

**EXISTENCE OF STRONG MILD SOLUTION OF THE
NAVIER-STOKES EQUATIONS IN THE HALF SPACE WITH
NONDECAYING INITIAL DATA**

HYEONG-OHK BAE AND BUM JA JIN

ABSTRACT. We construct a mild solutions of the Navier-Stokes equations in half spaces for nondecaying initial velocities. We also obtain the uniform bound of the velocity field and its derivatives.

1. Introduction

Consider the Navier-Stokes equations in the n -dimensional half space \mathbb{R}_+^n , $n \geq 2$;

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} u - \Delta u + \nabla p = -\operatorname{div}(u \otimes u) & \text{for } x \in \mathbb{R}_+^n, t > 0, \\ \nabla \cdot u = 0 & \text{for } x \in \mathbb{R}_+^n, t > 0, \\ u|_{t=0} = a & \text{for } x \in \mathbb{R}_+^n, \\ u|_{x_n=0} = 0 & \text{for } t > 0. \end{cases}$$

Here, $u = u(x, t) = (u_1(x, t), \dots, u_n(x, t))$ and $p = p(x, t)$ represent the unknown velocity vector field and the pressure, respectively, of the fluid, and a is a given initial velocity vector field, with $\nabla \cdot a = 0$ and $a|_{x_n=0} = 0$, which may not decay at infinity.

In this paper, we concentrate the existence and the uniqueness of solutions for nondecaying initial data a .

There are several results in the whole space \mathbb{R}^n for such a problem. It is well-known that for $a \in L^\infty(\mathbb{R}^n)$ there is a unique local in time solution u for the Navier-Stokes equations with

$$(1.2) \quad p = \sum_{i,j=1}^n R_i R_j u_i u_j,$$

Received August 30, 2010.

2010 *Mathematics Subject Classification.* 35Q30, 76D05.

Key words and phrases. Navier-Stokes, nondecaying data, existence, uniqueness.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology 2009-0072196.

where $R_j = (-\Delta)^{-1/2} \frac{\partial}{\partial x_j}$ is the Riesz operator. Refer to [1] and [7] for survey and references. In \mathbb{R}^2 , this solution can be extended globally in time [7].

In [10], it is shown that for linearly growing initial data of the form $a = b - Mx$, the Navier-Stokes equations admit a unique local in time solution in $L^p(\mathbb{R}^n)$, where M is an $n \times n$ matrix with constant real entries and $b \in L^p(\mathbb{R}^n)$.

As mentioned in [11], for $u = g(t)$, then (u, p) always solves (1.1) in \mathbb{R}^n with $p = -g'(t) \cdot x$, no matter what function g is. Therefore, solution u with a constant initial data is not unique without assuming (1.2). In [11], it is shown that for initial data $a \in L^\infty(\mathbb{R}^n)$ with $\nabla \cdot a = 0$, if (u, p) is a solution with

$$u \in L^\infty((0, T) \times \mathbb{R}^n), \quad p \in L^1_{\text{loc}}([0, T]; \text{BMO}),$$

then $(u, \nabla p)$ is uniquely determined by a , and $\nabla p = \sum \nabla R_i R_j u_i u_j$ in the distributional sense. Recently, in [14], the BMO space for the pressure is replaced by the generalized Campanato space.

Up to now, in the half space case, the local in time existence of mild solution of the Navier-Stokes problems is provided in [16] with continuous, bounded initial data. In this article, we also work on the existence and the uniqueness of (strong) mild solutions for the Navier-Stokes equations with nondecaying initial data in \mathbb{R}^n_+ . See also [9] for the rotating Navier-Stokes equations with nondecaying data.

Recently, we have an information of P. Maremonti [12]'s existence result on the Navier-Stokes equations with bounded and continuous initial data. However, our result is different in the sense that our initial data is assumed to be just bounded, and our estimate of the linear and nonlinear operator is from different technique.

We transform (1.1) into the abstract differential equations

$$(1.3) \quad \frac{\partial}{\partial t} u + Au = -\mathbb{P} \text{div}(u \otimes u),$$

where $A = -\mathbb{P}\Delta$ is the Stokes operator and \mathbb{P} is the Leray projection operator, which is represented as

$$\mathbb{P}f(x) \equiv f(x) + \nabla_x \int_{\mathbb{R}^n_+} \nabla_y \mathcal{G}(x, y) \cdot f(y) dy,$$

when $f_n|_{x_n=0} = 0$. Here,

$$(1.4) \quad \mathcal{G}(x, y) \equiv N(x - y) + N(x - y^*),$$

where $y^* \equiv (y_1, \dots, y_{n-1}, -y_n)$, $N(x) \equiv \frac{1}{n(2-n)\omega_n} |x|^{2-n}$, if $n \geq 3$, and $N(x) \equiv \frac{1}{2\pi} \ln |x|$ if $n = 2$, and ω_n denotes the surface area of the unit sphere in \mathbb{R}^n .

Using the solution operator of the Stokes equations in \mathbb{R}^n_+ , the solution of (1.3) is formally expressed in the integral form

$$(1.5) \quad u(t) = e^{-At} a - \int_0^t e^{-A(t-s)} \mathbb{P} \text{div}(u \otimes u)(s) ds.$$

We define a bilinear operator B by

$$B(u, v)(t) \equiv - \int_0^t e^{-A(t-s)} \mathbb{P} \operatorname{div}(u \otimes v)(s) ds.$$

A mild solution of the Navier-Stokes equations is defined as following.

Definition 1.1. Let $a \in S'$. A measurable function $u \in L_{\text{loc}}^\infty((0, T) \times \mathbb{R}_+^n)$ is called a mild solution of Navier-Stokes equations (1.1) on $(0, T)$, if u satisfies

$$u(t) = e^{-tA} a + B(u, u)(t) \text{ on } (0, T)$$

in the sense of distribution. Here, S' is the set of distributions.

Solonnikov [16] has expressed the solution operator of the Stokes equations in \mathbb{R}_+^n in the integral form

$$e^{-At} a \equiv \int_{\mathbb{R}_+^n} G(x, y, t) \cdot a(y) dy,$$

where $G = (G_{ij})_{i,j=1,\dots,n}$ is defined by

$$(1.6) \quad \begin{aligned} G_{ij}(x, y, t) &\equiv \delta_{ij}(\Gamma(x - y, t) - \Gamma(x - y^*, t)) \\ &+ 4(1 - \delta_{jn}) \frac{\partial}{\partial x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i} N(x - z) \Gamma(z - y^*, t) dz. \end{aligned}$$

The function $\Gamma(x, t)$ is the n dimensional Gaussian kernel defined by $\Gamma(x, t) \equiv \Gamma_t(x) \equiv \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$.

Stokes solutions in \mathbb{R}_+^n have also been derived in [17]. This solution formula have been used in the L^q framework, mainly for $1 < q < \infty$ (see [4, 6], see also [2, 3, 8, 15] for the L^1 or L^∞ estimates of the Stokes flow or its gradient). The solution formula in [16] has been used mainly for L^∞ framework (see [5, 13, 16]).

The goal of this paper is to construct strong mild solution of the Navier-Stokes equations in \mathbb{R}_+^n when the initial data is in $L^\infty(\mathbb{R}_+^n)$. For the construction of the solutions, the following estimate plays important role: for \mathcal{F} with 0 on $x_n = 0$,

$$(1.7) \quad \|e^{-At} \mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)} \leq C \|\mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)},$$

$$(1.8) \quad \|\nabla e^{-At} \mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)} \leq Ct^{-\frac{1}{2}} \|\mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)},$$

$$(1.9) \quad \|e^{-At} \mathbb{P} \operatorname{div} \mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)} \leq Ct^{-\frac{1}{2}} \|\mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)},$$

$$(1.10) \quad \|\nabla e^{-At} \mathbb{P} \operatorname{div} \mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)} \leq Ct^{-\frac{1}{2}} \|\nabla \mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)}.$$

Especially, the estimate (1.10) is used for the gradient of the velocity.

The followings are the main theorems.

Theorem 1.2. Let $a \in L^\infty(\mathbb{R}_+^n)$ be the given vector field $\operatorname{div} a = 0$ and $a|_{x_n=0} = 0$. Then there is a positive time $T > 0$, and there is a mild solution $u \in$

$L^\infty((0, T) \times \mathbb{R}_+^n)$ which satisfies (1.5). Moreover, it holds that for $0 < t < T$,

$$\|u(t)\|_{L^\infty(\mathbb{R}_+^n)} + t^{\frac{1}{2}} \|\nabla u(t)\|_{L^\infty(\mathbb{R}_+^n)} \leq C \|a\|_{L^\infty(\mathbb{R}_+^n)}.$$

Theorem 1.3. *Suppose that $u, v \in L_{\text{loc}}^\infty((0, T) \times \mathbb{R}_+^n)$ are mild solutions of the Navier-Stokes equations (1.5) with the same initial data. If both u and v satisfy the following conditions*

$$\sup_{0 < t < T} t^{\frac{\alpha}{2}} \|u(t)\|_{L^\infty} < \infty \text{ and } \sup_{0 < t < T} t^{\frac{\alpha}{2}} \|v(t)\|_{L^\infty} < \infty \text{ for some } 0 \leq \alpha < 1,$$

then $u \equiv v$ on $(0, T) \times \mathbb{R}_+^n$.

We compare our results with those in [16] with several points of view. In [16], to get the short time existence of mild solution, the estimates (1.7)–(1.9) are provided, but (1.10) is not.

In this paper as well as the boundedness of the velocity, we obtain a higher regularity, and hence our solution is not only mild solution but also strong solution for a short time. For the estimate of the gradient of the velocity, it is necessary to have the pointwise estimate of the kernels of the operator $\nabla e^{-tA} \mathbb{P} \text{div}$. Unfortunately, ∇ and $e^{-At} \mathbb{P}$ are not commutative, while $\nabla_{\bar{x}}$ and $e^{-At} \mathbb{P}$ are commutative. To obtain (1.10), we first treat the case $\nabla_{\bar{x}} e^{-At} \mathbb{P} \text{div} \mathcal{F}$, then $\frac{\partial}{\partial x_n} (e^{-At} \mathbb{P} \text{div} \mathcal{F})_n$, and finally $\frac{\partial}{\partial x_n} e^{-At} \mathbb{P} \text{div} \mathcal{F}$, separately. Here $\nabla_{\bar{x}} = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}})$, $\bar{f} = (f_1, \dots, f_{n-1})$.

We also compare our result with those in [12], where the classical solution u is constructed when the initial velocity a is bounded and continuous and the pointwise estimate of pressure function is also given. For the pointwise estimate of the pressure, the continuity assumption of the given initial data has been necessary.

Acknowledgement. We want to express our thanks to TongKeun Chang for valuable discussions, especially for the estimate of the Stokes operator and bilinear operators.

2. Preliminaries

Let us consider the Stokes problems in \mathbb{R}_+^n , $n \geq 2$:

$$(2.11) \quad \begin{cases} \operatorname{div} u = 0 & \text{for } x \in \mathbb{R}_+^n, t > 0, \\ u_t - \Delta u + \nabla p = \operatorname{div} \mathcal{F} & \text{for } x \in \mathbb{R}_+^n, t > 0, \\ u|_{t=0} = a & \text{for } x \in \mathbb{R}_+^n, \\ u|_{x_n=0} = 0 & \text{for } t > 0. \end{cases}$$

In [16] solution u of the Stokes problems (2.11) is formally given by

$$u(x, t) = \int_{\mathbb{R}_+^n} G(x, y, t) \cdot a(y) dy + \int_0^t d\tau \int_{\mathbb{R}_+^n} G(x, y, t - \tau) \cdot \mathbb{P} \operatorname{div} \mathcal{F}(y, \tau) dy.$$

For later usage, we use the notations:

$$\Gamma(x, t) \equiv \Gamma_t(x) \equiv \bar{\Gamma}_t(\bar{x})\Gamma_t^n(x_n) \equiv \left((2\pi t)^{-\frac{n-1}{2}} e^{-\frac{|\bar{x}|^2}{4t}} \right) \left((2\pi t)^{-\frac{1}{2}} e^{-\frac{|x_n|^2}{4t}} \right),$$

$$\|\cdot\|_\infty \equiv \|\cdot\|_{L^\infty(\mathbb{R}_+^n)}, \quad \partial_{x_i} \equiv \frac{\partial}{\partial x_i} \text{ for short.}$$

$$\nabla_{\bar{x}} \equiv (\partial_{x_1}, \dots, \partial_{x_{n-1}}), \quad \bar{x} = (x_1, \dots, x_{n-1}).$$

$1_{\mathbb{R}_+^n}$ denotes the characteristic function which is 1 on \mathbb{R}_+^n , and otherwise 0.

τ_a a translation operator defined by $\tau_a f(x) \equiv f(a - x)$.

Remark 2.1. Suppose that $\operatorname{div} f = 0$ with $f_n|_{x_n=0} = 0$. Then direct computation shows that

$$\operatorname{div} e^{-tA} f = 0.$$

Proposition 2.1. *If $k = 1, \dots, n-1$, then we have*

$$(2.12) \quad \frac{\partial}{\partial x_k} G_{ij}(x, y, t) = -\frac{\partial}{\partial y_k} G_{ij}(x, y, t).$$

And we have

$$\begin{aligned} \frac{\partial}{\partial x_n} G_{ij}(x, y, t) &= \frac{\partial}{\partial y_n} G_{ij}(x, y, t) - 2\delta_{ij} \frac{\partial}{\partial y_n} \Gamma(x - y, t) \\ &\quad - 4(1 - \delta_{jn}) \frac{\partial}{\partial x_j} \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i} N(\bar{x} - \bar{z}, x_n) \Gamma(\bar{z} - \bar{y}, y_n, t) d\bar{z}. \end{aligned}$$

Proof. Let us consider the case $k \neq n$. Differentiate G_{ij} in terms of x_k variable, then we have

$$\begin{aligned} \partial_{x_k} G_{ij}(x, y, t) &= \delta_{ij} \partial_{x_k} [\Gamma_t(x - y^*) - \Gamma_t(x - y^*)] \\ &\quad + 4(1 - \delta_{jn}) \partial_{x_k} \partial_{x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \Gamma_t(z - y^*) dz \\ &= \delta_{ij} \partial_{x_k} [\Gamma_t(x - y^*) - \Gamma_t(x - y^*)] \\ &\quad + 4(1 - \delta_{jn}) \partial_{x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \partial_{z_k} \Gamma_t(z - y^*) dz. \end{aligned}$$

The last term of the right hand side of the above identity comes from the fact that

$$\partial_{x_k} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) g(z) dz = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \partial_{z_k} g(z) dz.$$

Note that $\partial_{y_k} \Gamma_t(x - y) = -\partial_{x_k} \Gamma_t(x - y)$ and $\partial_{y_k} \Gamma_t(x - y^*) = -\partial_{x_k} \Gamma_t(x - y^*)$ if $k \neq n$. Hence we obtain the identity (2.12).

Now consider the case $k = n$. Notice that

$$\begin{aligned} &\partial_{x_n} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) g(z) dz \\ &= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{x_n} N(x - z) g(z) dz \end{aligned}$$

$$+ \frac{\delta_{in}}{2n} g(x) + \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x} - \bar{z}, 0) g(\bar{z}, x_n) d\bar{z}$$

and

$$\begin{aligned} & - \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{x_n} N(x - z) g(z) dz \\ &= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{z_n} N(x - z) g(z) dz \\ &= - \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \partial_{z_n} g(z) dz \\ &+ \frac{\delta_{in}}{2n} g(x) + \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x} - \bar{z}, 0) g(\bar{z}, x_n) d\bar{z} \\ &- \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x} - \bar{z}, z_n) g(\bar{z}, 0) d\bar{z}. \end{aligned}$$

Hence, we obtain that

$$\begin{aligned} & \partial_{x_n} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \Gamma(z - y^*, t) dz \\ &= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \partial_{z_n} \Gamma(z - y^*, t) dz \\ &+ \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x} - \bar{z}, x_n) \Gamma(\bar{z} - \bar{y}, y_n, t) d\bar{z}. \end{aligned}$$

Note that $\partial_{y_n} \Gamma_t(x - y) = -\partial_{x_n} \Gamma_t(x - y)$ and $\partial_{y_n} \Gamma_t(x - y^*) = \partial_{x_n} \Gamma_t(x - y^*)$. Therefore if we differentiate G_{ij} in terms of x_n variables, then we have

$$\begin{aligned} \partial_{x_n} G_{ij}(x, y, t) &= -\delta_{ij} \partial_{y_n} [\Gamma(x - y, t) + \Gamma(x - y^*, t)] \\ &+ 4(1 - \delta_{jn}) \partial_{y_n} \partial_{x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(x - z) \Gamma(z - y^*, t) dz \\ &+ 4(1 - \delta_{jn}) \partial_{x_j} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x} - \bar{z}, x_n) \Gamma(\bar{z} - \bar{y}, y_n, t) d\bar{z} \\ &= \partial_{y_n} G_{ij}(x, y, t) - 2\delta_{ij} \partial_{y_n} \Gamma(x - y, t) \\ &+ 4(1 - \delta_{jn}) \partial_{x_j} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x} - \bar{z}, x_n) \Gamma(\bar{z} - \bar{y}, y_n, t) d\bar{z}. \quad \square \end{aligned}$$

Proposition 2.2. *Let $x \in \mathbb{R}_+^n$. Let f be a Hölder continuous function with the exponent $0 < \alpha < 1$,*

$$[f]_\alpha = \sup_{x, z} \frac{|f(x) - f(z)|}{|x - z|^\alpha} < \infty.$$

Then, for $i, j \neq n$ or $i, j = n$, we have

$$\frac{\partial}{\partial x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i} N(x - z) f(z) dz$$

$$\begin{aligned}
 &= -\frac{\delta_{ij}}{2n}f(x) + \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial^2}{\partial x_i \partial x_j} N(x-z)f(z)dz \\
 &\quad + \delta_{jn} \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i} N(\bar{x}-\bar{z})f(\bar{z}, x_n)d\bar{z}.
 \end{aligned}$$

For $i = n, j \neq n$ or $j = n, i \neq n$, we have

$$\frac{\partial}{\partial x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i} N(x-z)f(z)dz = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial^2}{\partial x_i \partial x_j} N(x-z)f(z)dz.$$

Proof. The proof is given in Appendix A. \square

Now we define the Hardy space \mathcal{H}^1 . Define $\mathcal{N}g(x) \equiv \sup_{s>0} |g * \Gamma_s(x)|$. Let $\mathcal{H}^1(\mathbb{R}^n)$ be the space of functions g so that $\mathcal{N}g \in L^1(\mathbb{R}^n)$ with the norm $\|g\|_{\mathcal{H}^1(\mathbb{R}^n)} \equiv \|\mathcal{N}g\|_{L^1(\mathbb{R}^n)}$. The dual space of $\mathcal{H}^1(\mathbb{R}^n)$ is defined by $\text{BMO}(\mathbb{R}^n)$. Define $\mathcal{H}^1(\mathbb{R}_+^n)$ be the space of functions f so that there is $\tilde{f} \in \mathcal{H}^1(\mathbb{R}^n)$ with $\tilde{f}|_{\mathbb{R}_+^n} = f$ with the norm $\|f\|_{\mathcal{H}^1(\mathbb{R}_+^n)} \equiv \inf\{\|\tilde{f}\|_{\mathcal{H}^1(\mathbb{R}^n)} : \tilde{f}|_{\mathbb{R}_+^n} = f\}$, and define $\text{BMO}(\mathbb{R}_+^n)$ be the space of functions f so that there is $\tilde{f} \in \text{BMO}(\mathbb{R}^n)$ with $\tilde{f}|_{\mathbb{R}_+^n} = f$ with the norm $\|f\|_{\text{BMO}(\mathbb{R}_+^n)} = \inf\{\|\tilde{f}\|_{\text{BMO}(\mathbb{R}^n)} : \tilde{f}|_{\mathbb{R}_+^n} = f\}$. It is well known fact that $\nabla\Gamma_t \in \mathcal{H}^1(\mathbb{R}^n)$. Moreover, in [8] it is shown that $1_{\mathbb{R}_+^n} \partial_{x_k} \Gamma_t \in \mathcal{H}^1(\mathbb{R}^n)$ for $k \neq n$.

Proposition 2.3. Fix $a \in \mathbb{R}^n$. Then $\nabla(\tau_a \Gamma_t) \in \mathcal{H}^1(\mathbb{R}^n)$ with

$$\|\nabla(\tau_a \Gamma_t)\|_{\mathcal{H}^1} \leq Ct^{-\frac{1}{2}}.$$

The above proposition is well known and we omit the proof. Instead, we give the proof that $1_{\mathbb{R}_+^n} \nabla_{\bar{x}}(\tau_a \Gamma_t) \in \mathcal{H}^1(\mathbb{R}^n)$.

Lemma 2.4. Fix $a \in \mathbb{R}^n$. Then

$$1_{\mathbb{R}_+^n} \nabla_{\bar{x}}(\tau_a \Gamma_t) \in \mathcal{H}^1(\mathbb{R}^n)$$

with

$$\|1_{\mathbb{R}_+^n} \nabla_{\bar{x}}(\tau_a \Gamma_t)\|_{\mathcal{H}^1} \leq Ct^{-\frac{1}{2}}.$$

Proof. The proof is given in Appendix B, where we follow the idea in [8]. \square

Let $a \in \mathbb{R}_+^n$. Then $\tau_{a^*} \Gamma_t(x) = \Gamma_t(a^* - x) = \bar{\Gamma}_t(\bar{a} - \bar{x})\Gamma_t^n(a_n + x_n)$. We observe that

$$\begin{aligned}
 \int_{-\infty}^{x_n} \Gamma_t^n(a_n + x_n - y_n)\Gamma_s^n(y_n)dy_n &\leq e^{-\frac{a_n^2}{4t}} \int_{-\infty}^{x_n} \Gamma_t^n(x_n - y_n)\Gamma_s^n(y_n)dy_n \\
 &\leq e^{-\frac{a_n^2}{4t}} \int_{-\infty}^{\infty} \Gamma_t^n(x_n - y_n)\Gamma_s^n(y_n)dy_n \\
 &= e^{-\frac{a_n^2}{4t}} \Gamma_{s+t}^n(x_n).
 \end{aligned}$$

Then, we also have the following estimate.

Corollary 2.5. Fix $a = (a_1, \dots, a_n) \in \mathbb{R}_+^n$. Then we have

$$\|1_{\mathbb{R}_+^n} \nabla_{\bar{x}}(\tau_{a^*} \Gamma_t)\|_{\mathcal{H}^1} \leq Ct^{-\frac{1}{2}} e^{-\frac{a_n^2}{4t}}.$$

Lemma 2.6. Fix $a = (\bar{a}, a_n), b = (\bar{b}, a_n) \in \mathbb{R}_+^n$. Let $0 < \alpha < 1$. Then we have

$$\|1_{\mathbb{R}_+^n} [\tau_{a^*} \Gamma_t - \tau_{b^*} \Gamma_t]\|_{\mathcal{H}^1} \leq Ct^{-\alpha/2} e^{-\frac{a_n^2}{4t}} |\bar{a} - \bar{b}|^\alpha.$$

Proof. The proof is given in Appendix C. \square

Remark 2.2. In Lemma 2.4, Corollary 2.5, and Lemma 2.6, the results hold true even if we replace $1_{\mathbb{R}_+^n}$ by $\mathbb{R}^{n-1} \times K = \{x_n \in K\}$ for any one dimensional open set $K \subset \mathbb{R}$. Moreover, making use of the properties of the Gaussian kernels, we have

$$\begin{aligned} \|1_{\{x_n \in K\}} \nabla_x^k \nabla_{\bar{x}}(\tau_a \Gamma_t)\|_{\mathcal{H}^1} &\leq Ct^{-\frac{k+1}{2}}, \\ \|1_{\{x_n \in K\}} \nabla_x^k \nabla_{\bar{x}}(\tau_{a^*} \Gamma_t)\|_{\mathcal{H}^1} &\leq Ct^{-\frac{k+1}{2}} e^{-\frac{a_n^2}{4t}}, \end{aligned}$$

and

$$\|1_{\{x_n \in K\}} \nabla_x^k [\tau_{a^*} \Gamma_t - \tau_{b^*} \Gamma_t]\|_{\mathcal{H}^1} \leq Ct^{-\frac{k+\alpha}{2}} e^{-\frac{a_n^2}{4t}} |\bar{a} - \bar{b}|^\alpha.$$

Lemma 2.7. Let $j = 1, \dots, n-1$ and $i = 1, \dots, n$. Then we have

$$\begin{aligned} &\int_{\mathbb{R}_+^n} \left| \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial^2}{\partial z_i \partial z_j} N(x-z) f(z, y) dz \right| dy \\ &\leq C \sup_{x \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} |f(x, y)| dy \\ &\quad + C x_n^\alpha \sup_{x_n/2 < z_n < x_n} \sup_{\bar{z} \in \mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}_+^n} |f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)| dy}{|\bar{x} - \bar{z}|^\alpha}. \end{aligned}$$

Proof. The proof is given in Appendix D. \square

3. Estimate of $\nabla e^{-tA} \mathbb{P} \nabla \cdot \mathcal{F}$.

The following is well known (see Solonnikov [16] and Shimizu [15]).

Proposition 3.1.

$$(3.13) \quad \begin{aligned} \|\nabla^k e^{-At} f\|_{L^\infty(\mathbb{R}_+^n)} &\leq Ct^{-\frac{k}{2}} \|f\|_{L^\infty(\mathbb{R}_+^n)}, \\ \|e^{-At} \mathbb{P} \operatorname{div} \mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)} &\leq Ct^{-\frac{1}{2}} \|\mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)}. \end{aligned}$$

In this section we would like to show that

$$(3.14) \quad \|\nabla e^{-At} \mathbb{P} \operatorname{div} \mathcal{F}\|_\infty \leq Ct^{-\frac{1}{2}} \|\nabla \mathcal{F}\|_\infty.$$

Recall Proposition 2.1 that

$$\nabla_{\bar{x}} e^{-tA} \mathbb{P} \nabla \cdot \mathcal{F} = e^{-tA} \nabla_{\bar{x}} \mathbb{P} \nabla \cdot \mathcal{F}.$$

We have the identity

$$\nabla_{\bar{x}} \mathbb{P} \nabla \cdot \mathcal{F} = \mathbb{P} \nabla \cdot [\nabla_{\bar{x}} \mathcal{F}],$$

since for $k \neq n$

$$\begin{aligned}\partial_{x_k}(\mathbb{P}f)_j(x) &= \partial_{x_k}f_j(x) + \partial_{x_k}\partial_{x_j}\int_{\mathbb{R}_+^n}\nabla_y\mathcal{G}(x,y)\cdot f(y)dy \\ &= \partial_{x_k}f_j(x) + \partial_{x_j}\int_{\mathbb{R}_+^n}\nabla_y\mathcal{G}(x,y)\cdot\partial_{y_k}f(y)dy.\end{aligned}$$

Applying the known result (3.14) of Proposition 3.1, we have the following estimate.

Lemma 3.2.

$$\|\nabla_{\bar{x}}e^{-At}\mathbb{P}\operatorname{div}\mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)} = \|e^{-At}\mathbb{P}\operatorname{div}\nabla_{\bar{x}}\mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)} \leq ct^{-\frac{1}{2}}\|\nabla_{\bar{x}}\mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)}.$$

From the divergence free property of $e^{-At}\mathbb{P}\operatorname{div}\mathcal{F}$, we have that

$$\partial_{x_n}(e^{-At}\mathbb{P}\operatorname{div}\mathcal{F})_n = -\sum_{j=1}^{n-1}\partial_{x_j}(e^{-At}\mathbb{P}\operatorname{div}\mathcal{F})_j.$$

Hence we have the following estimate.

Lemma 3.3.

$$\|\partial_{x_n}(e^{-At}\mathbb{P}\operatorname{div}\mathcal{F})_n\|_{L^\infty(\mathbb{R}_+^n)} \leq ct^{-\frac{1}{2}}\|\nabla_{\bar{x}}\mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)}.$$

Now let us consider the case $i \neq n$. Denote by $H = (H_{ijk})_{i,j,k=1}^n$ the kernel tensors of the operator $e^{-tA}\mathbb{P}\operatorname{div}$. Then

$$[e^{-tA}\mathbb{P}\operatorname{div}\mathcal{F}]_i(x) = \int_{\mathbb{R}_+^n}H_{ijk}(x,y,t)F_{jk}(y)dy.$$

Direct computation shows that

$$H_{ijk}(x,y,t) = -\partial_{y_k}G_{ij}(x,y,t) + \sum_{l=1}^n\partial_{y_j}\int_{\mathbb{R}_+^n}\partial_{y_k}\mathcal{G}(z,y)\partial_{z_l}G_{il}(x,z,t)dz.$$

Moreover, since $G_{in} = 0$ if $i \neq n$, we have

(3.15)

$$H_{ijk}(x,y,t) = -\partial_{y_k}G_{ij}(x,y,t) + \sum_{l=1}^{n-1}\partial_{y_j}\int_{\mathbb{R}_+^n}\partial_{y_k}\mathcal{G}(z,y)\partial_{z_l}G_{il}(x,z,t)dz.$$

From Proposition 2.1, we have

$$\begin{aligned}\partial_{x_n}G_{il}(x,z,t) &= \partial_{z_n}G_{il}(x,z,t) - 2\delta_{il}\partial_{z_n}\Gamma(x-z,t) \\ &\quad - 4(1-\delta_{ln})\partial_{x_l}\int_{\mathbb{R}^{n-1}}(\partial_{x_i}N)(\bar{x}-\bar{w},x_n)\Gamma(\bar{w}-\bar{z},z_n,t)d\bar{w}.\end{aligned}$$

Apply Proposition 2.2 for $i, l \neq n$, then we have

$$G_{il}(x,z,t) = \delta_{il}(\Gamma(x-z,t) - \Gamma(x-z^*,t)) - \frac{2\delta_{il}}{n}\Gamma(x-z^*,t)$$

$$+ 4 \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{x_l} N(x-w) \Gamma(w-z^*, t) dw.$$

Hence we have that for $i, l \neq n$,

$$\begin{aligned} & \partial_{x_n} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{z_l} G_{il}(x, z, t) dz \\ &= \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{z_l} \partial_{x_n} G_{il}(x, z, t) dz \\ &= 4 \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{z_n} \partial_{z_l} \left[\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{x_l} N(x-w) \Gamma(w-z^*, t) dw \right] dz \\ &\quad - \delta_{il} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{z_l} \partial_{z_n} \left[\Gamma(x-z, t) + \left(1 + \frac{2}{n}\right) \Gamma(x-z^*, t) \right] dz \\ &\quad - 4 \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{z_l} \left[\partial_{x_i} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x}-\bar{w}, x_n) \Gamma(\bar{w}-\bar{z}, z_n, t) d\bar{w} \right] dz. \end{aligned}$$

Note that $\int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{z_n} f(z) dz = \partial_{y_n} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}^-(z, y) f(z) dz$, where

$$\mathcal{G}^-(z, y) \equiv N(z-y) - N(z-y^*),$$

since $\partial_{x_n} [\Gamma(x-z, t) - \Gamma(x-z^*, t)] = -\partial_{z_n} [\Gamma(x-z, t) + \Gamma(x-z^*, t)]$. Thus, we obtain that for $i, l \neq n$,

(3.16)

$$\begin{aligned} & \partial_{x_n} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{z_l} G_{il}(x, z, t) dz \\ &= 4 \partial_{y_n} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}^-(z, y) \left[\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{x_l} N(x-w) \partial_{w_l} \Gamma(w-z^*, t) dw \right] dz \\ &\quad - \delta_{il} \partial_{y_n} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}^-(z, y) \partial_{z_l} \left[\Gamma(x-z, t) + \left(1 + \frac{2}{n}\right) \Gamma(x-z^*, t) \right] dz \\ &\quad - 4 \partial_{y_l} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \left[\partial_{x_i} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x}-\bar{w}, x_n) \Gamma(\bar{w}-\bar{z}, z_n, t) d\bar{w} \right] dz. \end{aligned}$$

Combining the above estimates (3.15) and (3.16), we obtain

$$\begin{aligned} \partial_{x_n} H_{ijk}(x, y, t) &= -\partial_{x_n} \partial_{y_k} G_{ij}(x, y, t) \\ &\quad + \sum_{l=1}^{n-1} \partial_{y_j} (\tilde{H}_{ijk,l}^1(x, z, t) + \tilde{H}_{ijk,l}^2(x, y, t) + \tilde{H}_{ijk,l}^3(x, z, t)), \end{aligned}$$

where

$$\begin{aligned} & \tilde{H}_{ijk,l}^1(x, y, t) \\ &\equiv -\delta_{il} \partial_{y_n} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}^-(z, y) \partial_{z_l} \left[\Gamma(x-z, t) + \left(1 + \frac{2}{n}\right) \Gamma(x-z^*, t) \right] dz, \end{aligned}$$

$$\begin{aligned}
 & \tilde{H}_{ijk,l}^2(x, y, t) \\
 \equiv & -4\partial_{y_n} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}^-(z, y) \partial_{z_l} \left[\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{x_l} N(x-w) \Gamma(w-z^*, t) dw \right] dz, \\
 & \tilde{H}_{ijk,l}^3(x, y, t) \\
 \equiv & -4\partial_{y_l} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \left[\partial_{x_i} \int_{\mathbb{R}^{n-1}} \partial_{x_i} N(\bar{x}-\bar{w}, x_n) \Gamma(\bar{w}-\bar{z}, z_n, t) d\bar{w} \right] dz.
 \end{aligned}$$

If $\mathcal{F} = (F_{jk}) = 0$ on $x_n = 0$, taking integrations by parts, $\partial_{x_n}(e^{-At}\mathbb{P}\operatorname{div}\mathcal{F})_i$ can be written by

$$\begin{aligned}
 & \partial_{x_n}(e^{-At}\mathbb{P}\operatorname{div}\mathcal{F})_i(x, t) \\
 = & \int_{\mathbb{R}_+^n} \partial_{x_n} H_{ijk}(x, y, t) F_{jk}(y) dy \\
 = & \int_{\mathbb{R}_+^n} \partial_{x_n} G_{ij}(x, y, t) \partial_{y_k} F_{jk}(y) dy \\
 & - \sum_{l=1}^{n-1} \int_{\mathbb{R}_+^n} (\tilde{H}_{ijk,l}^1 + \tilde{H}_{ijk,l}^2 + \tilde{H}_{ijk,l}^3)(x, y, t) \partial_{y_j} F_{jk}(y) dy \\
 = & \partial_{x_n}(e^{-tA}\operatorname{div}\mathcal{F})_i(x) - \sum_{l=1}^{n-1} \int_{\mathbb{R}_+^n} (\tilde{H}_{ijk,l}^1 + \tilde{H}_{ijk,l}^2 + \tilde{H}_{ijk,l}^3)(x, y, t) \partial_{y_j} F_{jk}(y) dy.
 \end{aligned}$$

Hence

$$\begin{aligned}
 & \left\| \partial_{x_n}(e^{-At}\mathbb{P}\operatorname{div}\mathcal{F})_i \right\|_{L^\infty} \\
 \leq & C \|\partial_{x_n} e^{-At}\operatorname{div}\mathcal{F}\|_{L^\infty} + C \max_{1 \leq j, k \leq n} \int_{\mathbb{R}_+^n} (|\tilde{H}_{ijk}^1(x, y, t)| + |\tilde{H}_{ijk}^2(x, y, t)| \\
 & + |\tilde{H}_{ijk}^3(x, y, t)|) dy \left\| \partial_{y_k} F_{jk} \right\|_{L^\infty}.
 \end{aligned}$$

From the known result (3.13) of Proposition 3.1, we have

$$\|\partial_{x_n} e^{-At}\operatorname{div}\mathcal{F}\|_{L^\infty} \leq Ct^{-1/2} \|\operatorname{div}\mathcal{F}\|_{L^\infty}.$$

Hence, we have only to estimate

$$\int_{\mathbb{R}_+^n} (|\tilde{H}_{ijk}^1(x, y, t)| + |\tilde{H}_{ijk}^2(x, y, t)| + |\tilde{H}_{ijk}^3(x, y, t)|) dy$$

for $i, l \neq n$. For the estimation, we will make use of the Hardy space theory.

First let us estimate $\int_{\mathbb{R}_+^n} |\tilde{H}_{ijk}^1(x, y, t)| dy$. Note that $\tilde{H}_{ijk}^1(x, y, t)$ is the linear sum of the terms

$$\begin{aligned}
 & \partial_{y_n} \int_{\mathbb{R}^n} \partial_{y_k} N(z-y) 1_{\mathbb{R}_+^n} \partial_{z_l} \Gamma(x-z, t) dz, \\
 & \partial_{y_n} \int_{\mathbb{R}^n} \partial_{y_k} N(z-y) 1_{\mathbb{R}_-^n} \partial_{z_l} \Gamma(x-z, t) dz,
 \end{aligned}$$

$$\partial_{y_n} \int_{\mathbb{R}^n} \partial_{y_k} N(z-y) 1_{\mathbb{R}_+^n} \partial_{z_l} \Gamma(x-z^*, t) dz,$$

and

$$\partial_{y_n} \int_{\mathbb{R}^n} \partial_{y_k} N(z-y) 1_{\mathbb{R}_-^n} \partial_{z_l} \Gamma(x-z^*, t) dz.$$

Recall the well known result that $1_{\{z_n > 0\}} \partial_{z_l} \Gamma(x-z, t)$, $1_{\{z_n < 0\}} \partial_{z_l} \Gamma(x-z, t)$ are in Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ for any fixed $x \in \mathbb{R}^n$. By the boundedness of the Calderon-Zygmund type transforms in Hardy spaces, we obtain that

$$\begin{aligned} \int_{\mathbb{R}_+^n} |\tilde{H}_{ijk}^1(x, y, t)| dy &\leq C \left\| 1_{\mathbb{R}_+^n} \partial_{z_i} \Gamma(x-\cdot, t) \right\|_{\mathcal{H}^1} + C \left\| 1_{\mathbb{R}_-^n} \partial_{z_i} \Gamma(x^*-\cdot, t) \right\|_{\mathcal{H}^1} \\ &\quad + C \left\| 1_{\mathbb{R}_+^n} \partial_{z_i} \Gamma(x^*-\cdot, t) \right\|_{\mathcal{H}^1} + C \left\| 1_{\mathbb{R}_-^n} \partial_{z_i} \Gamma(x-\cdot, t) \right\|_{\mathcal{H}^1} \\ &\leq Ct^{-1/2}. \end{aligned}$$

We rewrite $\tilde{H}_{ijk,l}^2$ by

$$\begin{aligned} &\tilde{H}_{ijk,l}^2(x, y, t) \\ &= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial^2}{\partial x_n \partial x_i} N(x-w) \left[\partial_{y_l} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{w_l} \Gamma(w-z^*, t) dz \right] dw. \end{aligned}$$

Set

$$J_{kl}(w, y, t) \equiv \partial_{y_l} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \partial_{w_l} \Gamma(w-z^*, t) dz.$$

By Lemma 2.7 we have

$$\begin{aligned} (3.17) \quad &\int_{\mathbb{R}_+^n} |\tilde{H}_{ijk,l}^2(x, y, t)| dy \\ &\leq C \sup_{y \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} |J_{kl}(w, y, t)| dy \\ &\quad + C x_n^\alpha \sup_{x_n/2 < z_n < x_n} \sup_{\bar{w} \in \mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}_+^n} |J_{kl}(w, y, t) - J_{kl}(\bar{x}, w_n, y, t)| dy}{|\bar{x} - \bar{w}|^\alpha}. \end{aligned}$$

Note that

$$\begin{aligned} J_{kl}(w, y, t) &= \partial_{y_l} \int_{\mathbb{R}_+^n} \partial_{y_k} [N(z-y) + N(z-y^*)] \partial_{w_l} \Gamma(w-z^*, t) dz \\ &= \partial_{y_l} \int_{\mathbb{R}^n} \partial_{y_k} N(z-y) 1_{\{z_n > 0\}} \partial_{w_l} \Gamma(w-z^*, t) dz \\ &\quad + \partial_{y_l} \int_{\mathbb{R}^n} \partial_{y_k} N(z-y) 1_{\{z_n < 0\}} \partial_{w_l} \Gamma(w-z, t) dz. \end{aligned}$$

Recall the well known result that $1_{\mathbb{R}_+^n} \partial_{w_l} \Gamma(w-z^*, t)$, $1_{\mathbb{R}_-^n} \partial_{w_l} \Gamma(w-z, t)$ are in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ for any fixed $w \in \mathbb{R}^n$. Since Calderon-Zygmund type

transforms are bounded in Hardy spaces, we obtain that

$$(3.18) \quad \int_{\mathbb{R}_+^n} |J_{kl}(w, y, t)| dy \leq C \left\| 1_{\mathbb{R}_+^n} \partial_{w_l} \Gamma(w^* - \cdot, t) \right\|_{\mathcal{H}^1} + \left\| 1_{\mathbb{R}_+^n} \partial_{w_l} \Gamma(w - \cdot, t) \right\|_{\mathcal{H}^1} \leq Ct^{-1/2}.$$

Note that

$$\begin{aligned} & J_{kl}(w, y, t) - J_{kl}(\bar{x}, w_n, y, t) \\ &= \partial_{y_l} \int_{\mathbb{R}_+^n} \partial_{y_k} [N(z - y) + N(z - y^*)] \partial_{w_l} [\Gamma(w - z^*, t) - \Gamma(\bar{x} - \bar{z}, w_n + z_n, t)] dz. \end{aligned}$$

Recall Lemma 2.6 that for any $a = (\bar{w}, w_n)$, $b = (\bar{x}, w_n) \in \mathbb{R}_+^n$,

$$1_{\mathbb{R}_+^n} \left[\partial_{z_l} (\tau_{a^*} \Gamma_t) - \partial_{z_l} (\tau_{b^*} \Gamma_t) \right] \in \mathcal{H}^1(\mathbb{R}^n),$$

of which norm is bounded by $Ct^{-(1+\alpha)/2} e^{-\frac{w_n^2}{4t}} |\bar{w} - \bar{x}|^\alpha$ for $0 < \alpha < 1$. Hence by the same reasoning for J_{kl} , we conclude that

$$(3.19) \quad \int_{\mathbb{R}_+^n} |J_{kl}(w, y, t) - J_{kl}(\bar{x}, w_n, y, t)| dy \leq Ct^{-(1+\alpha)/2} e^{-\frac{w_n^2}{4t}} |\bar{w} - \bar{x}|^\alpha.$$

Apply the estimates (3.18) and (3.19) to (3.17) we have

$$\int_{\mathbb{R}_+^n} |\tilde{H}_{ijk,l}^2(x, y, t)| dy \leq Ct^{-1/2} + Cx_n^\alpha t^{-(1+\alpha)/2} e^{-\frac{x_n^2}{4t}} \leq Ct^{-1/2}.$$

Divide the domain of integration into two parts that

$$\begin{aligned} \tilde{H}_{ijk,l}^3(x, y, t) &= p.v. \int_{|\bar{x}-\bar{w}| \leq \sqrt{t}} \partial_{x_i} N(\bar{x} - \bar{w}, x_n) J_{kl}((\bar{w}, 0), y, t) d\bar{w} \\ &\quad + p.v. \int_{|\bar{x}-\bar{w}| > \sqrt{t}} \partial_{x_i} N(\bar{x} - \bar{w}, x_n) J_{kl}((\bar{w}, 0), y, t) d\bar{w} \\ &\equiv I + II. \end{aligned}$$

By integrations by parts

$$\begin{aligned} I &= -p.v. \int_{|\bar{x}-\bar{w}| \leq \sqrt{t}} N(\bar{x} - \bar{w}, x_n) \partial_{y_i} J_{kl}((\bar{w}, 0), y, t) d\bar{w} \\ &\quad + J_{kl}((\bar{x}, 0), y, t) \lim_{\epsilon \rightarrow 0} \int_{|\bar{x}-\bar{w}|=\epsilon} \frac{x_i - y_i}{|(\bar{x} - \bar{w})^2 + x_n^2|^{(n-1)/2}} dS_{\bar{w}} \\ &\quad - \int_{|\bar{x}-\bar{w}|=\sqrt{t}} \frac{x_i - y_i}{|\bar{x} - \bar{w}|^{n-1}} J_{kl}((\bar{w}, 0), y, t) dS_{\bar{w}}. \end{aligned}$$

By the anti-symmetry, the second term is zero, hence we have

$$I = -p.v. \int_{|\bar{x}-\bar{w}| \leq \sqrt{t}} N(\bar{x} - \bar{w}, x_n) \partial_{y_i} J_{kl}((\bar{w}, 0), y, t) d\bar{w}$$

$$\begin{aligned}
& - \int_{|\bar{x}-\bar{w}|=\sqrt{t}} \frac{x_i - y_i}{|(\bar{x} - \bar{w})^2 + x_n^2|^{(n-1)/2}} J_{kl}((\bar{w}, 0), y, t) dS_{\bar{w}} \\
& \equiv I_1 + I_2.
\end{aligned}$$

Recall the fact that $\partial_{y_i} J_{kl}$ is in Hardy space whose norm is bounded by Ct^{-1} . If we integrate I over \mathbb{R}_+^n in terms of y variables, then we have

$$\begin{aligned}
\int_{\mathbb{R}_+^n} |I_1(x, y, t)| dy & \leq \int_{|\bar{x}-\bar{w}| \leq \sqrt{t}} \frac{d\bar{w}}{|\bar{x} - \bar{w}|^{n-2}} \sup_{\bar{w} \in \mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} |\partial_{y_i} J_{kl}((\bar{w}, 0), y, t)| dy \\
& \leq C\sqrt{t} \sup_{\bar{w} \in \mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} |\partial_{y_i} J_{kl}((\bar{w}, 0), y, t)| dy \leq Ct^{-1/2},
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}_+^n} |I_2(x, y, t)| dy & \leq \int_{|\bar{x}-\bar{w}|=\sqrt{t}} \frac{dS_{\bar{w}}}{|\bar{x} - \bar{w}|^{n-2}} \left(\sup_{\bar{w} \in \mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} |J_{kl}((\bar{w}, 0), y, t)| dy \right) \\
& \leq C \sup_{\bar{w} \in \mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} |J_{kl}((\bar{w}, 0), y, t)| dy \leq Ct^{-1/2}.
\end{aligned}$$

Set

$$\tilde{J}_{kl}(\bar{w}, y, t) \equiv \partial_{y_l} \int_{\mathbb{R}_+^n} \partial_{y_k} \mathcal{G}(z, y) \Gamma((\bar{w} - \bar{z}, 0), t) dz.$$

Note that $J_{kl}((\bar{w}, 0), y, t) = \partial_{w_l} \tilde{J}_{kl}(\bar{w}, y, t)$. Since $\partial_{w_l} \tilde{J}_{kl}(\bar{x}, y, t) = 0$, by integration by parts, we obtain that

$$\begin{aligned}
II & = p.v. \int_{|\bar{x}-\bar{w}| > \sqrt{t}} \partial_{x_i} \partial_{x_l} N(\bar{x} - \bar{w}, x_n) [\tilde{J}_{kl}(\bar{w}, y, t) - \tilde{J}_{kl}(\bar{x}, y, t)] d\bar{w} \\
& - \lim_{R \rightarrow \infty} \int_{|\bar{x}-\bar{w}|=R} \frac{x_i - y_i}{|\bar{x} - \bar{w}|^2 + x_n^2|^{(n-1)/2}} \frac{y_l - x_l}{|\bar{x} - \bar{y}|} [\tilde{J}_{kl}(\bar{w}, y, t) - \tilde{J}_{kl}(\bar{x}, y, t)] dS_{\bar{w}} \\
& + \int_{|\bar{x}-\bar{w}|=\sqrt{t}} \frac{x_i - y_i}{|\bar{x} - \bar{w}|^2 + x_n^2|^{(n-1)/2}} \frac{y_l - x_l}{|\bar{x} - \bar{y}|} [\tilde{J}_{kl}(\bar{w}, y, t) - \tilde{J}_{kl}(\bar{x}, y, t)] dS_{\bar{w}}.
\end{aligned}$$

Since

$$\lim_{R \rightarrow \infty} \int_{|\bar{x}-\bar{w}|=R} \frac{R}{|\bar{x} - \bar{w}|^2 + x_n^2|^{n-1}} dS_{\bar{w}} \leq \lim_{R \rightarrow \infty} \frac{1}{R^{n-1}} \int_{|\bar{x}-\bar{w}|=R} dS_{\bar{w}} = 0,$$

we have

$$\begin{aligned}
II & = p.v. \int_{|\bar{x}-\bar{w}| \geq \sqrt{t}} \partial_{x_i} \partial_{x_l} N(\bar{x} - \bar{w}, x_n) [\tilde{J}_{kl}(\bar{w}, y, t) - \tilde{J}_{kl}(\bar{x}, y, t)] d\bar{w} \\
& + \int_{|\bar{x}-\bar{w}|=\sqrt{t}} \frac{(x_i - y_i)(x_l - y_l)}{|\bar{x} - \bar{w}|^2 + x_n^2|^{(n+1)/2}} [\tilde{J}_{kl}(\bar{w}, y, t) - \tilde{J}_{kl}(\bar{x}, y, t)] dS_{\bar{w}} \\
& \equiv II_1 + II_2.
\end{aligned}$$

If we integrate II over \mathbb{R}_+^n in terms of y variables, then we have

$$\int_{\mathbb{R}_+^n} |II_1(x, y, t)| dy$$

$$\begin{aligned}
&\leq \int_{|\bar{x}-\bar{w}|\geq\sqrt{t}} \frac{d\bar{w}}{|\bar{x}-\bar{w}|^{n-\alpha}} \sup_{\bar{w}\in\mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}_+^n} |\tilde{J}_{kl}(\bar{w}, y, t) - \tilde{J}_{kl}(\bar{x}, y, t)| dy}{|\bar{x}-\bar{w}|^\alpha} \\
&\leq C(\sqrt{t} + x_n)^{-1+\alpha} \sup_{\bar{w}\in\mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}_+^n} |\tilde{J}_{kl}(\bar{w}, y, t) - \tilde{J}_{kl}(\bar{x}, y, t)| dy}{|\bar{x}-\bar{w}|^\alpha}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} |II_2(x, y, t)| dy \\
&\leq \int_{|\bar{x}-\bar{w}|=\sqrt{t}} \frac{dS_{\bar{w}}}{|\bar{x}-\bar{w}|^{n-1-\alpha}} \sup_{\bar{w}\in\mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}_+^n} |\tilde{J}_{kl}(\bar{w}, y, t) - \tilde{J}_{kl}(\bar{x}, y, t)| dy}{|x-\bar{w}|^\alpha} \\
&\leq C(\sqrt{t} + x_n)^{-1+\alpha} \sup_{\bar{w}\in\mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}_+^n} |\tilde{J}_{kl}(\bar{w}, y, t) - \tilde{J}_{kl}(\bar{x}, y, t)| dy}{|x-\bar{w}|^\alpha}.
\end{aligned}$$

Now recall that $\Gamma(\bar{w} - \bar{z}, z_n, t) - \Gamma(\bar{x} - \bar{z}, z_n, t) \in \mathcal{H}^1(\mathbb{R}_+^n)$ for any fixed $\bar{w}, \bar{x} \in \mathbb{R}^{n-1}$ whose norm is bounded by $Ct^{-\alpha/2}$ and \tilde{J}_{kl} is linear sum of the terms

$$\begin{aligned}
&\partial_{y_n} \int_{\mathbb{R}^n} \partial_{y_k} N(z-y) \mathbf{1}_{\mathbb{R}_+^n} \partial_{z_l} [\Gamma(\bar{w} - \bar{z}, z_n, t) - \Gamma(\bar{x} - \bar{z}, z_n, t)] dz, \\
&\partial_{y_n} \int_{\mathbb{R}^n} \partial_{y_k} N(z-y) \mathbf{1}_{\mathbb{R}_-^n} [\Gamma(\bar{w} - \bar{z}, z_n, t) - \Gamma(\bar{x} - \bar{z}, z_n, t)] dz.
\end{aligned}$$

By the boundedness of Calderon-Zygmund type transforms in Hardy spaces, we obtain that

$$\int_{\mathbb{R}_+^n} |\tilde{J}_{kl}(\bar{w}, y, t) - \tilde{J}_{kl}(\bar{x}, y, t)| dy \leq Ct^{-\alpha/2} |\bar{x} - \bar{w}|^\alpha, \quad 0 < \alpha < 1.$$

Hence we have

$$\int_{\mathbb{R}_+^n} |II(x, y, t)| dy \leq Ct^{-1/2}.$$

Therefore we conclude that:

Lemma 3.4.

$$\|\partial_{x_n} e^{-tA} \mathbb{P} \operatorname{div} \mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)} \leq Ct^{-1/2} \|\partial_{x_n} \mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)}.$$

Combining Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have the following estimate for $\nabla e^{-tA} \mathbb{P} \operatorname{div} \mathcal{F}$.

Theorem 3.5.

$$\|\nabla e^{-tA} \mathbb{P} \operatorname{div} \mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)} \leq Ct^{-1/2} \|\nabla \mathcal{F}\|_{L^\infty(\mathbb{R}_+^n)}.$$

4. Proof of Theorem 1.2

Set $u_1 = e^{-tA}a$ and $u_0 = 0$. For given u_{n-1} , define u_n recursively by

$$u_n = e^{-tA}a + B(u_{n-1}, u_{n-1}).$$

In [16], it is well known that

$$\|e^{-tA}a\|_\infty \leq C_0\|a\|_\infty,$$

and from Proposition 3.1 we have the estimate

$$\|e^{-(t-s)A}\mathbb{P}\operatorname{div}(u_{n-1} \otimes u_{n-1})\|_\infty \leq C_1(t-s)^{-\frac{1}{2}}\|(u_{n-1} \otimes u_{n-1})\|_\infty.$$

Hence we have the following estimate for $\|u_n\|_\infty$.

$$(4.20) \quad \|u_n\|_\infty \leq C_0\|u_0\|_\infty + C_1 \int_0^t (t-s)^{-\frac{1}{2}} \|u_{n-1}(s)\|_\infty^2 ds.$$

Take large $M \geq 2C_0\|a\|_{L^\infty}$. Suppose $\sup_{0 < t < T} \|u_{n-1}(t)\|_\infty \leq M$, then from (4.20) we have

$$\begin{aligned} \|u_n(t)\|_\infty &\leq C_0\|u_0\|_\infty + C_1 \left[\int_0^t (t-s)^{-\frac{1}{2}} ds \right] \left[\sup_{0 < s < t} \|u_{n-1}(t)\|_\infty \right]^2 \\ &\leq \frac{M}{2} + C_1 C_2 \sqrt{t} M^2, \end{aligned}$$

where $\int_0^t (t-s)^{-\frac{1}{2}} ds = C_2\sqrt{t}$. Hence we have the inequality

$$\sup_{0 < t < T} \|u_n(t)\|_\infty \leq \frac{M}{2} + C_1 C_2 \sqrt{T} M^2.$$

Take $T > 0$ small enough that $2C_1 C_2 \sqrt{T} M < 1$, then we have

$$\sup_{0 < t < T} \|u_n(t)\|_\infty \leq M.$$

By (3.13) of Proposition 3.1, we have the estimate

$$\|\nabla e^{-tA}a\|_\infty \leq C_3 t^{-1/2} \|a\|_\infty,$$

and by Theorem 3.5, we have the estimate

$$\|\nabla e^{-(t-s)A}\mathbb{P}\operatorname{div}(u_{n-1} \otimes u_{n-1})\|_\infty \leq C_4(t-s)^{-\frac{1}{2}}\|\nabla(u_{n-1} \otimes u_{n-1})\|_\infty.$$

Hence we have the following estimate for $\|\nabla u_n\|_{L^\infty(\mathbb{R}_+^n)}$.

$$(4.21) \quad \|\nabla u_n\|_\infty \leq C_3 t^{-1/2} \|u_0\|_\infty + \int_0^t (t-s)^{-\frac{1}{2}} C_4 \|u_{n-1}(s)\|_\infty \|\nabla u_{n-1}(s)\|_\infty ds.$$

Take large $M_1 \geq 2C_3\|a\|_\infty$. Suppose $\sup_{0 < t < T} t^{1/2} \|\nabla u_{n-1}(t)\|_\infty \leq M_1$, then from (4.21) we have

$$\|\nabla u_n(t)\|_\infty \leq C_3 t^{-1/2} \|u_0\|_\infty$$

$$\begin{aligned}
 & + C_4 \left[\int_0^t (t-s)^{-\frac{1}{2}} s^{-1/2} ds \right] \left[\sup_{0 < s < t} \|u_{n-1}(s)\|_\infty \right] \left[\sup_{0 < s < t} s^{1/2} \|\nabla u_{n-1}(s)\|_\infty \right] \\
 & \leq t^{-1/2} \frac{M_1}{2} + C_4 C_5 M M_1,
 \end{aligned}$$

where $\int_0^t (t-s)^{-\frac{1}{2}} s^{-1/2} ds = C_5$. Hence we have the inequality

$$\sup_{0 < t < T} t^{1/2} \|\nabla u_n\|_\infty \leq \frac{M_1}{2} + C_4 C_5 M M_1 \sqrt{T}.$$

Take $T > 0$ small enough that $4C_4 C_5 M \sqrt{T} < 1$, then we have

$$\sup_{0 < t < T} t^{1/2} \|\nabla u_n(t)\|_\infty \leq M_1.$$

This implies that $\{u_n\}_{n=1}^\infty$ is uniformly bounded in $L^\infty((0, T); W^{1,\infty}(\mathbb{R}_+^n))$, if T is so small that

$$2C_1 C_2 \sqrt{T} M < 1, \quad 4C_4 C_5 M \sqrt{T} < 1,$$

where $M = 2C_0 \|a\|_\infty$ and $M_1 = 2C_3 \|a\|_\infty$.

Set $w_n = u_{n+1} - u_n$, $n = 1, 2, \dots$. Note that

$$\sup_{0 < t < T} \|w_0(t)\|_\infty = \sup_{0 < t < T} \|e^{-tA} u_0\|_\infty \leq C_0 \|a\|_\infty.$$

Note

$$B(u_n, u_n) - B(u_{n-1}, u_{n-1}) = B(u_n, u_n - u_{n-1}) + B(u_n - u_{n-1}, u_{n-1}).$$

Then we also have the following estimate for $\|w_n\|_\infty$.

$$(4.22) \quad \|w_n(t)\|_\infty \leq C_1 \int_0^t (t-s)^{-\frac{1}{2}} (\|u_n(s)\|_\infty + \|u_{n-1}(s)\|_\infty) \|w_{n-1}(s)\|_\infty ds.$$

Recall that if $M \geq 2C_0 \|a\|_{L^\infty(\mathbb{R}_+^n)}$ and $2C_1 C_2 M \sqrt{T} < 1$, then

$$\sup_{0 < t < T} \|u_n\|_{L^\infty(\mathbb{R}_+^n)} \leq M$$

for all $n = 1, 2, \dots$. From (4.22) we have

$$\begin{aligned}
 \sup_{0 < t < T} \|w_n(t)\|_\infty & \leq 2C_1 C_2 M \sqrt{T} \sup_{0 < t < T} \|w_{n-1}(t)\|_\infty \\
 & \leq (2C_1 C_2 M \sqrt{T})^n C_0 \|a\|_\infty.
 \end{aligned}$$

Since $2C_1 C_2 M \sqrt{T} < 1$, the series $\sum_{n=1}^\infty w_n(t)$ converges to some $u \in L^\infty((0, T) \times \mathbb{R}_+^n)$. This implies u_n converges to some u in $L^\infty((0, T) \times \mathbb{R}_+^n)$, since $\sum_{k=0}^{n-1} w_k(t) = u_n(t)$. By the lower semi-continuity, we have

$$\sup_{0 < t < T} \|u(t)\|_\infty \leq M, \quad \sup_{0 < t < T} t^{1/2} \|\nabla u(t)\|_\infty \leq M_1.$$

5. Proof of Theorem 1.3

Let u and v be the mild solution of the Navier-Stokes equations in the class $L_{\text{loc}}^\infty(0, T; L^\infty(\mathbb{R}_+^n))$ with the property

$$\sup_{0 < t < T} t^{\frac{\alpha}{2}} \|u(t)\|_\infty = M_1 \quad \text{and} \quad \sup_{0 < t < T} t^{\frac{\alpha}{2}} \|v(t)\|_\infty = M_2.$$

Here, $0 < \alpha < 1$.

Set $w = u - v$, then w satisfies the integral equation

$$w = B(u, u) - B(v, v) = B(w, u) + B(v, w).$$

Apply Proposition 3.1, then we obtain the following Gronwall inequality

$$\begin{aligned} \|w(t)\|_\infty &\leq C_1 \int_0^t (t-s)^{-\frac{1}{2}} (\|u(s)\|_\infty + \|v(s)\|_\infty) \|w(s)\|_\infty ds \\ (5.23) \quad &\leq C_1(M_1 + M_2) \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{\alpha}{2}} \|w(s)\|_\infty ds. \end{aligned}$$

Set $X(t) = \|w(t)\|_\infty$. Then X satisfies inequality

$$(5.24) \quad X(t) \leq C_1(M_1 + M_2) \left[\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{\alpha}{2}} X(s) ds \right].$$

Note that $\int_0^t (t-s)^{-\frac{1}{2}} s^{-\alpha} ds = C_2 t^{\frac{1}{2}-\alpha}$ if $0 \leq \alpha < 1$. Then

$$(5.25) \quad \sup_{0 < t \leq t_0} t^{\frac{\alpha}{2}} X(t) \leq C_1 C_2 (M_1 + M_2) t_0^{\frac{1}{2}-\frac{\alpha}{2}} \left(\sup_{0 < s \leq t_0} s^{\frac{\alpha}{2}} X(s) \right).$$

Choose t_0 small enough that $C_1 C_2 (M_1 + M_2) t_0^{\frac{1}{2}-\frac{\alpha}{2}} < 1$, then (5.25) implies that

$$\sup_{0 < t \leq t_0} t^{\frac{\alpha}{2}} X(t) = 0.$$

This again implies that $X(t) = 0$ for all $0 < t \leq t_0$. Then from (5.24) we have

$$\begin{aligned} X(t) &\leq C_1(M_1 + M_2) \left[\int_{t_0}^t (t-s)^{-\frac{1}{2}} s^{-\frac{\alpha}{2}} X(s) ds \right] \\ (5.26) \quad &\leq C_1(M_1 + M_2) t_0^{-\frac{\alpha}{2}} \int_{t_0}^t (t-s)^{-\frac{1}{2}} X(s) ds. \end{aligned}$$

Iterating the estimate (5.26), we obtain

$$\begin{aligned} X(t) &\leq C_1^2 (M_1 + M_2)^2 t_0^{-\alpha} \int_{t_0}^t (t-s)^{-\frac{1}{2}} \left[\int_{t_0}^s (s-\tau)^{-\frac{1}{2}} X(\tau) d\tau \right] ds \\ &= C_1^2 (M_1 + M_2)^2 t_0^{-\alpha} \int_{t_0}^t \left[\int_\tau^t (t-s)^{-\frac{1}{2}} (s-\tau)^{-\frac{1}{2}} ds \right] X(\tau) d\tau \\ (5.27) \quad &\leq C_1^2 C_3 (M_1 + M_2)^2 t_0^{-\alpha} \int_{t_0}^t X(\tau) d\tau. \end{aligned}$$

Here, we noted that $\int_\tau^t (t-s)^{-\frac{1}{2}} (s-\tau)^{-\frac{1}{2}} ds = C_3 < \infty$.

Set $Y(t) = \int_{t_0}^t X(\tau) d\tau$. Then from (5.27) we have the Gronwall inequality

$$Y'(t) \leq C_1^2 C_3 (M_1 + M_2)^2 t_0^{-\alpha} Y(t) \quad \text{and} \quad Y(t_0) = 0.$$

Solving the above Gronwall inequality we conclude that $Y(t) \equiv 0$ for $t_0 < t < T$. Applying this result to (5.27), then we have $X(t) \leq C_1^2 C_3 (M_1 + M_2)^2 t_0^{-\alpha} Y(t) = 0$ for $t_0 < t < T$.

Therefore we conclude that $\|w(t)\|_\infty = X(t) = 0$ for all $0 < t < T$, that is, $u(t) = v(t)$ for all $0 < t < T$.

Appendix A. Proof of Proposition 2.2.

Let $x \in \mathbb{R}_+^n$. Choose a nonnegative smooth function $\phi \in C_0^\infty([0, 2])$ with $\phi(t) = 1$ for $0 \leq t \leq 1$ and $\int_0^2 \phi(t) dt = 1$. For $\epsilon < \frac{x_n}{2}$, define a cut-off function $\phi_{\epsilon, x}$ by $\phi_{\epsilon, x}(z) = \phi(\frac{|x-z|}{\epsilon})$. Then $\phi_{\epsilon, x}$ is compactly supported in \mathbb{R}_+^n .

Define a function v_ϵ by

$$v_\epsilon(x) = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \phi_{\epsilon, x}(z) \partial_{z_i} N(x-z) f(z) dz.$$

We differentiate v_ϵ by x_j variable. If $j \neq n$, we have

$$\partial_{x_j} v_\epsilon(x) = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \left[\partial_{x_j} \phi_{\epsilon, x}(z) \partial_{z_i} N(x-z) + \phi_{\epsilon, x}(z) \partial_{z_i} \partial_{x_j} N(x-z) \right] f(z) dz.$$

If $j = n$, we have

$$\begin{aligned} \partial_{x_n} v_\epsilon(x) &= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \left[\partial_{x_n} \phi_{\epsilon, x}(z) \partial_{z_i} N(x-z) + \phi_{\epsilon, x}(z) \partial_{z_i} \partial_{x_n} N(x-z) \right] f(z) dz \\ &\quad + \int_{\mathbb{R}^{n-1}} \phi_{\epsilon, x}(\bar{z}, x_n) \partial_{x_i} N(\bar{x} - \bar{z}, 0) f(\bar{z}, x_n) d\bar{z}. \end{aligned}$$

Set

$$\begin{aligned} I_\epsilon(x) &= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_j} \phi_{\epsilon, x}(z) \partial_{x_i} N(x-z) f(z) dz, \quad \text{and} \\ II_\epsilon(x) &= \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \phi_{\epsilon, x}(z) \frac{\partial^2}{\partial x_i \partial x_j} N(x-z) f(z) dz. \end{aligned}$$

Obviously,

$$II_\epsilon(x) \rightarrow \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial^2}{\partial x_i \partial x_j} N(x-z) f(z) dz \quad \text{as } \epsilon \rightarrow 0.$$

Divide I_ϵ by I_ϵ^1 and I_ϵ^2 defined by

$$I_\epsilon^1(x) = \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_j} \phi_{\epsilon, x}(z) \partial_{x_i} N(x-z) [f(z) - f(x)] dz$$

and

$$I_\epsilon^2(x) = f(x) \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_j} \phi_{\epsilon, x}(z) \partial_{x_i} N(x-z) dz.$$

Note that $I_\epsilon^1(x) \rightarrow 0$ as $\epsilon \rightarrow 0$, since

$$\begin{aligned} |I_\epsilon^1(x)| &\leq \int_{|x-z|\leq\epsilon} \left| \partial_{x_j} \phi_{\epsilon,x}(z) \right| \left| \partial_{x_i} N(x-z) \right| |f(z) - f(x)| dz \\ &\leq \int_{|x-z|\leq\epsilon} \frac{1}{\epsilon} \frac{1}{|x-z|^{n-1}} |x-z|^\alpha dz [f]_\alpha \leq \epsilon^\alpha [f]_\alpha, \end{aligned}$$

where

$$[f]_\alpha = \sup_{x,z,|x-z|\leq\epsilon} \frac{|f(x) - f(z)|}{|x-z|^\alpha}.$$

Now it remains to show that

$$\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_j} \phi_{\epsilon,x}(z) \partial_{x_i} N(x-z) dz = \begin{cases} -\frac{\delta_{ij}}{2n} & \text{if } i, j \neq n \text{ or } i = j = n \\ 0 & \text{if } i = n, j \neq n \text{ or } j = n, i \neq n. \end{cases}$$

Taking change of variables and sending ϵ to the zero, we obtain

$$\begin{aligned} &\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_j} \phi_{\epsilon,x}(z) \partial_{x_i} N(x-z) dz \\ &= \frac{1}{\epsilon} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{(z_i - x_j)(z_j - x_j)}{n\omega_n |x-z|^{n+1}} \phi'\left(\frac{|x-z|}{\epsilon}\right) dz \\ &\rightarrow \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{w_i w_j}{n\omega_n |w|^{n+1}} \phi'(|w|) dw. \end{aligned}$$

If $i, j \neq n$ or $i = j = n$, then by the symmetry of $\frac{w_i w_j}{n\omega_n |w|^{n+1}} \phi'(|w|)$ in terms of w_n variables, we have

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{w_i w_j}{n\omega_n |w|^{n+1}} \phi'(|w|) dw = \frac{1}{2} \int_{\mathbb{R}^n} \frac{w_i w_j}{n\omega_n |w|^n} \phi'(|w|) dw \\ &= \frac{1}{2n\omega_n} \int_0^2 \int_{S^{n-1}} r w_i w_j \phi'(r) dS_w dr = -\frac{1}{2n} \delta_{ij} \int_0^2 \phi(r) dr = -\frac{1}{2n} \delta_{ij}. \end{aligned}$$

If $i = n, j \neq n$ or $j = n, i \neq n$, then by the anti-symmetry of $\frac{w_i w_j}{n\omega_n |w|^{n+1}} \phi'(|w|)$ in terms of w_n variables, we have

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{w_i w_j}{n\omega_n |w|^{n+1}} \phi'(|w|) dw = 0.$$

Appendix B. Proof of Lemma 2.4

Let $k = 1, \dots, n-1$. Set $f_t(x) = f(x, t) = 1_{\mathbb{R}_+^n} \partial_{x_k} (\tau_a \Gamma_t)$. We would like to show that $\mathcal{N} f_t(x) \in L^1(\mathbb{R}^n)$. For $s > 0$, we have

$$\begin{aligned} &f_t * \Gamma_s(x) \\ &= \int_{\mathbb{R}^n} f(x-y, t) \Gamma_s(y) dy \\ &= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{x_n} \partial_{x_k} \Gamma(a-x+y, t) \Gamma(y, s) dy \end{aligned}$$

$$\begin{aligned}
 &= \left[\int_{\mathbb{R}^{n-1}} \partial_{x_k} \bar{\Gamma}(\bar{a} - \bar{x} + \bar{y}, t) \bar{\Gamma}(\bar{y}, s) d\bar{y} \right] \left[\int_{-\infty}^{x_n} \Gamma_t^n(a_n - x_n + y_n) \Gamma_s^n(y_n) dy_n \right] \\
 &= \left(\partial_{x_k} \bar{\Gamma}_{s+t} \right) (\bar{a} - \bar{x}) \left[\int_{-\infty}^{x_n} \Gamma_t^n(a_n - x_n + y_n) \Gamma_s^n(y_n) dy_n \right].
 \end{aligned}$$

Note that

$$\left| \partial_{x_k} \bar{\Gamma}_{s+t}(\bar{a} - \bar{x}) \right| \leq C(s+t)^{-\frac{1}{2}} \bar{\Gamma}_{2(s+t)}(\bar{a} - \bar{x}),$$

and

$$\begin{aligned}
 \int_{-\infty}^{x_n} \Gamma_t^n(a_n - x_n + y_n) \Gamma_s^n(y_n) dy_n &\leq \int_{-\infty}^{\infty} \Gamma_t^n(a_n - x_n + y_n) \Gamma_s^n(y_n) dy_n \\
 &= \Gamma_{s+t}^n(a_n - x_n).
 \end{aligned}$$

Hence we obtain that

$$|f_t * \Gamma_s(x)| \leq C(s+t)^{-\frac{1}{2}} \Gamma_{2(t+s)}(a-x).$$

Note that $\Gamma_{2(t+s)}(a-x) \leq C(s+t)^{-n/2}$ and $\Gamma_{2(t+s)}(a-x) \leq C(s+t)^{\frac{1}{2}} |a-x|^{-n-1}$. Hence we have that

$$\begin{aligned}
 \mathcal{N}f_t(x) &= \sup_{s>0} |\Gamma_s * f_t(x)| \leq \sup_{s>0} \min\{(s+t)^{-\frac{n+1}{2}}, |x-a|^{-n-1}\} \\
 &\leq \min\{t^{-\frac{n+1}{2}}, |x-a|^{-n-1}\}.
 \end{aligned}$$

Now we estimate the L^1 norm of $\mathcal{N}f_t$.

$$\begin{aligned}
 \int_{\mathbb{R}^n} \mathcal{N}f_t(x) dx &= \int_{|x-a| \geq \sqrt{t}} \mathcal{N}f_t(x) dx + \int_{|x-a| \leq \sqrt{t}} \mathcal{N}f_t(x) dx \\
 &\leq \int_{|x-a| \geq \sqrt{t}} |x-a|^{-n-1} dx + \int_{|x-a| \leq \sqrt{t}} t^{-\frac{n+1}{2}} dx \\
 &\leq Ct^{-\frac{1}{2}}.
 \end{aligned}$$

This leads to the conclusion that

$$f_t = 1_{\mathbb{R}_+^n} \tau_a \partial_{y_k} \Gamma_t \in \mathcal{H}^1(\mathbb{R}^n).$$

Appendix C. Proof of Lemma 2.6

Let $a = (\bar{a}, a_n)$ and $b = (\bar{b}, a_n)$. Set

$$f_t(x) = f(x, t) = 1_{\mathbb{R}_+^n} \left[(\tau_a * \Gamma_t)(x) - (\tau_b * \Gamma_t)(x) \right].$$

We would like to estimate that the L^1 norm of $\mathcal{N}f_t(x) = \sup_{s>0} |f_t * \Gamma_s(x)|$. In the proof of Lemma 2.4, we recall that

$$(1_{\mathbb{R}_+^n} \Gamma_t^{a*}) * \Gamma_s(x) = \bar{\Gamma}_{t+s}(\bar{a} - \bar{x}) \left[\int_{-\infty}^{x_n} \Gamma_t^n(a_n + x_n - y_n) \Gamma_s^n(y_n) dy_n \right].$$

Hence we have that

$$f_t * \Gamma_s(x) = \left[\bar{\Gamma}_{t+s}(\bar{a} - \bar{x}) - \bar{\Gamma}_{t+s}(\bar{b} - \bar{x}) \right] \left[\int_{-\infty}^{x_n} \Gamma_t^n(a_n + x_n - y_n) \Gamma_s^n(y_n) dy_n \right].$$

Recall that

$$\int_{-\infty}^{x_n} \Gamma_t^n(a_n + x_n - y_n) \Gamma_s^n(y_n) dy_n \leq C e^{-\frac{\alpha_n^2}{4t}} \Gamma_{s+t}^n(x_n).$$

If $|\bar{a} - \bar{b}| \leq \frac{|\bar{x} - \bar{a}|}{2}$, then by the mean value theorem we have that

$$\begin{aligned} & \left| \bar{\Gamma}_{t+s}(\bar{a} - \bar{x}) - \bar{\Gamma}_{t+s}(\bar{b} - \bar{x}) \right| \\ & \leq |\bar{b} - \bar{a}| |\nabla_{\bar{x}} \bar{\Gamma}_{s+t}(\bar{a} - \bar{x} + \theta(\bar{b} - \bar{a}))| \\ & \leq C(s+t)^{-1/2} |\bar{b} - \bar{a}| \bar{\Gamma}_{4(s+t)}(\bar{a} - \bar{x}). \end{aligned}$$

Therefore, if $|\bar{a} - \bar{b}| \leq \frac{|\bar{x} - \bar{a}|}{2}$, then

$$|f_t * \Gamma_s(x)| \leq C(s+t)^{-1/2} |\bar{b} - \bar{a}| \bar{\Gamma}_{4(s+t)}(\bar{a} - \bar{x}) e^{-\frac{\alpha_n^2}{4t}} \Gamma_{s+t}^n(x_n).$$

Choose $0 < \alpha < 1$. Recall that

$$\bar{\Gamma}_{4(t+s)}(\bar{a} - \bar{x}) \leq C(s+t)^{(1-\alpha)/2} |\bar{a} - \bar{x}|^{-n+\alpha},$$

and

$$\Gamma_{s+t}^n(x_n) \leq \min\{C(s+t)^{-\frac{1}{2}}, C(s+t)^{\frac{\alpha}{2}} x_n^{-1-\alpha}\}.$$

Hence if $|\bar{a} - \bar{b}| \leq \frac{|\bar{x} - \bar{a}|}{2}$, then

$$|f_t * \Gamma_s(x)| \leq C |\bar{b} - \bar{a}| e^{-\frac{\alpha_n^2}{4t}} \begin{cases} (s+t)^{-\frac{1}{2}-\frac{\alpha}{2}} |\bar{a} - \bar{x}|^{-n+\alpha}, \\ |\bar{a} - \bar{x}|^{-n+\alpha} x_n^{-1-\alpha}, \end{cases}$$

which implies that

$$(C.28) \quad \mathcal{N}f_t(x) \leq C |\bar{b} - \bar{a}| e^{-\frac{\alpha_n^2}{4t}} \begin{cases} t^{-\frac{1}{2}+\frac{\alpha}{2}} |\bar{a} - \bar{x}|^{-n+\alpha}, \\ |\bar{a} - \bar{x}|^{-n+\alpha} x_n^{-1-\alpha}. \end{cases}$$

We also note that

$$\begin{aligned} |f_t * \Gamma_s(x)| & \leq C \left| \bar{\Gamma}_{t+s}(\bar{a} - \bar{x}) - \bar{\Gamma}_{t+s}(\bar{b} - \bar{x}) \right| e^{-\frac{\alpha_n^2}{4t}} \Gamma_{s+t}^n(x_n) \\ & \leq C \left(\left| \bar{\Gamma}_{t+s}(\bar{a} - \bar{x}) \right| + \left| \bar{\Gamma}_{t+s}(\bar{b} - \bar{x}) \right| \right) e^{-\frac{\alpha_n^2}{4t}} \Gamma_{s+t}^n(x_n) \\ & \leq C \left[\bar{\Gamma}_{2(s+t)}(\bar{a} - \bar{x}) + \bar{\Gamma}_{2(s+t)}(\bar{b} - \bar{x}) \right] e^{-\frac{\alpha_n^2}{4t}} \Gamma_{s+t}^n(x_n). \end{aligned}$$

Note that

$$\bar{\Gamma}_{2(t+s)}(\bar{a} - \bar{x}) \leq C(s+t)^{-\frac{\alpha}{2}} |\bar{a} - \bar{x}|^{-n+1+\alpha}$$

and

$$\Gamma_{t+s}^n(x_n) \leq \min\{C(s+t)^{-\frac{1}{2}}, C(t+s)^{\frac{\alpha}{2}} x_n^{-1-\alpha}\}.$$

Hence, we also have

$$|f_t * \Gamma_s(x)| \leq C e^{-\frac{\alpha_n^2}{4t}} \begin{cases} (s+t)^{-\frac{1}{2}-\frac{\alpha}{2}} |\bar{a} - \bar{x}|^{-n+1+\alpha} \\ |\bar{a} - \bar{x}|^{-n+1+\alpha} x_n^{-1-\alpha}, \end{cases}$$

which implies that

$$(C.29) \quad \mathcal{N}f_t(x) \leq Ce^{-\frac{\alpha^2}{4t}} \begin{cases} t^{-\frac{\alpha}{2}} |\bar{a} - \bar{x}|^{-n+1+\alpha} \\ |\bar{a} - \bar{x}|^{-n+1+\alpha} x_n^{-1-\alpha}. \end{cases}$$

Now we estimate the L^1 norm of $\mathcal{N}f_t$. For $|\bar{a} - \bar{b}| \leq \frac{|\bar{x} - \bar{a}|}{2}$, we make use of the estimate (C.28) and for $|\bar{a} - \bar{b}| \geq \frac{|\bar{x} - \bar{a}|}{2}$ we make use of the estimate (C.29).

Decompose the domain \mathbb{R}_+^n into four parts $D_i, i = 1, \dots, 4$, where

$$\begin{aligned} D_1 &= \{(\bar{z}, x_n) : |\bar{x} - \bar{a}| \geq 2|\bar{a} - \bar{b}|, |x_n| \geq \sqrt{t}\}, \\ D_2 &= \{(\bar{z}, x_n) : |\bar{x} - \bar{a}| \geq 2|\bar{a} - \bar{b}|, |x_n| \leq \sqrt{t}\}, \\ D_3 &= \{(\bar{z}, x_n) : |\bar{x} - \bar{a}| \leq 2|\bar{a} - \bar{b}|, |x_n| \geq \sqrt{t}\}, \\ D_4 &= \{(\bar{z}, x_n) : |\bar{x} - \bar{a}| \leq 2|\bar{a} - \bar{b}|, |x_n| \leq \sqrt{t}\}. \end{aligned}$$

Then we have that

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{N}f_t(x) dx &= \sum_{i=1}^4 \int_{D_i} \mathcal{N}f_t(x) dx \\ &\leq Ce^{-\frac{\alpha^2}{4t}} \left[\int_{D_1} |\bar{a} - \bar{b}| |\bar{a} - \bar{x}|^{-n+\alpha} x_n^{-1-\alpha} dx \right. \\ &\quad + \int_{D_2} |\bar{a} - \bar{b}| t^{-\frac{1}{2} - \frac{\alpha}{2}} |\bar{a} - \bar{x}|^{-n+\alpha} dx \\ &\quad + \int_{D_3} |\bar{a} - \bar{x}|^{-n+1+\alpha} x_n^{-1-\alpha} dx \\ &\quad \left. + \int_{D_4} t^{-\frac{1}{2} - \frac{\alpha}{2}} |\bar{a} - \bar{x}|^{-n+1+\alpha} dx \right] \\ &\leq Ce^{-\frac{\alpha^2}{4t}} |\bar{a} - \bar{b}|^\alpha t^{-\frac{\alpha}{2}}. \end{aligned}$$

Therefore, we conclude that

$$f_t = 1_{\mathbb{R}_+^n} [\tau_{a^*} \Gamma_t - \tau_{b^*} \Gamma_t] \in \mathcal{H}^1(\mathbb{R}^n).$$

Appendix D. Proof of Lemma 2.7.

Set

$$\begin{aligned} (a) &= \int_0^{x_n/2} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{x_j} N(x-z) f(z, y) dz, \\ (b) &= \int_{x_n/2}^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{x_j} N(x-z) [f(z, y) - f(\bar{x}, z_n, y)] dz, \quad \text{and} \\ (c) &= \int_{x_n/2}^{x_n} f(\bar{x}, z_n, y) \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{x_j} N(x-z) d\bar{z} dz_n. \end{aligned}$$

Then

$$\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \partial_{x_i} \partial_{x_j} N(x-z) f(z, y) dz = (a) + (b) + (c).$$

Since $|\partial_{x_i} \partial_{x_j} N(x-z)| \leq C|x-z|^{-n}$ and $\int_{\mathbb{R}^{n-1}} |x-z|^{-n} d\bar{z} = \frac{C}{|x_n - z_n|}$, we have

$$\begin{aligned} \int_{\mathbb{R}_+^n} (a) dy &\leq C \left(\int_0^{x_n/2} \frac{1}{x_n - z_n} dz_n \right) \left(\sup_{z \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} |f(z, y)| dy \right) \\ &\leq C \sup_{z \in \mathbb{R}_+^n} \int_{\mathbb{R}_+^n} |f(z, y)| dy. \end{aligned}$$

Since, for $0 < \alpha < 1$,

$$\begin{aligned} &|\partial_{x_i} \partial_{x_j} N(x-z)[f(z, y) - f(\bar{x}, z_n, y)]| \\ &\leq C|x-z|^\alpha |x-z|^{-n} \frac{|f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)|}{|x-z|^\alpha} \\ &\leq C|x_n - z_n|^\alpha |x-z|^{-n} \frac{|f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)|}{|\bar{x} - \bar{z}|^\alpha}, \end{aligned}$$

we obtain that

$$\begin{aligned} &\int_{\mathbb{R}_+^n} (b) dy \\ &\leq C \int_{\frac{x_n}{2}}^{x_n} (x_n - z_n)^\alpha \int_{\mathbb{R}^{n-1}} |x-z|^{-n} \left[\sup_{\bar{z} \in \mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} \frac{|f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)|}{|\bar{x} - \bar{z}|^\alpha} dy \right] d\bar{z} dz_n \\ &\leq Cx_n^{\alpha+1} \left(\int_{\mathbb{R}^{n-1}} |x-z|^{-n} d\bar{z} \right) \left(\sup_{x_n/2 < z_n < x_n} \sup_{\bar{z} \in \mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} \frac{|f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)|}{|\bar{x} - \bar{z}|^\alpha} dy \right) \\ &\leq Cx_n^\alpha \left(\sup_{x_n/2 < z_n < x_n} \sup_{\bar{z} \in \mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} \frac{|f(\bar{z}, z_n, y) - f(\bar{x}, z_n, y)|}{|\bar{x} - \bar{z}|^\alpha} dy \right). \end{aligned}$$

Now we would like to show that $(c) = 0$. Note that

$$p.v. \int_{\mathbb{R}^{n-1}} \frac{\partial^2}{\partial y_i \partial y_j} N(x-y) d\bar{y} = \lim_{\epsilon \rightarrow 0, R \rightarrow \infty} \int_{\epsilon \leq |\bar{x} - \bar{y}| \leq R} \frac{\partial^2}{\partial y_i \partial y_j} N(x-y) dS_{\bar{y}}$$

and

$$\begin{aligned} &\int_{\epsilon \leq |\bar{x} - \bar{y}| \leq R} \frac{\partial^2}{\partial y_i \partial y_j} N(x-y) d\bar{y} \\ &= \int_{S_R(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} - \int_{S_\epsilon(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}}. \end{aligned}$$

Here $S_R(\bar{x}) = \{\bar{y} \in \mathbb{R}^{n-1} : |\bar{x} - \bar{y}| = R\}$ and $n_j = \frac{y_j - x_j}{|\bar{y} - \bar{x}|}$ is the j -th component of the unit outer normal vector. If $i = 1, \dots, n-1$, then

$$\left| \int_{S_R(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} \right| = \left| \int_{S_R(\bar{x})} \frac{y_i - x_i}{|x-y|^n} \frac{y_j - x_j}{|\bar{y} - \bar{x}|} dS_{\bar{y}} \right|$$

$$= \left| \int_{S_{n-2}} \frac{w_i w_j}{n\omega_n(R^2 + (x_n - y_n)^2)^{n/2}} R^{n-1} dS_w \right| \leq \frac{C}{R} \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

where S_{n-2} is the unit sphere in \mathbb{R}^{n-1} , $w_i = \frac{y_i}{|\bar{y}|}$ is the outward unit normal vector to S_{n-2} , and

$$\begin{aligned} & \left| \int_{S_\epsilon(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} \right| \\ &= \left| \int_{S_{n-2}} \frac{w_i w_j}{n\omega_n(\epsilon^2 + (x_n - y_n)^2)^{n/2}} \epsilon^{n-1} dS_{\bar{y}} \right| \leq C \frac{\epsilon^{n-1}}{|x_n - y_n|^n} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \end{aligned}$$

If $i = n$, then

$$\int_{S_R(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} = 0,$$

since

$$\begin{aligned} \int_{S_R(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} &= \int_{S_{n-2}} \frac{(x_n - y_n) w_j}{n\omega_n(R^2 + (x_n - y_n)^2)^{n/2}} R^{n-2} dS_{\bar{y}} \\ &= \frac{(x_n - y_n) R^{n-2}}{n\omega_n(R^2 + (x_n - y_n)^2)^{n/2}} \int_{S_{n-2}} w_j dS_{\bar{y}} = 0. \end{aligned}$$

Likewise,

$$\int_{S_\epsilon(\bar{x})} \partial_{y_i} N(x-y) n_j dS_{\bar{y}} = 0.$$

This implies that

$$\int_{\mathbb{R}^{n-1}} \partial_{y_i} \partial_{y_j} N(x-y) d\bar{y} = 0,$$

and this again implies that $(c) = 0$.

References

- [1] H. Amann, *On the strong solvability of the Navier-Stokes equations*, J. Math. Fluid Mech. **2** (2000), no. 1, 16–98.
- [2] H.-O. Bae, *Temporal and spatial decays for the Stokes flow*, J. Math. Fluid Mech. **10** (2008), no. 4, 503–530.
- [3] ———, *Temporal decays in L^1 and L^∞ for the Stokes flow*, J. Differential Equations **222** (2006), no. 1, 1–20.
- [4] W. Borchers and T. Miyakawa, *L^2 decay for the Navier-Stokes flow in halfspaces*, Math. Ann. **282** (1988), no. 1, 139–155.
- [5] F. Crispo and P. Maremonti, *On the (x, t) asymptotic properties of solutions of the Navier-Stokes equations in the half space*, Zap. Nauchn. Sem. POMI, 318 (2004); translation in J. Math. Sci. **136** (2006), no. 2, 3735–3767.
- [6] Y. Fujigaki and T. Miyakawa, *Asymptotic profiles of nonstationary incompressible Navier-Stokes flows in the half-space*, Methods Appl. Anal. **8** (2001), no. 1, 121–157.
- [7] Y. Giga, S. Matsui, and O. Sawada, *Global existence of two-dimensional Navier-Stokes flow with nondecaying initial velocity*, J. Math. Fluid Mech. **3** (2001), no. 3, 302–315.
- [8] Y. Giga, S. Matsui, and Y. Shimizu, *On estimates in Hardy spaces for the Stokes flow in a half-space*, Math. Z. **231** (1999), no. 2, 383–396.

- [9] Y. Giga, K. Inui, A. Mahalov, A. Matsui, and J. Saal, *Rotating Navier-Stokes equations in \mathbb{R}_+^3 with initial data nondecreasing at infinity: the Ekman boundary layer problem*, Arch. Ration. Mech. Anal. **186** (2007), no. 2, 177–224.
- [10] M. Hieber and O. Sawada, *The Navier-Stokes equations in \mathbb{R}^n with linearly growing initial data*, Arch. Ration. Mech. Anal. **175** (2005), no. 2, 269–285.
- [11] J. Kato, *The uniqueness of nondecaying solutions for the Navier-Stokes equations*, Arch. Ration. Mech. Anal. **169** (2003), no. 2, 159–175.
- [12] P. Maremonti, *Stokes and Navier-Stokes problems in a half space: the existence and uniqueness of solutions a priori nonconvergent to a limit at infinity*, Journal of Mathematical Sciences **159** (2009), no. 4, 486–523.
- [13] P. Maremonti and G. Starita, *Nonstationary Stokes equations in a half-space with continuous initial data*, Zap. Nauchn. Sem. POMI, 295; translation in J. Math. Sci. **127** (2005), no. 2, 1886–1914.
- [14] E. Nakai and T. Yoneda, *Generalized Campanato spaces and the uniqueness of nondecaying solutions for the Navier-Stokes equations*, preprint, 2009.
- [15] Y. Shimizu, *L^∞ estimate of the first-order space derivatives of Stokes flow in a half space*, Funkcial. Ekvac. **42** (1999), no. 2, 291–309.
- [16] V. A. Solonnikov, *On nonstationary Stokes problem and Navier-Stokes problem in a half-space with initial data nondecreasing at infinity*, Problemy Matematicheskogo Analiza, (2003) no. 25, 189–210; translation in J. Math. Sci. (N. Y.) **114** (2003), no. 5, 1726–1740.
- [17] S. Ukai, *A solution formula for the Stokes equation in \mathbb{R}_+^n* , Comm. Pure Appl. Math. **40** (1987), no. 5, 611–621.

HYEONG-OHK BAE
 DEPARTMENT OF FINANCIAL ENGINEERING
 AJOU UNIVERSITY
 SUWON 443-749, KOREA
E-mail address: hobae@ajou.ac.kr

BUM JA JIN
 DEPARTMENT OF MATHEMATICS
 MOKPO NATIONAL UNIVERSITY
 MUAN 534-729, KOREA
E-mail address: bumjajin@hanmail.net