounded by some random variable X. If $E|X|^{2p} < \infty$ for some $1 \leq p \leq 2$, then

$$n^{-1/p} \sum_{i=1}^{n} X_{ni} \to 0 \text{ completely as } n \to \infty \text{ if and only if } E|X_{11}|^{2p} < \infty.$$

In this paper, we discuss the strong law of large numbers for weighted sums of arrays of rowwise $LNQD$ random variables. The main purpose of this paper is to extend and generalize Theorem A to rowwise $LNQD$ r.v.'s with a below concepts. We first recall the definitions and lemmas of negatively associated, negative quadrant dependent and linearly negative quadrant dependent random variables.

Throughout this paper, $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$, where $a_{ni}^{+} = \max(a_{ni}, 0)$, $a_{ni}^{-} = \max(-a_{ni}, , 0)$ and c denote positive constant whose values are unimportant and mat vary at different place.

Definition 1.1 (Joag-Dev & Proschan (1983)). A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be negatively associated(NA) if for every pair of disjoint subset A_1, A_2 of $\{1, 2, \cdots, n\}$,

$$Cov\{f(X_i : i \in A_1), \; g(X_j : j \in A_2)\},$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{X_n, n \geq 1\}$ is NA if every finite subcollection is NA.

Definition 1.2 (Lehmann (1966)). Two random variables X and Y are said to be negative quadrant dependent(NQD) if for any $x, y \in \mathbb{R}$,

$$P(X < x, Y < y) \leq P(X < x)P(Y < y).$$

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be pairwise NQD if X_i and X_j are NQD for all $i, j \in \mathbb{N}^+$ and $i \neq j$.

Lemma 1.1 (Lehmann (1966)). Let X and Y be NQD random variables, then (a) $EXY \leq EXEY$, (b) $P(X > x, Y > y) \leq P(X > x)P(Y > y)$, and (c) If f and g are both nondecreasing (or both nonincreasing)functions, then $f(X)$ and $g(Y)$ are NQD.

Definition 1.3 (Newman (1984)). A sequence $\{X_n, n \geq 1\}$ of random variables is said to be linearly negative quadrant dependent ($LNQD$) if for any disjoint subsets $A, B \subset \mathbb{Z}^+$ and positive $r_j{}'s$,

$$\sum_{k \in A} r_k X_k \text{ and } \sum_{j \in B} r_j X_j \text{ are } NQD.$$

Lemma 1.2. Let $\{X_n, n \geq 1\}$ be a sequence of $LNQD$ random variables with $EX_n = 0$ for each $n \geq 1$, then for any $t > 0$,

$$Ee^{t\sum_{i=1}^{n} X_i} \leq \prod_{i=1}^{n} Ee^{tX_i} \leq e^{\frac{t^2}{2}\sum_{i=1}^{n} EX_i^2 e^{t|X_i|}}.$$

Proof. Noticing that tX_i and $\sum_{j=i+1}^{n} X_j$ are $LNQD$, we know by Definition 1.3, e^{tX_i} and $e^{t\sum_{j=i+1}^{n} X_j}$ are also NQD for $i = 1, 2, \cdots, n-1$. We will prove the first inequality by mathematical induction that

$$Ee^{t\sum_{i=1}^{n} X_i} \leq \prod_{i=1}^{n} Ee^{tX_i} \tag{1.2}$$

First, we observe that

$$\begin{aligned} Ee^{t(X_1 + X_2)} &\leq Ee^{tX_1} Ee^{tX_2} \\ &= \prod_{i=1}^{2} Ee^{tX_i}\text{i} , \end{aligned}$$

where the inequality follows from Lemma 1.1. Thus (1.2) is true for $i = 2$. Assume now that the statement is true for $i = k$. We will show that it is true for $i = k+1$.

$$\begin{aligned} Ee^{t\sum_{i=1}^{k+1}X_i} &= E(e^{t\sum_{i=1}^{k}X_i}e^{tX_{k+1}}) \\ &\le Ee^{t\sum_{i=1}^{k}X_i}Ee^{tX_{k+1}} \\ &\le \prod_{i=1}^{k}Ee^{tX_i}Ee^{tX_{k+1}} \\ &= \prod_{i=1}^{k+1}Ee^{tX_i} \quad . \end{aligned}$$

Next, we will prove the second inequality that

$$\prod_{i=1}^{n}Ee^{tX_i} \le e^{\frac{t^2}{2}\sum_{i=1}^{n}EX_i^2e^{t|X_i|}} .$$

For all $x \in \mathbb{R}$, taking $e^x \le 1 + x + \frac{1}{2}x^2e^{|x|}$ and $EX_i = 0$, we have

$$\begin{aligned} Ee^{tX_i} &\le 1 + tEX_i + \frac{1}{2}t^2EX_i^2e^{t|X_i|} \\ &\le 1 + \frac{1}{2}t^2EX_i^2e^{t|X_i|} \\ &\le e^{\frac{1}{2}t^2EX_i^2e^{t|X_i|}}, \text{ by } 1 + x \le e^x. \end{aligned}$$

Thus, we get that

$$\prod_{i=1}^{n}Ee^{tX_i} \le e^{\frac{t^2}{2}\sum_{i=1}^{n}EX_i^2e^{t|X_i|}} .$$

Newman(1984) introduced the concepts of $LNQD$ r.v.'s. Many authors derived several important properties about $LNQD$ r.v.'s and also discussed some applications in several areas(see Newman(1984), Cai and Roussas(1997), Wang and Zhang(2006), Ko et al.(2007), Wang et al.(2010) among others). The studying of limit theorems for $LNQD$ r.v.'s is of interest, since $LNQD$ r.v.'s are much weaker than independent and NA r.v.'s. Throughout this paper, $a = O(b)$ means that there exists some $C > 0$ such that $a \le C'b$.

2. Main results

Theorem 2.1. Let $\{X_{ni}|1 \le i \le n,\ n \ge 1\}$ be a sequence of rowwise $LNQD$ random variables such that $EX_{ni}=0$. Assume that $\{a_{ni}|1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying

$$\sup_i |a_{ni}| = o(1/\log n).$$

If $EX_{ni}^2 < \infty$ and $\sum_{i=1}^{n} a_{ni}^2 = o(1/\log n)$, then

$$\sum_{n=1}^{\infty} n^{\alpha} P(|\sum_{i=1}^{n} a_{ni}X_{ni}| > \varepsilon) < \infty \ \text{ for all } \varepsilon > 0 \text{ and } \alpha \ge 0 \ .$$

Proof. Since $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$, it suffices to show that

$$\sum_{n=1}^{\infty} n^{\alpha} P(|\sum_{i=1}^{n} a_{ni}^{+}X_{ni}| > \varepsilon) < \infty \ \text{ for any } \varepsilon > 0 \tag{2.1}$$

$$\sum_{n=1}^{\infty} n^{\alpha} P(|\sum_{i=1}^{n} a_{ni}^{-}X_{ni}| > \varepsilon) < \infty \ \text{ for any } \varepsilon > 0 \tag{2.2}$$

Since the proof of (2.2) is similar to (2.1), we prove only (2.1). To prove (2.1), we need only to prove that

$$\sum_{n=1}^{\infty} n^{\alpha} P(\sum_{i=1}^{n} a_{ni}^{+}X_{ni} > \varepsilon) < \infty \ \text{ for any } \varepsilon > 0 \tag{2.3}$$

$$\sum_{n=1}^{\infty} n^{\alpha} P(\sum_{i=1}^{n} a_{ni}^{+}X_{ni} < -\varepsilon) < \infty \ \text{ for any } \varepsilon > 0 \tag{2.4}$$

We first prove (2.3). By the definition of $LNQD$ random variables we know that $\{a_{ni}^{+}X_{ni}\,|1 \le i \le n,\ n \ge 1\}$ is still an array of rowwise $LNQD$ random variables.
Thus using Lemma 1.2, we obtain that for $t = \log n/\varepsilon$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\alpha} P(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} > \varepsilon) &\le \sum_{n=1}^{\infty} n^{\alpha} e^{-\varepsilon t} E e^{t\sum_{i=1}^{n} a_{ni}^{+} X_{ni}} \\ &\le \sum_{n=1}^{\infty} n^{\alpha-1} e^{t^2/2\sum_{i=1}^{n}(a_{ni}^{+})^2} E X_{ni}^2 e^{t\,|a_{ni}^{+} X_{ni}|} \\ &\le \sum_{n=1}^{\infty} n^{\alpha-1} e^{(\log n)^2/\varepsilon^2 o(1)(\log n)} e^{\log n/\varepsilon\ o(1)/\log n} \\ &\le \sum_{n=1}^{\infty} n^{\alpha-1} e^{o(1)\log n/\varepsilon^2 e^{o(1)}} \\ &\le \sum_{n=1}^{\infty} n^{\alpha-1} e^{c\log n} \\ &\le \sum_{n=1}^{\infty} n^{\alpha+c-1} < \infty\end{aligned}$$

provided $\alpha + c < 0$. Thus (2.3) is provided. By replacing X_{ni} by $-X_{ni}$ from the above statement and noticing $\{a_{ni}^{+}(-X_{ni}) | 1 \le i \le n,\ n \ge 1\}$ is still an array of rowwise $LNQD$ random variables, we know that

$$\sum_{n=1}^{\infty} n^{\alpha} P(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} < -\varepsilon) < \infty \ \text{ for any } \varepsilon > 0.$$

Hence, the result follows by (2.3) and (2.4). The proof is complete.

Theorem 2.2. Suppose that $p > 0$ and let $\{X_{ni} | 1 \le i \le n,\ n \ge 1\}$ be a sequence of rowwise $LNQD$ random variables. Suppose that there is a, X such that $P(|X_{ni}| > x) = O(1)P(|X| > x)$ for all i, n and $x \ge 0$. Assume that $\{a_{ni} | 1 \le i \le n,\ n \ge 1\}$ is an array of real numbers satisfying

$$\text{(a) } \max_{1 \le i \le n} |a_{ni}| = O(n^{-1/p}), \quad \text{(b)} \sum_{i=1}^{n} a_{ni}^2 = o(1/\log n).$$

If $E|X|^{2p} < \infty$, then

$$\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni} X_{ni} > \varepsilon) < \infty \ \text{ for any } \varepsilon > 0.$$

Proof. Since $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$, it suffices to show that

$$\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} a_{ni}^{+} X_{ni}| > \varepsilon) < \infty \ \text{ for any } \varepsilon > 0, \qquad (2.5)$$

$$\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} a_{ni}^{-} X_{ni}| < \varepsilon) < \infty \ \text{ for any } \varepsilon > 0. \qquad (2.6)$$

We prove only (2.5), the proof of (2.6) is similar. To prove (2.5), we need only prove that

$$\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} > \varepsilon) < \infty \text{ for any } \varepsilon > 0,$$

$$\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} < -\varepsilon) < \infty \text{ for any } \varepsilon < 0.$$

Without loss of generality, we can assume that $0 < a_{ni}^{+} \leq n^{1/p}$, for all $1 \leq i \leq n$, $n \geq 1$ and let q be a constants such that $0 < p < q < 4q/3 < 2$ and $\alpha = 1/p - 1/q$ for $\alpha > 0$. Let $(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} \geq \varepsilon) \subset \bigcup_{i=1}^{n} (a_{ni}^{+} X_{ni} I(|X_{ni}| \leq n^{1/q} > \varepsilon/2) \bigcup$(there exists i such that $a_{ni}^{+} X_{ni} > \varepsilon/2$ for some i, $1 \leq i \leq n) \bigcup (X_{ni} > n^{1/q}$, $X_{nj} > n^{1/q}$ for at least two different values of i, j, $1 \leq i < j \leq n$).

Then

$$\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} \geq \varepsilon) \leq \sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} I(|X_{ni}| \leq n^{1/q}) > \varepsilon/2)$$
$$+ \sum_{n=1}^{\infty} P(\bigcup_{i=1}^{n} a_{ni}^{+} X_{ni} > \varepsilon/2) + \sum_{n=1}^{\infty} P(\bigcup_{1 \leq i < j \leq n} X_{ni} > n^{1/q}, X_{nj} > n^{1/q})$$
$$=: I_1 + I_2 + I_3 .$$

To prove I_1, we first define that

$$Y_{ni} = X_{ni} I(|X_{ni}| \leq n^{1/q}) + n^{1/q} I(X_{ni} > n^{1/q}) - n^{1/q} I(X_{ni} < -n^{1/q}).$$

Then

$$\sum_{i=1}^{n} a_{ni}^{+} X_{ni} I(|X_{ni}| \leq n^{1/q})$$
$$= \sum_{i=1}^{n} a_{ni}^{+} X_{ni} (Y_{ni} - EY_{ni}) + n^{1/q} \sum_{i=1}^{n} a_{ni}^{+} (I(X_{ni} > n^{1/q}) - P(X_{ni} > n^{1/q}))$$
$$+ n^{1/q} \sum_{i=1}^{n} a_{ni}^{+} (I(X_{ni} < -n^{1/q}) - P(X_{ni} < -n^{1/q})) + \sum_{i=1}^{n} a_{ni}^{+} E X_{ni} I(|X_{ni}| \leq n^{1/q})$$
$$=: I_4 + I_5 + I_6 + I_7 .$$

As to I_4, we consider two cases of (a) $p \geq 1$ and $0 < p < 1$, and note that $\{a_{ni}^{+} | (Y_{ni} - EY_{ni}) | 1 \leq i \leq n, n \geq 1\}$ is still an array of rowwise $LNQD$ random variables by definition and $|a_{ni}^{+} (Y_{ni} - EY_{ni})| \leq 2n^{-\alpha}$.

(a) when $p \geq 1$, note that

$$E|Y_{ni}|^2 \leq E|X_{ni}|^2 I(|X_{ni}| \leq n^{1/q}) + n^{2/q} P(|X_{ni}| > n^{1/q})$$
$$\leq E|X|^2 I(|X| \leq n^{1/q}) + n^{2/q} P(|X| > n^{1/q})$$
$$\leq 2E|X|^2 < \infty$$

which implies that $E|Y_{ni}|^2 < \infty$ since $E|X|^{2p} < \infty$ implies $E|X|^2 < \infty$.

Hence, by using Lemma 1.2, we can obtain that $\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}^{+} X_{ni}(Y_{ni} - EY_{ni}) > \varepsilon) < \infty$.

(b) When $0 < p < 1$, taking $t = n^{\alpha}/2$, we get that

$$\begin{aligned}
I_4 = \ & P(\sum_{i=1}^{n} a_{ni}^{+}(Y_{ni} - EY_{ni}) > \varepsilon) \\
\leq & \ e^{-\varepsilon t} \prod_{i=1}^{n} e^{t a_{ni}^{+}(Y_{ni} - EY_{ni})} \\
\leq & \ e^{-\varepsilon t} e^{t^2/2 \sum_{i=1}^{n} (a_{ni}^{+})^2 EY_{ni}^2 e^{t|a_{ni}^{+}(Y_{ni} - EY_{ni})|}} \\
\leq & \ e^{-\varepsilon t} e^{cn^{2\alpha} \sum_{i=1}^{n} n^{-2/p E(X_{ni})^2 I(|X_{ni}| \leq n^{1/q}) + n^{2/q} P(|X_{ni}| > n^{1/q}) e^{tn^{-\alpha}}}} \\
\leq & \ e^{-\varepsilon t} e^{cn(2/p - 2/q) n^{(1-2/p)} c(1 + n^{2(1-p)/q}) e^{tn^{-\alpha}}} \\
\leq & \ e^{-cn^{\alpha}} e^{cn^{1-2/q+1-2p/q}}
\end{aligned}$$

Which is summable since $\alpha > 0$ and $0 < p < q < 4q/3 < 2$ implies that $1 - 2/q < 0$ and $1 - 2p/q < 0$. Hence, by (a) and (b), for all $0 < p < 2$.

$$\sum_{n=1}^{\infty} P(\sum_{i=1}^{n} a_{ni}^{+} X_{ni}(Y_{ni} - EY_{ni}) > \varepsilon) < \infty.$$

As to I_5, let $Z_{ni} = n^{1/q} a_{ni}^{+} I(X_{ni} > n^{1/q}) - P(X_{ni} > n^{1/q})$.

Then Z_{ni} is still an array of rowwise $LNQD$ random variables by definition and $|Z_{ni}| \leq n^{-\alpha}$ and

$$\begin{aligned}
E|Z_{ni}|^2 \leq & \ n^{2/q}(a_{ni}^{+})^2 P(X_{ni} > n^{1/q}) + P(X_{ni} > n^{1/q}) \\
\leq & \ cn^{2/q} n^{-2/p} E|X|^{2p}/n^{2p/q}.
\end{aligned}$$

So, we get that

$$\begin{aligned}
I_5 \leq & \ e^{-\varepsilon t} \prod_{i=1}^{n} Ee^{tZ_{ni}} \\
\leq & \ e^{-\varepsilon t} e^{t^2/2 \sum_{i=1}^{n} EZ_{ni}^2 e^{t|Z_{ni}|}} \\
\leq & \ e^{-\varepsilon n^{\alpha}} e^{n^{2\alpha}/2 \sum_{i=1}^{n} cn^{2/q - 2/p} E|X|^2/n^{2p/q} e^{n^{\alpha} n^{-\alpha}}} \\
\leq & \ e^{-\varepsilon n^{\alpha} n^{1-2p/q}} \to 0
\end{aligned}$$

Which is summable since $\alpha > 0$ and $1 - 2p/q < 0$. Hence

$$\sum_{n=1}^{\infty} P(n^{1/q}\sum_{i=1}^{n} a_{ni}^{+}(I(X_{ni} > n^{1/q}) - P(X_{ni} > n^{1/q})) > \varepsilon) < \infty.$$

As to I_6, the proof of I_6 is similar to I_5.

Finally, proof of rest I_7, I_2 and I_3 it follows the proof of paper of Baek(2009)

Corollary 2.1. Suppose that $\{X_{n_i} | 1 \le i \le n, n \ge 1\}$ be a sequence of rowwise $LNQD$ random variables such that $P(|X_{ni}| > x) = O(1)P(|X| > x)$ and let $EX_{ni} = 0$ and $E|X|^P < \infty$. Then

$$\sum_{i=1}^{n} X_{ni}/n^{1/2}(\log n)^{1/2+\gamma} \to 0 \quad \text{as } n \to \infty \text{ for any } \gamma > 0.$$

Proof. If $p = 2$ and $a_{ni} = n^{-1/2}(\log)^{-(1/2+\gamma)}$, for $1 \le i \le n$, $n \ge 1$, we can obtain the result of Corollary 2.1.

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