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Delta Closure and Delta Interior in Intuitionistic Fuzzy Topological Spaces

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Abstract

Due to importance of the concepts of θ-closure and δ-closure, it is natural to try for their extensions to fuzzy topological spaces. So, Ganguly and Saha introduced and investigated the concept of fuzzy δ-closure by using the concept of quasi-coincidence in fuzzy topological spaces. In this paper, we will introduce the concept of δ-closure in intuitionistic fuzzy topological spaces, which is a generalization of the δ-closure by Ganguly and Saha.

Key Words: intuitionistic fuzzy, δ-closure, δ-interior

1. Introduction and Preliminaries

The concepts of θ-closure and δ-closure are useful tools in standard topology in the study of H-closed spaces, Katetov's and H-closed extensions, generalizations of the Stone-Weierstrass theorem and others [1, 2, 3, 4, 5, 6]. Due to importance of these concepts, it is natural to try for their extensions to fuzzy topological spaces. So, Ganguly and Saha introduced and investigated the concept of fuzzy δ-closure by using the concept of quasi-coincidence in fuzzy topological spaces [7]. Many researchers investigated properties of closure operators and continuous mappings in the intuitionistic fuzzy topological spaces,[8, 9, 10, 11, 12].

In this paper, we will introduce the concept of δ-closure in intuitionistic fuzzy topological spaces, which is a generalization of the δ-closure by Ganguly and Saha.

Let X be a nonempty set and I the unit interval $[0, 1]$. An *intuitionistic fuzzy set* A in X is an object of the form

$$A = (\mu_A, \gamma_A),$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership and the degree of nonmembership, respectively, and $\mu_A + \gamma_A \leq 1$. Obviously, every fuzzy set μ_A in X is an intuitionistic fuzzy set of the form $(\mu_A, 1 - \mu_A)$.

Throughout this paper, $I(X)$ denotes the family of all intuitionistic fuzzy sets in X, and " IF " stands for " intuitionistic fuzzy. "

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Definition 1.1 ([13]). Let X be a nonempty set, and let the intuitionistic fuzzy sets A and B be of the form $A = (\mu_A, \gamma_A)$, $B = (\mu_B, \gamma_B)$. Then

(1) $A \leq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$,

(2) $A = B$ iff $A \leq B$ and $B \leq A$,

(3) $A^c = (\gamma_A, \mu_A)$,

(4) $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$,

(5) $A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$,

(6) $\underline{0} = (\tilde{0}, \tilde{1})$ and $\underline{1} = (\tilde{1}, \tilde{0})$.

Definition 1.2 ([14]). An *intuitionistic fuzzy topology* on a nonempty set X is a family $\mathcal{T}$ of intuitionistic fuzzy sets in X which satisfies the following axioms:

(1) $\underline{0}, \underline{1} \in \mathcal{T}$,

(2) $G_1 \cap G_2 \in \mathcal{T}$ for any $G_1, G_2 \in \mathcal{T}$,

(3) $\bigcup G_i \in \mathcal{T}$ for any $\{G_i : i \in J\} \subseteq \mathcal{T}$.

In this case the pair $(X, \mathcal{T})$ is called an *intuitionistic fuzzy topological space*, and any intuitionistic fuzzy set in $\mathcal{T}$ is known as an *intuitionistic fuzzy open set* in X.

Definition 1.3 ([14]). Let $(X, \mathcal{T})$ be an intuitionistic fuzzy topological space and A an intuitionistic fuzzy set in X. Then the *intuitionistic fuzzy interior* of A and the *intuitionistic fuzzy closure* of A are defined by

$$\mathrm{cl}(A) = \bigcap\{K \mid A \leq K, K^c \in \mathcal{T}\}$$

and

$$\mathrm{int}(A) = \bigcup\{G \mid G \leq A, G \in \mathcal{T}\}.$$

Theorem 1.4 ([14])**.** For any IF set A in an IF topological space $(X, \mathcal{T})$, we have

$$\text{cl}(A^c) = (\text{int}(A))^c \quad \text{and} \quad \text{int}(A^c) = (\text{cl}(A))^c.$$

Definition 1.5 ([15, 16])**.** Let $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. An *intuitionistic fuzzy point* $x_{(\alpha,\beta)}$ of X is an intuitionistic fuzzy set in X defined by

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) & \text{if} \quad y = x, \\ (0, 1) & \text{if} \quad y \neq x. \end{cases}$$

In this case, x is called the *support* of $x_{(\alpha,\beta)}$, α the *value* of $x_{(\alpha,\beta)}$ and β the *nonvalue* of $x_{(\alpha,\beta)}$. An intuitionistic fuzzy point $x_{(\alpha,\beta)}$ is said to *belong* to an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in X, denoted by $x_{(\alpha,\beta)} \in A$, if $\alpha \leq \mu_A(x)$ and $\beta \geq \gamma_A(x)$.

Remark 1.6. If we consider an IF point $x_{(\alpha,\beta)}$ as an IF set, then we have the relation $x_{(\alpha,\beta)} \in A$ if and only if $x_{(\alpha,\beta)} \leq A$.

Definition 1.7 ([15, 16])**.** Let $x_{(\alpha,\beta)}$ be an intuitionistic fuzzy point in X and $U = (\mu_U, \gamma_U)$ an intuitionistic fuzzy set in X. Suppose further that α and β are nonnegative real numbers with $\alpha + \beta \leq 1$. The intuitionistic fuzzy point $x_{(\alpha,\beta)}$ is said to be *properly contained* in U if $\alpha < \mu_U(x)$ and $\beta > \gamma_U(x)$.

Definition 1.8 ([16, 17])**.** Let $(X, \mathcal{T})$ be an intuitionistic fuzzy topological space.

(1) An intuitionistic fuzzy point $x_{(\alpha,\beta)}$ is said to be *quasi-coincident* with the intuitionistic fuzzy set $U = (\mu_U, \gamma_U)$, denoted by $x_{(\alpha,\beta)}qU$, if $\alpha > \gamma_U(x)$ or $\beta < \mu_U(x)$.

(2) Let $U = (\mu_U, \gamma_U)$ and $V = (\mu_V, \gamma_V)$ be two intuitionistic fuzzy sets in X. Then U and V are said to be *quasi-coincident*, denoted by UqV, if there exists an element $x \in X$ such that $\mu_U(x) > \gamma_V(x)$ or $\gamma_U(x) < \mu_V(x)$.

The word 'not quasi-coincident' will be abbreviated as $\widetilde{q}$.

Proposition 1.9 ([16])**.** Let U, V be IF sets and $x_{(\alpha,\beta)}$ an IF point in X. Then

(1) $U\widetilde{q}V^c \iff U \leq V$,

(2) $UqV \iff U \not\leq V^c$,

(3) $x_{(\alpha,\beta)} \leq U \iff x_{(\alpha,\beta)}\widetilde{q}U^c$,

(4) $x_{(\alpha,\beta)}qU \iff x_{(\alpha,\beta)} \not\leq U^c$.

Theorem 1.10 ([17])**.** Let $x_{(\alpha,\beta)}$ be an IF point in X, and $U = (\mu_U, \gamma_U)$ an IF set in X. Then $x_{(\alpha,\beta)} \in \text{cl}(U)$ if and only if UqN, for any IF q-neighborhood N of $x_{(\alpha,\beta)}$.

Definition 1.11 ([18])**.** Let A be an intuitionistic fuzzy set in an intuitionistic fuzzy topological space $(X, \mathcal{T})$. A is said to be

(1) an *intuitionistic fuzzy semi-open set* of X, if there exists an intuitionistic fuzzy open set B of X such that $B \leq A \leq \text{cl}(B)$.

(2) an *intuitionistic fuzzy regular open set* of X, if $\text{int}(\text{cl}(A)) = A$. The complement of an intuitionistic fuzzy regular open set is said to be an *intuitionistic fuzzy regular closed set*.

Theorem 1.12 ([18])**.** The following are equivalent:

(1) An IF set A is IF semi-open in X.

(2) $A \leq \text{cl}(\text{int}(A))$.

Remark 1.13. Comparing to fuzzy sets, intuitionistic fuzzy sets have some different properties as follows, which are shown in the next examples.

1. $x_{(\alpha,\beta)} \in A \cup B \not\Rightarrow x_{(\alpha,\beta)} \in A$ or $x_{(\alpha,\beta)} \in B$.
2. $x_{(\alpha,\beta)}qA$ and $x_{(\alpha,\beta)}qB \not\Rightarrow x_{(\alpha,\beta)}q(A \cap B)$.

Thus we have a little different results in generalizing the concepts of the fuzzy topological spaces to the intuitionistic fuzzy topological space.

Example 1.14. Let A, B be IF sets on the unit interval $[0, 1]$ defined by

$$\mu_A = \frac{1}{3}\chi_{[0,\frac{1}{2}]}, \quad \gamma_A = \frac{2}{3}\chi_{[0,1]},$$

$$\mu_B = \frac{1}{3}\chi_{[\frac{1}{2},1]}, \quad \gamma_B = \frac{1}{3}\chi_{[0,1]}.$$

Also let $x = \frac{1}{4}$, $(\alpha, \beta) = (\frac{1}{4}, \frac{1}{2})$. Then $x_{(\alpha,\beta)} \in A \cup B$. However $x_{(\alpha,\beta)} \notin A$ and $x_{(\alpha,\beta)} \notin B$.

Example 1.15. Let A, B be IF sets on the unit interval $[0, 1]$ defined by

$$\mu_A = \frac{1}{3}\chi_{[0,\frac{1}{2}]}, \quad \gamma_A = \frac{2}{3}\chi_{[0,1]},$$

$$\mu_B = \frac{1}{3}\chi_{[\frac{1}{2},1]}, \quad \gamma_B = \frac{1}{3}\chi_{[0,1]}.$$

Also let $x = \frac{1}{4}$, $(\alpha, \beta) = (\frac{1}{2}, \frac{1}{4})$. Then $x_{(\alpha,\beta)}qA$ and $x_{(\alpha,\beta)}qB$, but $x_{(\alpha,\beta)}\tilde{q}(A \cap B)$

2. Intuitionistic Fuzzy δ-closure and δ-interior

We introduce the concept of intuitionistic fuzzy δ-closure in intuitionistic fuzzy topological spaces, and compare this concept with the intuitionistic fuzzy θ-closure introduced by Hanafy [8].

Definition 2.1. Let $(X, \mathcal{T})$ be an intuitionistic fuzzy topological space. An intuitionistic fuzzy point $x_{(\alpha,\beta)}$ is said to be an *intuitionistic fuzzy δ-cluster point* of an intuitionistic fuzzy set U if AqU for each intuitionistic fuzzy regular open q-neighborhood A of $x_{(\alpha,\beta)}$. The set of all intuitionistic fuzzy δ-cluster points of U is called the *intuitionistic fuzzy δ-closure* of U and denoted by $\text{cl}_\delta(U)$. An intuitionistic fuzzy set U is said to be an *intuitionistic fuzzy δ-closed set* if $U = \text{cl}_\delta(U)$. The complement of an intuitionistic fuzzy δ-closed set is said to be an *intuitionistic fuzzy δ-open set*.

Definition 2.2. Let $(X, \mathcal{T})$ be an intuitionistic fuzzy topological space, and let U be an intuitionistic fuzzy set in X. The *intuitionistic fuzzy δ-interior* of U is denoted and defined by

$$\text{int}_\delta(U) = (\text{cl}_\delta(U^c))^c.$$

From the above definition, we have the following relations:

(1) $\text{cl}_\delta(U^c) = (\text{int}_\delta(U))^c$,

(2) $(\text{cl}_\delta(U))^c = \text{int}_\delta(U^c)$.

Remark 2.3. For any IF set U, U is an IF δ-open set if and only if $\text{int}_\delta(U) = U$ because U is IF δ-open if and only if U^c is IF δ-closed if and only if $U^c = \text{cl}_\delta(U^c)$ if and only if $U = (\text{cl}_\delta(U^c))^c = \text{int}_\delta(U)$.

Lemma 2.4. (1) For any IF set U in an IF topological space $(X, \mathcal{T})$, $\text{int}(\text{cl}(U))$ is an IF regular open set.

(2) For any IF open set U in an IF topological space $(X, \mathcal{T})$ such that $x_{(\alpha,\beta)}qU$, $\text{int}(\text{cl}(U))$ is an IF regular open q-neighborhood of $x_{(\alpha,\beta)}$.

Proof. (1) Enough to show that $\text{int}(\text{cl}(U)) = \text{int}(\text{cl}(\text{int}(\text{cl}(U))))$. Since $\text{int}(\text{cl}(U)) \leq \text{cl}(\text{int}(\text{cl}(U))$, we have $\text{int}(\text{int}(\text{cl}(U))) \leq \text{int}(\text{cl}(\text{int}(\text{cl}(U))))$. Thus $\text{int}(\text{cl}(U)) \leq \text{int}(\text{cl}(\text{int}(\text{cl}(U))))$. Conversely, since $\text{int}(\text{cl}(U)) \leq \text{cl}(U)$, we have $\text{cl}(\text{int}(\text{cl}(U))) \leq \text{cl}(\text{cl}(U)) = \text{cl}(U)$. Thus $\text{int}(\text{cl}(\text{int}(\text{cl}(U)))) \leq \text{int}(\text{cl}(U))$. Hence $\text{int}(\text{cl}(U))$ is an IF regular open set.

(2) Clearly, $\text{int}(U) \leq \text{int}(\text{cl}(U))$. Since U is an IF open set, we have

$$U = \text{int}(U) \leq \text{int}(\text{cl}(U)).$$

By (1), $\text{int}(\text{cl}(U))$ is an IF regular open set. Therefore $\text{int}(\text{cl}(U))$ is an IF regular open q-neighborhood of $x_{(\alpha,\beta)}$. □

Hanafy [8] showed that $\text{cl}(U) \leq \text{cl}_\theta(U)$ for each IF set U. The next theorem shows that the IF δ-closure defined above is the refined concept which goes between $\text{cl}(U)$ and $\text{cl}_\theta(U)$. However, it has a little different properties compared to the IF θ-closure as in Remark 2.17.

Theorem 2.5. For any IF set U in an IF topological space $(X, \mathcal{T})$,

$$\text{cl}(U) \leq \text{cl}_\delta(U) \leq \text{cl}_\theta(U).$$

Proof. Let $x_{(\alpha,\beta)} \notin \text{cl}_\delta(U)$. Then there exists an IF regular open q-neighborhood A of $x_{(\alpha,\beta)}$ such that $A\widetilde{q}U$. Then A is an IF q-neighborhood of $x_{(\alpha,\beta)}$ such that $A\widetilde{q}U$. By Theorem 1.10, $x_{(\alpha,\beta)} \notin \text{cl}(U)$. Thus $\text{cl}(U) \leq \text{cl}_\delta(U)$.

Let $x_{(\alpha,\beta)} \in \text{cl}_\delta(U)$. Then for each IF regular open q-neighborhood A of $x_{(\alpha,\beta)}$, AqU. Suppose that there exists an IF open q-neighborhood B of $x_{(\alpha,\beta)}$ such that $\text{cl}(B)\widetilde{q}U$. Put $\text{int}(\text{cl}(B)) = G$. By Lemma 2.4, G is an IF regular open q-neighborhood of $x_{(\alpha,\beta)}$. Since $G = \text{int}(\text{cl}(B)) \leq \text{cl}(B)\widetilde{q}U$, $G = \text{int}(\text{cl}(B)) \leq \text{cl}(B) \leq U^c$. G is an IF regular open q-neighborhood of $x_{(\alpha,\beta)}$ such that $G\widetilde{q}U$. This is a contradiction. Therefore, for any IF open q-neighborhood B of $x_{(\alpha,\beta)}$, $\text{cl}(B)qU$. Hence $x_{(\alpha,\beta)} \in \text{cl}_\theta(U)$. □

Corollary 2.6. (1) If U is an IF δ-closed set in an IF topological space $(X, \mathcal{T})$, then U is an IF closed set.

(2) If U is an IF θ-closed set in an IF topological space $(X, \mathcal{T})$, then U is an IF δ-closed set.

Remark 2.7. The converses of Corollary 2.6 do not hold. We will give counterexamples in Example 2.14 and 2.15.

Theorem 2.8. If U is an IF open set in an IF topological space $(X, \mathcal{T})$, then the IF closure and IF δ-closure are the same, i.e. $\text{cl}(U) = \text{cl}_\delta(U)$.

Proof. By Theorem 2.5, it is sufficient to show that $\text{cl}_\delta(U) \leq \text{cl}(U)$. Take any $x_{(\alpha,\beta)} \in \text{cl}_\delta(U)$. Suppose that $x_{(\alpha,\beta)} \notin \text{cl}(U)$. By Theorem 1.10, there exists an IF q-neighborhood G of $x_{(\alpha,\beta)}$ such that $G\widetilde{q}U$. Since $G\widetilde{q}U$, we have $G \leq U^c$. Since U^c is an IF closed set, $\text{cl}(G) \leq \text{cl}(U^c) = U^c$. Therefore, $\text{int}(\text{cl}(G)) \leq \text{int}(U^c) \leq U^c$, i.e. $\text{int}(\text{cl}(G))\widetilde{q}U$. By Lemma 2.4, $\text{int}(\text{cl}(U))$ is an IF regular open q-neighborhood of $x_{(\alpha,\beta)}$ such that $\text{int}(\text{cl}(U))\widetilde{q}U$. Hence $x_{(\alpha,\beta)} \notin \text{cl}_\delta(U)$. □

In fact, the IF closure and the IF δ-closure are the same for any IF semi-open set as follows.

Theorem 2.9. For any IF semi-open set A, $\text{cl}(A) = \text{cl}_\delta(A)$.

Proof. Enough to show that $\text{cl}_\delta(A) \leq \text{cl}(A)$. Take any $x_{(\alpha,\beta)} \in \text{cl}_\delta(A)$. Suppose that $x_{(\alpha,\beta)} \notin \text{cl}(A)$. Then there exists an IF open q-neighborhood V of $x_{(\alpha,\beta)}$ such that $V\widetilde{q}A$. By definition of semi-open set, there exists an IF open set G such that $G \leq A \leq \text{cl}(G)$. Thus $V \leq$

$A^c \leq G^c$. Hence $\text{cl}(V) \leq \text{cl}(A^c) \leq \text{cl}(G^c) = G^c$. Also, $\text{int}(\text{cl}(V)) \leq \text{int}(\text{cl}(A^c)) \leq \text{int}(\text{cl}(G^c)) = \text{int}(G^c) \leq G^c$, i.e. $\text{int}(\text{cl}(V)) \leq G^c$. Therefore $G \leq (\text{int}(\text{cl}(V)))^c$. Hence $A \leq \text{cl}(G) \leq \text{cl}((\text{int}(\text{cl}(V)))^c) = (\text{int}(\text{cl}(V)))^c$ because $(\text{int}(\text{cl}(V)))^c$ is an IF closed set. Thus $\text{int}(\text{cl}(V))\widetilde{q}A$. By Lemma 2.4, $\text{int}(\text{cl}(V))$ is an IF regular open q-neighborhood of $x_{(\alpha,\beta)}$ such that $\text{int}(\text{cl}(V))\widetilde{q}A$. Hence $x_{(\alpha,\beta)} \notin \text{cl}_\delta(A)$. □

Theorem 2.10. Let U and V be two IF sets in an IF topological space $(X, \mathcal{T})$. Then we have the following properties:

(1) $\text{cl}_\delta(\underline{0}) = \underline{0}$,

(2) $U \leq \text{cl}_\delta(U)$,

(3) $U \leq V \Rightarrow \text{cl}_\delta(U) \leq \text{cl}_\delta(V)$,

(4) $\text{cl}_\delta(U) \cup \text{cl}_\delta(V) = \text{cl}_\delta(U \cup V)$,

(5) $\text{cl}_\delta(U \cap V) \leq \text{cl}_\delta(U) \cap \text{cl}_\delta(V)$.

Proof. (1) Obvious.

(2) Since $U \leq \text{cl}(U) \leq \text{cl}_\delta(U)$, $U \leq \text{cl}_\delta(U)$.

(3) Let $x_{(\alpha,\beta)}$ be an IF point in X such that $x_{(\alpha,\beta)} \notin \text{cl}_\delta(V)$. Then there is an IF regular open q-neighborhood A of $x_{(\alpha,\beta)}$ such that $A\widetilde{q}V$. Since $U \leq V$, we have $A\widetilde{q}U$. Therefore $x_{(\alpha,\beta)} \notin \text{cl}_\delta(U)$.

(4) Since $U \leq U \cup V$, $\text{cl}_\delta(U) \leq \text{cl}_\delta(U \cup V)$. Similarly, $\text{cl}_\delta(V) \leq \text{cl}_\delta(U \cup V)$. Hence $\text{cl}_\delta(U) \cup \text{cl}_\delta(V) \leq \text{cl}_\delta(U \cup V)$. To show that $\text{cl}_\delta(U \cup V) \leq \text{cl}_\delta(U) \cup \text{cl}_\delta(V)$, take any $x_{(\alpha,\beta)} \in \text{cl}_\delta(U \cup V)$. Then for any IF regular open q-neighborhood A of $x_{(\alpha,\beta)}$, $Aq(U \cup V)$. Hence, AqU or AqV. Therefore $x_{(\alpha,\beta)} \in \text{cl}_\delta(U)$ or $x_{(\alpha,\beta)} \in \text{cl}_\delta(V)$. Hence $x_{(\alpha,\beta)} \in \text{cl}_\delta(U) \cup \text{cl}_\delta(V)$.

(5) Since $U \cap V \leq U$, $\text{cl}_\delta(U \cap V) \leq \text{cl}_\delta(U)$. Similarly, $\text{cl}_\delta(U \cap V) \leq \text{cl}_\delta(V)$. Therefore $\text{cl}_\delta(U \cap V) \leq \text{cl}_\delta(U) \cap \text{cl}_\delta(V)$. □

In general, finite intersection of IF regular closed sets is not IF regular closed. However, IF δ-closed sets have a nice properties as in the following theorem.

Theorem 2.11. Let $(X, \mathcal{T})$ be an IF topological space. Then the following hold:

(1) Finite union of IF δ-closed sets in X is an IF δ-closed set in X.

(2) Arbitrary intersection of IF δ-closed sets in X is an IF δ-closed set in X.

Proof. (1) Let G_1 and G_2 be IF δ-closed sets. Then $\text{cl}_\delta(G_1 \cup G_2) = \text{cl}_\delta(G_1) \cup \text{cl}_\delta(G_2) = G_1 \cup G_2$. Thus $G_1 \cup G_2$ is an IF δ-closed set.

(2) Let G_i be an IF δ-closed set, for each $i \in I$. To show that $\text{cl}_\delta(\cap G_i) \leq \cap G_i$, take any $x_{(\alpha,\beta)} \in \text{cl}_\delta(\cap G_i)$. Suppose that $x_{(\alpha,\beta)} \notin \cap G_i$. Then there exists an $i_0 \in I$ such that $x_{(\alpha,\beta)} \notin G_{i_0}$. Since G_{i_0} is an IF δ-closed set, $x_{(\alpha,\beta)} \notin \text{cl}_\delta(G_{i_0})$. Therefore, there exists an IF regular open q-neighborhood A of $x_{(\alpha,\beta)}$ such that $A\widetilde{q}G_{i_0}$. Since $A\widetilde{q}G_{i_0}$ and $\cap G_i \leq G_{i_0}$, we have $A\widetilde{q}(\cap G_i)$. Thus $x_{(\alpha,\beta)} \notin \text{cl}_\delta(\cap G_i)$. This is a contradiction. Hence $\text{cl}_\delta(\cap G_i) \leq \cap G_i$. □

Theorem 2.12. Let A be an IF set in an IF $(X, \mathcal{T})$, then $\text{cl}_\delta(A)$ is the intersection of all IF regular closed supersets of A, or

$$\text{cl}_\delta(A) = \bigwedge\{F \mid A \leq F = \text{cl}(\text{int}(F))\}.$$

Proof. Suppose that $x_{(\alpha,\beta)} \notin \bigwedge\{F \mid A \leq F = \text{cl}(\text{int}(F))\}$. Then there exists an IF regular closed set F such that $x_{(\alpha,\beta)} \notin F$ and $A \leq F$. Since $x_{(\alpha,\beta)} \notin F$, $x_{(\alpha,\beta)}qF^c$. Note that $A \leq F$ if and only if $A\widetilde{q}F^c$. Thus F^c is an IF regular open q-neighborhood of $x_{(\alpha,\beta)}$ such that $A\widetilde{q}F^c$. Hence $x_{(\alpha,\beta)} \notin \text{cl}_\delta(A)$.

Let $x_{(\alpha,\beta)} \in \bigwedge\{F \mid A \leq F = \text{cl}(\text{int}(F))\}$. Suppose that $x_{(\alpha,\beta)} \notin \text{cl}_\delta(A)$. Then there exists an IF regular open q-neighborhood U of $x_{(\alpha,\beta)}$ such that $A\widetilde{q}U$. So, $A \leq U^c$. Since $x_{(\alpha,\beta)}qU$, $x_{(\alpha,\beta)} \notin U^c$. Therefore, there exists an IF regular closed set U^c such that $x_{(\alpha,\beta)} \notin U^c$ and $A \leq U^c$. Hence $x_{(\alpha,\beta)} \notin \bigwedge\{F \mid A \leq F = \text{cl}(\text{int}(F))\}$. This is a contradiction. Thus $x_{(\alpha,\beta)} \in \text{cl}_\delta(A)$. □

Remark 2.13. From the above theorem, for any IF set A, the IF δ-closure $\text{cl}_\delta(A)$ is an IF closed set. Moreover, $\text{cl}_\delta(A)$ becomes IF δ-closed, which will be shown in Theorem 2.18.

Now, we are ready to make counterexamples mentioned in Remark 2.7.

Example 2.14. Let $X = \{a, b\}$, and A the IF set defined by

$$A =< (\frac{a}{0.5}, \frac{b}{0.3}), (\frac{a}{0.3}, \frac{b}{0.5}) >$$

Let $\mathcal{T} = \{\underline{0}, \underline{1}, A\}$. Then $\mathcal{T}$ is an IF topology on X. Clearly, A^c is an IF closed set. Since $\text{cl}(\text{int}(A^c)) = \text{cl}(\underline{0}) = \underline{0} \neq A^c$, A^c is not an IF regular closed set. Hence $\underline{0}$ and $\underline{1}$ are the only regular closed sets. Thus $\text{cl}_\delta(A^c) = \bigcap\{F \mid A^c \leq F, F \text{ is regular closed }\} = \underline{1} \neq A^c$. Hence A^c is not IF δ-closed. Therefore, A^c is an IF closed set which is not IF δ-closed.

Example 2.15. Let $X = \{a, b\}$, and A the IF set defined by

$$A =< (\frac{a}{0.5}, \frac{b}{0.3}), (\frac{a}{0.5}, \frac{b}{0.4}) >$$

Let $\mathcal{T} = \{\underline{0}, \underline{1}, A\}$. Then $\mathcal{T}$ is an IF topology on X. Since $\text{int}(\text{cl}(A)) = \text{int}(A^c) = A$, A is an IF regular open set. Thus A^c is an IF regular closed set, and consequently $\text{cl}_\delta(A^c) = \bigcap\{F \mid A^c \leq F, F \text{ is regular closed }\} = A^c$. Hence A^c is an IF δ-closed set. But $\text{cl}_\theta(A^c) = \bigcap\{\text{cl}(F) \mid$

$F \in \mathcal{T}, A^c \leq F\} = \underline{1} \neq A^c$, and hence A^c is not IF θ-closed. Therefore, A^c is an IF δ-closed set which is not IF θ-closed.

Theorem 2.16. If U is an IF regular closed set, then U is an IF δ-closed set.

Proof. Let U be an IF regular closed set. Then $\text{cl}(\text{int}(U)) = U$. By Theorem 2.12, $\text{cl}_\delta(U) = \bigwedge\{F \mid U \leq F = \text{cl}(\text{int}(F))\} = U$. Thus U is IF δ-closed. □

Remark 2.17. Note that $\text{cl}_\theta(U)$ is not IF θ-closed in general (See [8]). But any IF δ-closure of an IF set is IF δ-closed as in the following theorem.

Theorem 2.18. For any IF set U, $\text{cl}_\delta(U)$ is an IF δ-closed set.

Proof. By Theorem 2.11, 2.12 and 2.16. □

Theorem 2.19. IF δ-closure satisfies the Kuratowski closure axioms.

From the results of IF δ-closure which are obtained above, we have following properties of IF δ-interior.

Theorem 2.20. Let U and V be two IF sets in an IF topological space $(X, \mathcal{T})$. Then we have the following:

(1) $\text{int}_\delta(\underline{1}) = \underline{1}$,

(2) $\text{int}_\delta(U) \leq U$,

(3) $U \leq V \Rightarrow \text{int}_\delta(U) \leq \text{int}_\delta(V)$,

(4) $\text{int}_\delta(U \cap V) = \text{int}_\delta(U) \cap \text{int}_\delta(V)$,

(5) $\text{int}_\delta(V) \cup \text{int}_\delta(U) \leq \text{int}_\delta(U \cup V)$.

Theorem 2.21. Let $(X, \mathcal{T})$ be an IF topological space. Then the following hold:

(1) Finite intersection of IF δ-open sets in X is an IF δ-open set in X.

(2) Arbitrary union of IF δ-open sets in X is an IF δ-open set in X.

Theorem 2.22. Let U be an IF set in an IF topological space $(X, \mathcal{T})$. Then

$$\text{int}_\delta(U) = \bigvee\{G \mid \text{int}(\text{cl}(G)) = G \leq U\}.$$

As a result, $\text{int}_\delta(U)$ is an IF open set.

Corollary 2.23. (1) If U is an IF δ-open set in an IF topological space $(X, \mathcal{T})$, then U is an IF open set.

(2) If U is an IF θ-open set in an IF topological space $(X, \mathcal{T})$, then U is an IF δ-open set.

Theorem 2.24. For any IF set U in an IF topological space $(X, \mathcal{T})$, $\text{int}_\theta(U) \leq \text{int}_\delta(U) \leq \text{int}(U)$. In particular, for any IF closed set U, $\text{int}_\theta(U) = \text{int}_\delta(U) = \text{int}(U)$.

Corollary 2.25. If U is an IF regular open set, then U is an IF δ-open set.

Corollary 2.26. For any IF set U, $\text{int}_\delta(U)$ is an IF δ-open set.

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