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Validity of Ocean Wave Spectrum Using Rayleigh Probability Density Function

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Abstract

The distribution of wave heights is assumed to be a Rayleigh distribution, based on the assumption of a narrow band and Gaussian distribution of wave elevation. The present study was started with doubts about the narrow band assumption. We selected the wave spectra widely used to simulate irregular random waves. The wave spectra used in this study included the Pierson-Moskowitz spectrum, Bretschneider-Mitsuyasu spectrum, and JONSWAP spectrum. The directionality of the waves was considered. The cosine 2-l type directional spreading function and mixed form of the half-cosine 2-s type with Mitsuyasu type directional spreading are considered here to investigate the effects of a directional spreading function on random waves. The simulated wave height distribution is compared with a Rayleigh distribution.

Keywords: Wave spectrum, Pierson-Moskowitz spectrum, Bretchneider and Mitsuyasu spectrum, JONSWAP spectrum, Directional spreading function, Cosine 2-l type, Mitsuyasu type, Rayleigh distribution

1. Introduction

The design of marine vehicles and structures operated far from shore requires information about the external environments. To ensure their design performance, we need to investigate the environmental conditions. However, the environment involves waves, which are random in nature. Neglecting the randomness of waves in a design could lead to a fatal situation. Thus, the presence of waves makes the design of a marine structure slightly different from that of an onshore structure.

In the early stage of the research on waves, sea waves were measured. This statistical data have been accumulated for several decades. For example, Putz announced in 1952 that the mean ratio of the significant to mean wave height is 1.57 and ratio of the one tenth to significant wave height is 1.29 (Putz, 1952).

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As the data accumulated, Longuet–Higgins announced that the wave height distribution follows the Rayleigh distribution (Longuet–Higgins, 1952). The basic assumption of a Rayleigh distribution is a narrow frequency band. Many researchers have come up with a wave spectrum and directional spreading function or random sea waves.

The present study was motivated by the question of whether the Rayleigh distribution fits well with the distribution of ocean wave heights. To determine this, we investigated the wave spectrum and directional spreading function widely used. The time domain wave elevation was simulated. The distribution law for the wave height from the simulated wave height was compared with a Rayleigh distribution.

2. Wave Spectrum

Many researchers have studied the wave spectrum from recorded wave profiles covering several decades. For the case of wind generated waves, the

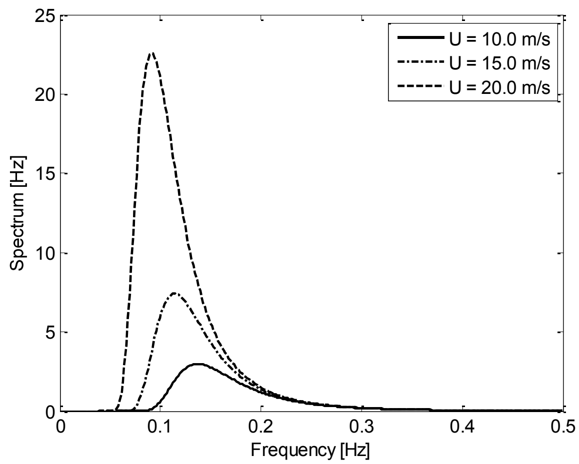

Fig. 1. Pierson-Moskowitz Spectrum.

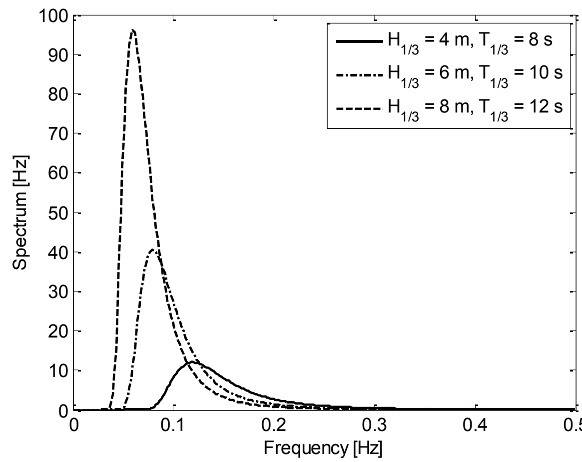

Fig. 2. Bretschneider-Mitsuyasu spectrum.

spectrum has a single-peak when waves travel so far from the disturbed or generated region. Many researchers has suggested single-peaked wave spectrum which are suggested in subsection.

2.1 Pierson-Moskowitz Spectrum

The Pierson-Moskowitz spectrum is a well-known spectrum for fully developed wind waves. The suggested formula has the following form.

$$S(f) = 8.10\times10^{-3}\left(2\pi\right)^{-4} g^2 f^{-5}\, \mathrm{e}^{-0.74\left(g/2\pi fU\right)^4} \tag{1}$$

where g is gravitational acceleration and U is the wind speed at 19.5m above the mean free surface. Fig. 1 shows the spectral shape of the Pierson-Moskowitz spectrum.

2.2 Bretschneider-Mitsuyasu Model

The major factor of the P-M spectrum is the wind speed. This invokes inconvenience from the engineers' point of view. Thus, Bretschneider (1959) suggested a modified formula. The coefficients were adjusted by Mitsuyasu (1970). The modified formula is called the Bretschneider-Mitsuyasu spectrum and is expressed in terms of the wave height and its period.

$$S(f) = 0.257 H_{1/3}^2 T_{1/3}^{-4} f^{-5} e^{-1.03\left(T_{1/3} f\right)^{-4}} \tag{2}$$

where $H_{1/3}$ and $T_{1/3}$ are the significant wave height and its period. The Bretschneider-Mitsuyasu spectrum is featured in Fig. 2.

In addition, the spectral peak period, T_p, is expressed by the following form (Mitsuyasu,1970).

$$T_p = 1.05 T_{1/3} \tag{3}$$

2.3 JONSWAP

The Pierson-Moskowitz spectrum is used for fully developed wind waves. The waves developed in a limited fetch under strong winds have a sharp and narrow peak in the spectrum. This characteristic has been verified by a wave observation program called the Joint North Sea Wave Project (JONSWAP). To describe a sharp and narrow peak in the spectrum, the new wave spectrum is introduced by Hasselmann et al., (1973). The modified spectrum is called the JONSWAP spectrum. A dimensionless fetch length is introduced in the formula.

$$S(f) = \frac{\alpha g^2}{\left(2\pi f\right)^5} \exp\left[-\beta \frac{f_p^4}{f^4}\right] \gamma^a \tag{4}$$

where

$$a = \exp\left[-\frac{\left(f - f_p\right)^2}{2\omega_p^2 \sigma^2}\right] \tag{5}$$

$$\sigma = \begin{cases} 0.07 & : \quad f \le f_p \\ 0.09 & : \quad f > f_p \end{cases} \tag{6}$$

$$\beta = 1.25 \tag{7}$$

where α is a constant relate to the wind speed and fetch length. Typical values in the northern North Sea are in the range of 0.0081 to 0.01. The f_p represents the peak wave frequency.

Equations (4) to (7) have the same problem as the Pierson-Moskowitz spectrum from the engineers' point of view. Therefore, Goda introduced an adjusted JONSWAP spectrum, which is expressed with the wave height and period (Goda, 1988).

$$S(f) = \beta_J H_{1/3}^2 T_p^{-4} f^{-5} e^{\left(-1.25\left(T_p f\right)^{-4}\right)} \gamma^{\exp\left[-\left(T_p f - 1\right)^2 / 2\sigma^2\right]} \tag{8}$$

where

$$\beta_J \approx \frac{0.06238\left(1.094 - 0.01915 \ln \gamma\right)}{0.230 + 0.0336\gamma - 0.185\left(1.9 + \gamma\right)^{-1}} \tag{9}$$

$$T_{1/3} \approx \left[1 - 0.132\left(\gamma + 0.2\right)^{-0.559}\right] T_p \tag{10}$$

$$\bar{T} \approx \left[1 - 0.532\left(\gamma + 2.5\right)^{-0.569}\right] T_p \tag{11}$$

$$\sigma = \begin{cases} 0.07 & : \quad f \le f_p \\ 0.09 & : \quad f > f_p \end{cases} \tag{12}$$

where $\bar{T}$ denotes the mean wave period defined as the zero-upcrossing period, and γ is the peak enhancement factor. The range of γ is 1 to 7. with a mean value of 3.3 (Hasselmann et al., 1973). The typical spectral shape of the JONSWAP spectrum is shown in Fig. 3.

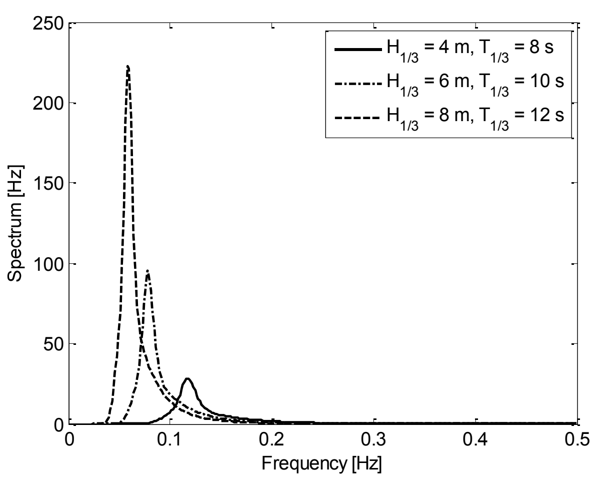

Fig. 3. JONSWAP spectrum (γ = 3.3).

3. Directional Spreading Function

Sea waves cannot be depicted by using the wave spectrum alone. Waves have their own direction. The concept of a directional spectrum is therefore introduced to describe wave fields. A wave spectrum that involves the direction could be expressed as below for the sake of convenience.

$$S\left(f,\theta\right) = S\left(f\right) G\left(\theta \middle| f\right) \tag{13}$$

where $S\left(f,\theta\right)$ is the directional wave spectral density function, or simply the directional wave spectrum, and $G\left(\theta \middle| f\right)$ is the directional spreading function, or the spreading function.

The function $G\left(\theta \middle| f\right)$, which represents the directional distribution of wave energy, has been found to vary with the frequency. In addition, the directional spreading function should have no dimension and be normalized.

$$\int_{-\pi}^{\pi} G\left(\theta \middle| f\right) d\theta = 1 \tag{14}$$

3.1 Cosine 2l-power type

The earliest directional spreading function was the cosine-squared type, which was used by Pierson, Neumann, and James (1955). Later, the function was evolved into the cosine 2l-power type. The cosine 2l-power spreading function is expressed as below.

$$G_1\left(\theta\right) = \begin{cases} \dfrac{2l!!}{\pi\left(2l-1\right)!!} \cos^{2l}\left(\theta - \theta_0\right) & : \quad \left|\theta - \theta_0\right| \le \dfrac{\pi}{2} \\ 0 & : \quad \left|\theta - \theta_0\right| > \dfrac{\pi}{2} \end{cases} \tag{15}$$

where θ_0 denotes the main wave direction. The cosine 2l-power type spreading function is not a function of the wave frequency. The directional distribution of the cosine squared directional spreading function is shown in Fig. 4.

The cosine 2l-power type directional spreading function has a limited range. As plotted in Fig. 4, the wave direction is only $-\pi/2$ to $\pi/2$.

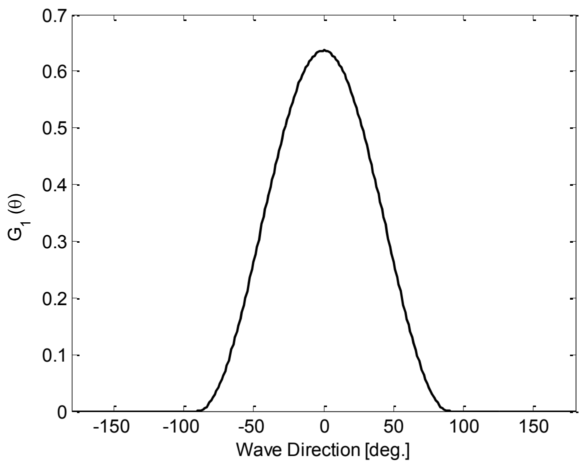

Fig. 4. Cosine-squared directional spreading function with main direction $\theta_0 = 0$.

3.2 Half-cosine 2s-power type

The cosine 2l-power type spreading function is applied for the half plane of $|\theta| \le \pi/2$. However, the wave direction should involve all of the directions. Thus, as initiated by Longuet-Higginis et al. (1963), many researchers have adopted the following spreading function for data analysis.

$$G_2(\theta) = \frac{2^{2s-1}}{\pi} \frac{\Gamma^2(s+1)}{\Gamma(2s+1)} \cos^{2s}\left(\frac{\theta}{2}\right) \tag{16}$$

The spreading parameter, s , has been found to be dependent on the frequency, as will be introduced in the subsection on the Mitsuyasu type.

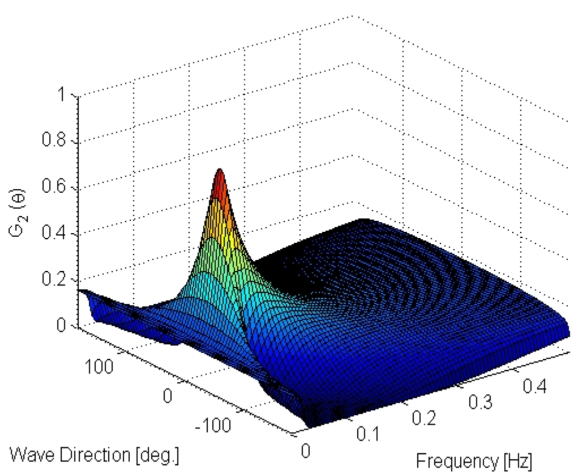

Fig. 5. Mitsuyasu-type directional spreading function with $s_{\max} = 10$.

3.3 Mitsuyasu-type

Mitsuyasu et al. (1975) introduced a spreading function that included the frequency. The introduced directional spreading is the narrowest around the spectral peak frequency and becomes wider as the frequency moves farther from the peak. Equation 17 shows this effect clearly.

$$s = \begin{cases} s_{\max}\left(f/f_p\right)^5 & : \ f \le f_p \\ s_{\max}\left(f/f_p\right)^{-2.5} & : \ f > f_p \end{cases} \tag{17}$$

Mitsuyasu et al. set spreading parameter $s_{\max}$ in the range of 5 to 30, with a mean of about 10 for wind generated waves.

Goda and Suzuki (1975) proposed a spreading parameter for the purpose of engineering. They suggested the use of fixed values for wind waves and swell.

$$s_{\max} = \begin{cases} 10 & : \quad \textit{wind waves} \\ 25 & : \quad \textit{swell with short decay dis}\tan\textit{ces} \\ 75 & : \quad \textit{swell with long decay dis}\tan\textit{ces} \end{cases} \tag{18}$$

Fig. 5 shows a spreading function with $s_{\max} = 10$. Its contour plot is shown in Fig. 6. We can easily understand that the directional spreading function has a heavy density near the spectral peak frequency. And it becomes lower farther from the peak.

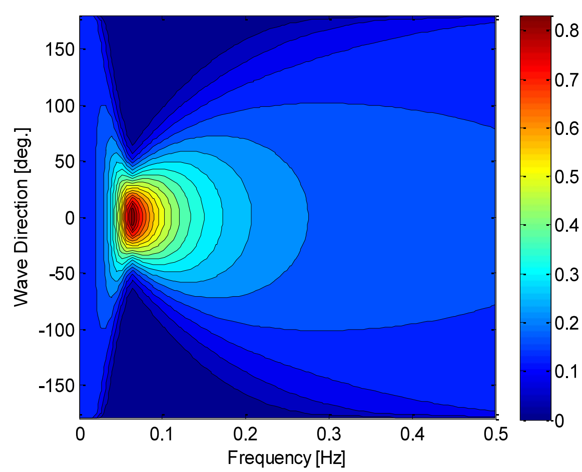

Fig. 6. Mitsuyasu-type directional spreading function with $s_{\max} = 10$.

4. Description of Random Sea

A random sea field can be expressed as the decomposition of various frequency components, wave directions, and phases.

$$\zeta=\zeta(x,y,t)=\sum_{n=1}^{\infty}a_n\cos\left(k_n x\cos\theta_n+k_n y\sin\theta_n-2\pi f_n t+\varepsilon_n\right) \tag{19}$$

where k_n and ε_n are the wave number and random phase, respectively. The wave number could be computed from the wave frequency by using the dispersion relation. A random phase is selected from the uniform distribution in the range of 0 to 2π. And wave amplitudes are computed from the directional spectrum using the equation shown below.

$$\sum_{f}^{f+df}\sum_{\theta}^{\theta+d\theta}\frac{1}{2}a_n=S(f,\theta)dfd\theta \tag{20}$$

For a fixed position, the time history of the wave profile could be expressed in a simple form.

$$\zeta(t)=\sum_{n=1}^{\infty}a_n\cos\left(2\pi f_n t+\varepsilon_n\right) \tag{21}$$

where

$$\sum_{f}^{f+df}\frac{1}{2}a_n=S(f)df \tag{22}$$

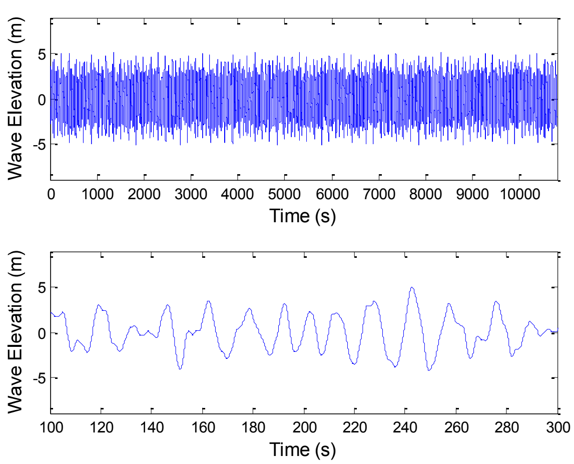

Fig. 7. Time simulated wave profile with JONSWAP.

Examples of simulated wave profiles for a fixed position and fixed time are plotted in Figs. 7 and 8. From the random simulation, we can obtain a discrete time history.

5. Stochastic Characteristics and Analysis

Let us consider the case in which the energy of the wave field is concentrated within a narrow frequency band. Actually, ocean waves have periods in the range of 3~15s. Such a spectrum is said to be narrow-banded. It is said that the individual waves have almost the same period but slowly varying amplitudes.

From the assumption of a narrow band process, we can assume that the number of peaks is almost the same as the number of zero up-crossings. From this assumption, the Rayleigh distribution concerning the peak distribution could be deduced when some mathematical concepts are involved (Newland, 1993). The Rayleigh distribution has the following form.

$$f_p(H)=\frac{H}{4m_0}\exp\left[-\frac{H^2}{8m_0}\right] \tag{23}$$

where m_0 is the zeroth moment of the wave spectrum, or the area of the wave spectrum. The mean height and root mean square height are calculated in the function of the zeroth moment.

$$\bar{H}=\int_0^\infty Hf_p(H)dH=\left(2\pi m_0\right)^{1/2} \tag{24}$$

$$H_{rms}^2=\int_0^\infty H^2 f_p(H)dH=8m_0 \tag{25}$$

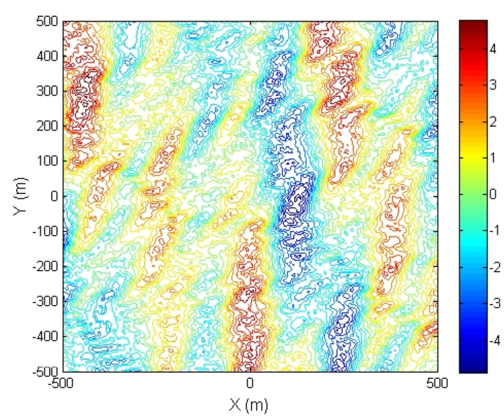

Fig. 8. Random waves field using JONWAP and Mitsuyasu type spreading function.

Table 1. Characteristic wave height of Rayleigh distribution.

N	$H_{1/N} / H_{rms}$	Remarks
100	2.359	-
50	2.207	-
20	1.986	-
10	1.800	Highest 1/10th wave
5	1.591	-
3	1.416	Significant wave
2	1.256	-
1	0.886	Mean wave

From the Rayleigh distribution, we can compute the most probable and n-th highest wave amplitude (Goda, 1984). Therefore, we can compute the N-th highest wave height from the wave spectrum.

5.1 Direct approach to discrete wave profile

Returning to the definition of the up-crossing approach, we can count the number of up-crossing points by increasing the criterion. The concept of counting is illustrated in Fig. 9. In Fig. 9, we can count 5 points with threshold a_N . However, with threshold a_{N+1}, the up-crossing number is just 3. From this simple assumption, the relation shown below can be derived.

$$\int_a^\infty f_p(a)\,da = \frac{N_a^+(T)}{N_0^+(T)}$$

where

$$f_p(a) = f_p(H/2)$$

where $N_a^+(T)$ and $N_0^+(T)$ are the numbers of up-crossing points with criterion a and $a=0$ during the simulation time. Then, $f_p(a)$ can be derived.

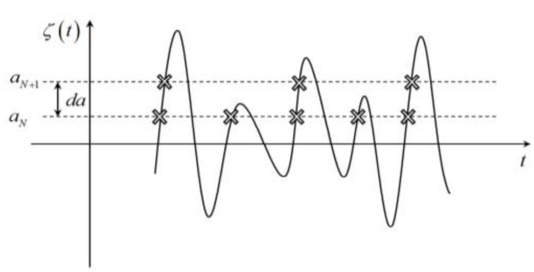

Fig. 9 Description of counting up-crossing points.

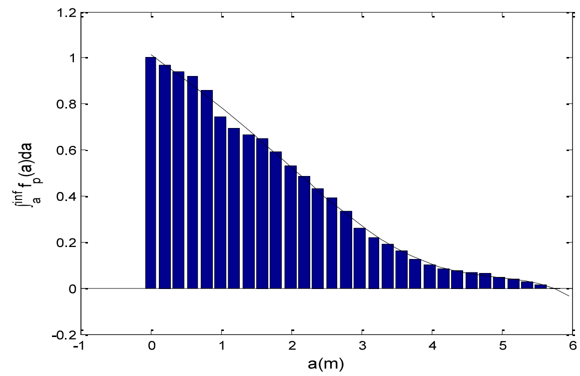

Fig 10. Numerically computed integration of Rayleigh distribution from simulated time history.

$$f_p(a) = -\frac{d}{da}\int_a^\infty f_p(a)\,da$$

To get $f_p(a)$ using the direct approach, we first obtain the discrete form of the integration of $f_p(a)$. Fig. 10 illustrates the integration performed using the numerical scheme. Then, a numerical differential scheme is adopted. However, the smooth scheme is also adopted because the differential data are so unstable. A robust version of the LOESS smooth scheme is used for data smoothing with a factor of 0.5.

6. Results

Because of the random process, the simulation of the wave profile in the time domain for a fixed position is repeated 30 times. The duration of each time simulation is 3 h. From the repeated simulations, the numerically computed Rayleigh distribution is found by taking the mean value. Fig. 11 represents an example of the numerically computed Rayleigh distribution with an analytic solution.

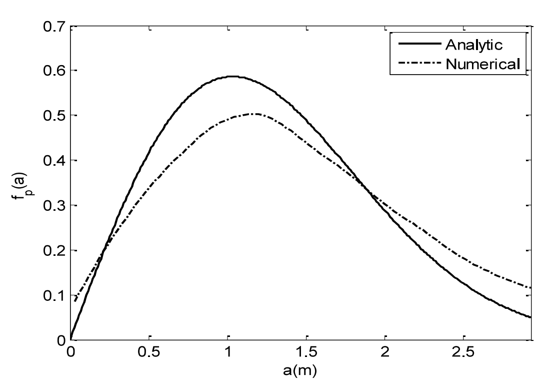

Fig. 11 Probability density function of wave height..

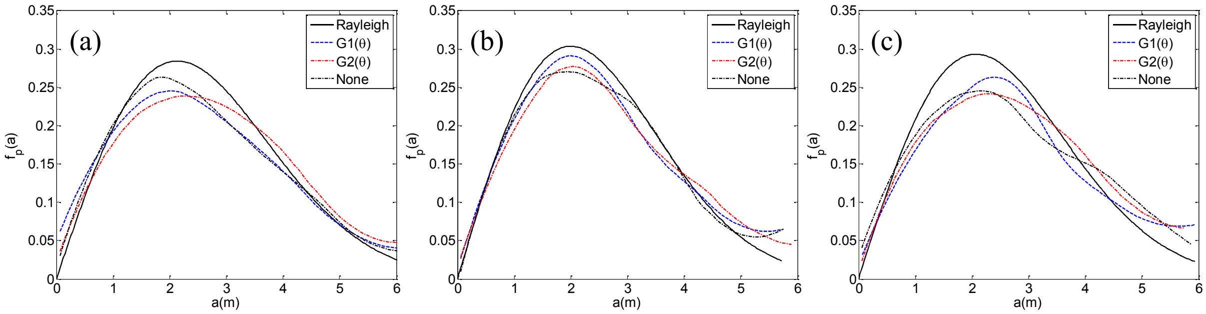

Fig. 12 Comparison of probability density function of wave amplitude with various directional spreading functions: (a): Pierson-Moskowitz (U = 20m/s), (b,c): Bretschneider-Mitsuyasu, JONSWAP ($H_{1/3}$ = 8 m, $T_{1/3}$ = 15 s, $\gamma = 3.3$).

The numerically computed probability density functions (PDFs) for the wave amplitude are plotted in Fig. 12 for various wave spectra and directional spreading functions. Most of the PDFs show distribution behaviors similar to that of a Rayleigh distribution for a given zero-th moment. The zero-th moment used for the Rayleigh distribution in Fig. 12 is numerically computed from the wave spectrum.

A numerically computed PDF has a lower value than those with the Rayleigh distribution at the peak, but the most probable wave height is almost the same. The most probable wave height is shown in Table 2. From this table, the PDF for the JONSWAP spectrum has more error than the other spectra.

The PDFs for the JONSWAP spectra have the tendency of a higher probability when wave height has a large value. This means that the JONSWAP spectrum has a higher wave height than the other wave spectra. The JONSWAP spectrum has a high density near the peak. This means that high energy components related to the wave height are close to the specific frequency.

Table 2. Most probable wave heights from probability density functions.

	Spreading Function	$\bar{H}_p$ [m]	Relative Error [%]
Pierson -Moskowith $\bar{H}_p$ = 8.53 m	None	7.30	-14.46
	Cosine 2l	7.96	-6.69
	Mitsuyasu	9.01	5.65
Bretschneider -Mitsuyasu $\bar{H}_p$ = 7.99 m	None	7.75	-3.06
	Cosine 2l	7.94	-0.60
	Mitsuyasu	8.17	2.31
JONSWAP	None	8.63	4.32

Then, wave components close to the specific components travel together, which are called group waves. Wave groups cause a higher wave height in the time simulation than other spectra. From Table 2, we can also deduce the same results because the most probable wave height is skewed to the high value.

We also compare the probability density function simulated from the Bretschneider-Mitsuyasu spectrum with that for the JONSWAP spectrum for the same significant wave height and period. The results are plotted in Fig. 13. For a lower wave height, JONSWAP has a lower value than the other, but for a high wave height, JONSWAP has a higher value. This phenomenon can also be explained by the spectrum, as previously mentioned.

The JONSWAP spectrum proposed by Goda (1988) has 3 parameters to evaluate the spectrum. Therefore, we plotted the results of the probability function for each of the various parameters in Figs. 14 to 16. We can also see that most of the probability density functions have behaviors similar to that of a Rayleigh distribution.

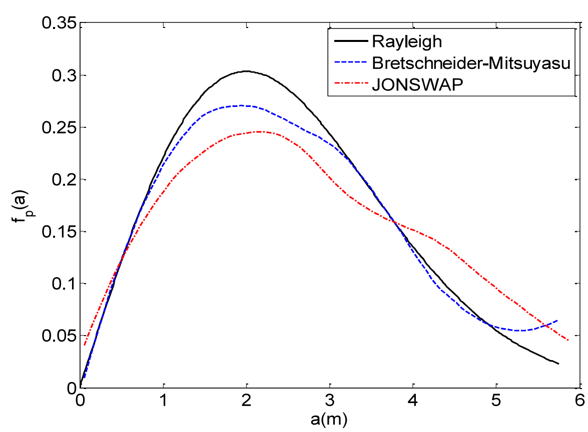

Fig. 13 Comparison of Bretschneider-Mitsuyasu with JONSWAP ($H_{1/3}$ = 8 m, $T_{1/3}$ = 15 s, $\gamma = 3.3$).

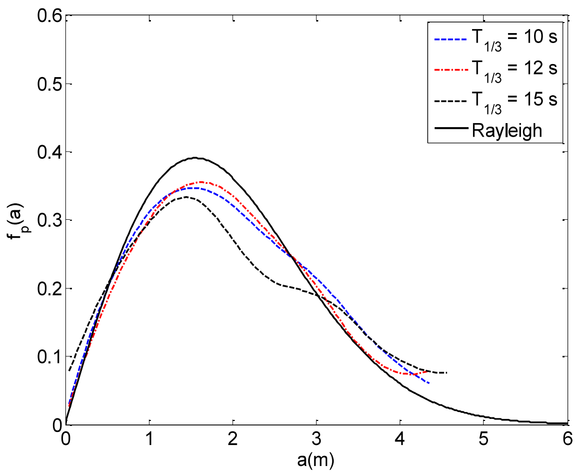

Fig. 14 JONSWAP simulation comparison with significant wave period variance ($H_{1/3}$ = 6 m, $\gamma = 3.3$).

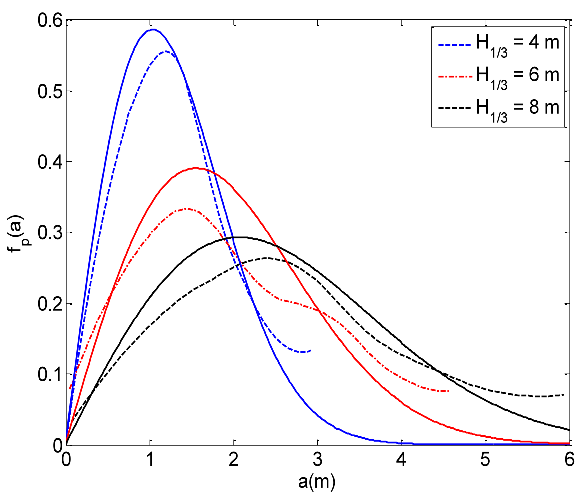

Fig. 15 JONSWAP simulation comparison with significant wave height variance ($T_{1/3}$ = 15 s, $\gamma = 3.3$).

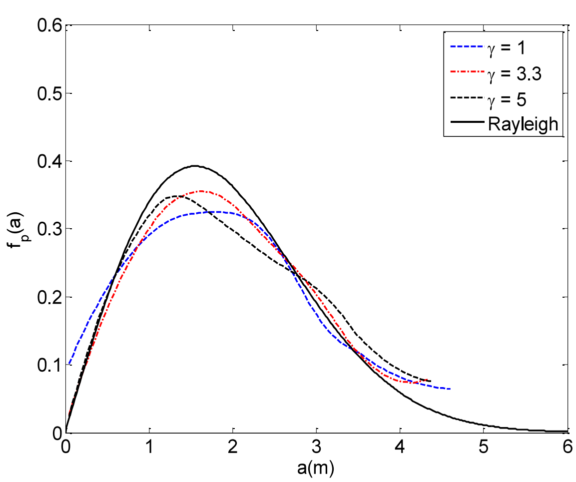

Fig. 16 JONSWAP simulation comparison with γ ($H_{1/3}$ = 6 m, $T_{1/3}$ = 12 s).

7. Conclusions

It is a common practice to assume that the wave height distribution in the ocean is a Rayleigh distribution. However the authors of the present study wanted to prove this through a simulation using well-known spectra. The characteristic of wave directionality was also included in the simulation. It was shown that the Rayleigh distribution fits well with the simulated results. Therefore, it can be concluded that the Rayleigh distribution can be safely introduced when the wave height distribution is modeled.

Acknowledgements

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