J. Korean Math. Soc. **49** (2012), No. 1, pp. 85–98
http://dx.doi.org/10.4134/JKMS.2012.49.1.085

THE TOTAL GRAPH OF A COMMUTATIVE RING WITH RESPECT TO PROPER IDEALS

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Abstract. Let R be a commutative ring and I its proper ideal, let $S(I)$ be the set of all elements of R that are not prime to I. Here we introduce and study the total graph of a commutative ring R with respect to proper ideal I, denoted by $T(\Gamma_I(R))$. It is the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in S(I)$. The total graph of a commutative ring, that denoted by $T(\Gamma(R))$, is the graph where the vertices are all elements of R and where there is an undirected edge between two distinct vertices x and y if and only if $x + y \in Z(R)$ which is due to Anderson and Badawi [2]. In the case $I = \{0\}$, $T(\Gamma_I(R)) = T(\Gamma(R))$; this is an important result on the definition.

1. Introduction

The concept of total graph of a commutative ring R, one of the most interesting concept of the algebraic structures in graph theory denoted by $T(\Gamma(R))$, was first introduced by Anderson and Badawi in [2], such that the vertex set is R and the distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$ where $Z(R)$ is the zero divisors of R. Throughout this work all rings are assumed to be commutative with non-zero identity. Let I be a proper ideal of R. The total graph of a commutative ring R with respect to proper ideal I, denoted by $T(\Gamma_I(R))$, is the graph which vertices are all elements of R and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in S(I)$. We use the notation $S(I)$ to refer to the set of elements of R that are not prime to I, we say that $a \in R$ is prime to I, if $ra \in I$ (where $r \in R$) implies that $r \in I$ (see [6, 7]). Clearly, $S(I)$ is not empty since I is a proper ideal of R. It is easy to check that, when $I = \{0\}$, $T(\Gamma_I(R)) = T(\Gamma(R))$. The zero-divisor graph of R, denoted $\Gamma(R)$, is the graph whose vertices are $Z(R)^*$ (the non-zero zero-divisors of R) with two distinct vertices joined by an edge when the product of the vertices is zero (c.f. [3]). In [8], Redmond introduced the zero divisor graph with respect to proper ideal I, denoted by $\Gamma_I(R)$, as the graph

Received August 24, 2010.
2010 *Mathematics Subject Classification.* 13D45, 13E10, 13C05.
Key words and phrases. commutative rings, zero divisor, total graph.

with vertices $\{x \in R - I : xy \in I \text{ for some } y \in R - I\}$ where distinct vertices x and y are adjacent if and only if $xy \in I$. If $I = \{0\}$, then $\Gamma_I(R) = \Gamma(R)$. Redmond explored the relationship between $\Gamma_I(R)$ and $\Gamma(R)$. He gave an example of rings R, S and ideals $I \trianglelefteq R$, $J \trianglelefteq S$, where $\Gamma(R/I) \cong \Gamma(S/J)$ but $\Gamma_I(R) \ncong \Gamma_J(S)$. Similarly, in this paper we give an example (see Example 2.2) such that $T(\Gamma_I(R)) \cong T(\Gamma_J(S))$ but $T(\Gamma(R/I)) \ncong T(\Gamma(S/J))$ and some basic results on the relationship between $T(\Gamma_I(R))$ and $T(\Gamma(R/I))$ in Section 2.

The set $S(I)$ is not necessarily an ideal of R (not always closed under addition) and since $S(I)$ is a union of prime ideals of R containing I (see [4, Exe. 3.9] and note that 2.1), whenever $xy \in S(I)$ for $x, y \in R$, then $x \in S(I)$ or $y \in S(I)$. So, if $S(I)$ is an ideal of R, then it is actually a prime ideal of R; hence the study of $T(\Gamma_I(R))$ breaks naturally into two cases depending on whether or not $S(I)$ is an ideal of R and in Sections $3, 4$, we state several results about the relationship between diameter and girth of $T(\Gamma_I(R))$ and $T(\Gamma(R/I))$. The proper ideal I is said to be P-primal ideal of R when $P = S(I)$ forms an ideal; then P is said to be the adjoint ideal of I. It is easy to see that, $S(I) = I$ when I is a prime ideal R (see [6, 7]). Let $S(\Gamma_I(R))$ be the (induced) subgraph of $T(\Gamma_I(R))$ with vertices $S(I)$, and let $\overline{S}(\Gamma_I(R))$ be the (induced) subgraph $T(\Gamma_I(R))$ with vertices $R - S(I)$.

Let G be a graph with vertex set $V(G)$. Recall that G is connected if there is a path between any two distinct vertices of G. At the other extreme, we say that G is totally disconnected if no two vertices of G are adjacent. For vertices x and y of G, $d(x, y)$ be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The diameter of a graph G, denoted by $\text{diam}(G)$, is the supremum of the distances between vertices. The girth of G, denoted by $\text{gr}(G)$, is the length of a shortest cycle in G ($\text{gr}(G) = \infty$ if G contains no cycles). A graph G is said to be complete bipartite if $V(G)$ can be partitioned into two disjoint sets V_1, V_2 such that no two vertices within any V_1 or V_2 are adjacent, but for every $u \in V_1, v \in V_2$, u, v are adjacent. Then we use the symbol $K^{m,n}$ for the complete bipartite graph where the cardinal numbers of V_1 and V_2 are m, n, respectively (we allow m and n to be infinite cardinals). A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. Let K_n denote the complete graph with n vertices.

In Section 2, we obtain an identity between completeness of $\overline{S}(\Gamma_I(R))$ and $Reg\Gamma(R/I)$. We study the Graphs $T(\Gamma_I(R))$, $S(\Gamma_I(R))$ and $\overline{S}(\Gamma_I(R))$ for the case when $S(I)$ is an ideal in Section 3 and for the case $S(I)$ is not an ideal in Section 4. Though our definition of total graph of a commutative ring is a generalization of the definition given in [2], we would like to point out that many of the proofs provided in this paper are essentially the same as the proofs provided in [2].

2. Example and basic structure

In this section, we explore the relationship between $T(\Gamma_I(R))$ and $T(\Gamma(R/I))$ on basic structure.

Note 2.1. We can easily show that $Z(R/I) = \{a + I : a \in S(I)\}$ and $Reg(R/I) = \{a + I : a \notin S(I)\}$. Thus $Z(R/I)$ is an ideal R/I if and only if $S(I)$ is an ideal R.

Let $Reg(\Gamma(R/I))$ be the (induced) subgraph of $T(\Gamma(R/I))$ with vertices $Reg(R/I)$, the set of regular elements of R/I, let $Z(\Gamma(R/I))$ be the (induced) subgraph of $T(\Gamma(R/I))$ with vertices $Z(R/I)$.

Example 2.2. Let $R = \mathbb{Z}_8$, $S = \mathbb{Z}_4 \times \mathbb{Z}_2$ and $I = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \trianglelefteq R$, $J = \{\bar{0}\} \times \mathbb{Z}_2 \trianglelefteq S$. It is easy to check that $S(I) = I$ and $S(J) = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{1}), (\bar{2}, \bar{0}), (\bar{2}, \bar{1})\}$. $T(\Gamma_I(R))$ and $T(\Gamma_J(S))$ are the union of 2 disjoint K^4's. Now, $T(\Gamma(R/I))$ is a graph with two vertices but $T(\Gamma(S/J))$ is a graph with four vertices.

Theorem 2.3. *Let R be a commutative ring with the proper ideal I, and let $x, y \in R$. Then*

(1) *If $x + I$ and $y + I$ are (distinct) adjacent vertices in $T(\Gamma(R/I))$, then x is adjacent to y in $T(\Gamma_I(R))$.*

(2) *If x and y are (distinct) adjacent vertices in $T(\Gamma_I(R))$ and $x + I \neq y + I$, then $x + I$ is adjacent to $y + I$ in $T(\Gamma(R/I))$.*

(3) *If x is adjacent to y in $T(\Gamma_I(R))$ and $x + I = y + I$, then $2x, 2y \in S(I)$ and all distinct elements of $x + I$ are adjacent in $T(\Gamma_I(R))$.*

Proof. It is clear. □

According to the following corollary and remark, there is a strong relationship between $T(\Gamma(R/I))$ and $T(\Gamma_I(R))$.

Note that for a graph G, we say that $\{G_\theta\}_{\theta \in \Theta}$ is a collection of disjoint subgraphs of G if all vertices and edges of each G_θ are contained in G and no two of these G_θ contain a common vertex.

Corollary 2.4. *Let R be a commutative ring with the proper ideal I. Then $T(\Gamma_I(R))$ contains $|I|$ disjoint subgraphs isomorphic to $T(\Gamma(R/I))$.*

Proof. Let $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq R$ be a set of distinct representatives of the vertices of $T(\Gamma(R/I))$. Define a graph G_i, for each $i \in I$, with vertices $\{a_\lambda + i | \lambda \in \Lambda\}$, where $a_\lambda + i$ is adjacent to $a_\beta + i$ in G_i whenever $a_\lambda + I$ is adjacent to $a_\beta + I$ in $T(\Gamma(R/I))$; i.e., whenever $a_\lambda + a_\beta \in S(I)$. By the above theorem, G_i is a subgraph of $T(\Gamma_I(R))$. Also, each $G_i \cong T(\Gamma(R/I))$, and G_i and G_j contains no common vertices if $i \neq j$. □

Remark 2.5. It follows from the above corollary that $S(\Gamma_I(R))$ contains $|I|$ disjoint subgraphs isomorphic to $Z(\Gamma(R/I))$ and $\overline{S}(\Gamma_I(R))$ contains $|I|$ disjoint subgraphs isomorphic to $Reg(\Gamma(R/I))$; since for each $a \in S(I)$ and $b \in R - S(I)$, and $i \in I$; $a + i \in S(I)$ (for some $r \in R - I$, $ar \in I$; hence $(a + i)r \in I$) and

$b+i \in R-S(I)$. So a graph G_i with vertices $\{a_\lambda + i \,|\, \lambda \in \Lambda\}$ such that $a_\lambda \in S(I)$ is a subgraph $S(\Gamma_I(R))$ and a graph G_i with vertices $\{a_\beta + i \,|\, \beta \in \Lambda\}$ such that $a_\beta \notin S(I)$ is a subgraph $\bar{S}(\Gamma_I(R))$.

One can verify that the following method can be used to construct a graph $T(\Gamma_I(R))$.

Remark 2.6. Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq R$ be a set of representatives of the vertices of $T(\Gamma(R/I))$. For each $i \in I$, define a graph G_i with vertices $\{a_\lambda + i \,|\, \lambda \in \Lambda\}$, where edges are defined by the relationship $a_\lambda + i$ is adjacent to $a_\beta + i$ in G_i if and only if $a_\lambda + I$ is adjacent to $a_\beta + I$ in $T(\Gamma(R/I))$ (i.e., $a_\lambda + a_\beta \in S(I)$). Define the graph G to have as its vertex set $V = \bigcup_{i\in I} G_i$. We define the edge set of G to be:

(1) all edges contained in G_i for each $i \in I$,

(2) for distinct $\lambda, \beta \in \Lambda$ and for any $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\beta + j$ if and only if $a_\lambda + I$ is adjacent to $a_\beta + I$ in $T(\Gamma(R/I))$ (i.e., $a_\lambda + a_\beta \in S(I)$),

(3) for $\lambda \in \Lambda$ and distinct $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\lambda + j$ if and only if $2a_\lambda \in S(I)$.

It follows that if $T(\Gamma(R/I))$ is a graph on $N = |R/I|$ vertices, then $T(\Gamma_I(R))$ is a graph on $N.|I|$ vertices.

Proposition 2.7. *Let R be a commutative ring with the proper ideal I. Then*

(1) *$S(\Gamma_I(R))$ is complete (connected) if and only if $Z(\Gamma(R/I))$ is complete (connected).*

(2) *If $\bar{S}(\Gamma_I(R))$ is complete, then $Reg(\Gamma(R/I))$ is complete.*

(3) *$\bar{S}(\Gamma_I(R))$ is connected if and only if $Reg(\Gamma(R/I))$ is connected.*

Proof. (1) Let $S(\Gamma_I(R))$ be a complete subgraph $T(\Gamma_I(R))$ and $x + I \neq y + I$ are distinct elements of $Z(\Gamma(R/I))$. So x and y are adjacent in $S(\Gamma_I(R))$; hence $x + I$ and $y + I$ are adjacent in $Z(\Gamma(R/I))$. Conversely, suppose x and y are distinct elements of $S(\Gamma_I(R))$. If $x + I = y + I$, then $x - y \in I$. There exists $r \in R - I$ such that $ry \in I$; hence $rx \in I$. It follows that $r(x + y) \in I$, thus x and y are adjacent in $S(\Gamma_I(R))$. If $x + I \neq y + I$, then $x + I$ and $y + I$ are adjacent in $Z(\Gamma(R/I))$. So $x + y \in S(I)$, as required.

(2) The proof is omitted. The converse is not necessarily true, for example consider $R = \mathbb{Z}_{18}$, and $I = \langle \bar{3} \rangle$ (it is easy to check that $S(I) = I$).

(3)The sufficiency implication is clear. Let $Reg(\Gamma(R/I))$ is connected. Suppose x and y are distinct elements of $\bar{S}(\Gamma_I(R))$. If $x+I = y+I$, then $x-(-y)-y$ is a path between x and y (if $x = -y$, then x and y are adjacent). If $x+I \neq y+I$, the proof is clear and omitted. □

Lemma 2.8. *Let R be a commutative ring with the proper ideal I. Then* $\mathrm{gr}(T(\Gamma_I(R))) \leq \mathrm{gr}(T(\Gamma(R/I)))$. *If $T(\Gamma(R/I))$ contains a cycle, then so does $T(\Gamma_I(R))$, and therefore* $\mathrm{gr}(T(\Gamma_I(R))) \leq \mathrm{gr}(T(\Gamma(R/I))) \leq 4$.

Proof. If $\mathrm{gr}(T(\Gamma(R/I))) = \infty$ we are done. Now suppose $\mathrm{gr}(T(\Gamma(R/I))) = k < \infty$. Let $x_1 + I - x_2 + I - \cdots - x_k + I - x_1 + I$ be a cycle in $T(\Gamma(R/I))$ through k

distinct vertices. Thus $x_1 - x_2 - \cdots - x_k - x_1$ is a cycle in $T(\Gamma_I(R))$ of length k. Hence, $\mathrm{gr}(T(\Gamma_I(R))) \leq k$. According to [2, Theorem 2.6(3), 3.15(2)], it follows that $\mathrm{gr}(T(\Gamma(R/I))) \leq 4$. □

3. The case when $S(I)$ is an ideal of R

In this section, we state a general structure for $\overline{S}(\Gamma_I(R))$ the (induced) subgraph $T(\Gamma_I(R))$ (see Theorem 3.5) and we investigate the relationship between $T(\Gamma_I(R))$ and $T(\Gamma(R/I))$ with assumption that, $S(I)$ be an ideal of R (i.e., I is a primal ideal of R). We begin with the following theorem.

Proposition 3.1. *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R. Then $S(\Gamma_I(R))$ is a complete (induced) subgraph $T(\Gamma_I(R))$ and is disjoint from $\overline{S}(\Gamma_I(R))$.*

Proof. This is clear according to definition. □

Theorem 3.2. *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R.*

(1) *The (induced) subgraph $S(\Gamma_I(R))$ with vertices $\sqrt{I}$ is complete and each vertex of this subgraph is adjacent to each vertex of $S(\Gamma_I(R))$ and is disjoint from $\overline{S}(\Gamma_I(R))$.*

(2) *If $\{0\} \neq \sqrt{I} \subset S(I)$, then $\mathrm{gr}(S(\Gamma_I(R))) = 3$.*

Proof. (1) Let $x \in \sqrt{I}$. If $x \in I$, then $x \in S(I)$; otherwise there is an integer $n \geq 2$ such that $x^n \in I$ and $x^{n-1} \notin I$. We have $x.x^{n-1} \in I$; hence $x \in S(I)$. So Part (1) follows since $\sqrt{I} \subseteq S(I)$ is an ideal and $\sqrt{I} + S(I) \subseteq S(I)$.

(2) Let $0 \neq x \in \sqrt{I}$ and $y \in S(I) \backslash \sqrt{I}$. Then $0 - x - y - 0$ is a 3-cycle in $S(\Gamma_I(R))$, as required. □

Theorem 3.3. *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R.*

(1) *Assume that Γ is an induced subgraph of $\overline{S}(\Gamma_I(R))$ and let x and y be distinct vertices of Γ such that are connected by a path in Γ. Then there exists a path in Γ of length 2 between x and y. In particular, if $\overline{S}(\Gamma_I(R))$ is connected, then $\mathrm{diam}(\overline{S}(\Gamma_I(R))) \leq 2$.*

(2) *Suppose x and y are distinct elements of $\overline{S}(\Gamma_I(R))$ that are connected by a path. If $x + y \notin S(I)$ (that is, if x and y are not adjacent), then $x - (-x) - y$ and $x - (-y) - y$ are paths of length 2 between x and y in $\overline{S}(\Gamma_I(R))$.*

Proof. (1) Let x_1, x_2, x_3, and x_4 are distinct vertices of Γ. It suffices to show that if there is a path $x_1 - x_2 - x_3 - x_4$ from x_1 to x_4, then x_1 and x_4 are adjacent. So $x_1 + x_2$, $x_2 + x_3$, $x_3 + x_4 \in S(I)$ gives $x_1 + x_4 = (x_1 + x_2) - (x_2 + x_3) + (x_3 + x_4) \in S(I)$ since $S(I)$ is an ideal of R. Thus x_1 and x_4 are adjacent. So if $\overline{S}(\Gamma_I(R))$ is connected, then $\mathrm{diam}(\overline{S}(\Gamma_I(R))) \leq 2$.

(2) Since $x, y \in R - S(I)$ and $x + y \notin S(I)$, there exists $z \in R - S(I)$ such that $x - z - y$ is a path of length 2 by part (1) above. Thus $x + z, z + y \in S(I)$,

and hence $x - y = (x + z) - (z + y) \in S(I)$. Also, since $x + y \notin S(I)$, we must have $x \neq -x$ and $y \neq -x$. Thus $x - (-x) - y$ and $x - (-y) - y$ are paths of length 2 between x and y in $\overline{S}(\Gamma_I(R))$. □

Theorem 3.4. *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R. Then the following statements are equivalent.*

(1) *$\overline{S}(\Gamma_I(R))$ is connected.*

(2) *Either $x + y \in S(I)$ or $x - y \in S(I)$ for all $x, y \in R - S(I)$.*

(3) *Either $x+y \in S(I)$ or $x+2y \in S(I)$ (but not both) for all $x, y \in R-S(I)$. In particular, either $2x \in S(I)$ or $3x \in S(I)$ for all $x \in R - S(I)$.*

Proof. (1)$\Longrightarrow$(2) Let $x, y \in R - S(I)$ be such that $x + y \notin S(I)$. If $x = y$, then $x - y \in S(I)$. Otherwise, $x - (-y) - y$ is a path from x to y by Theorem 3.3(2), and hence $x - y \in S(I)$.

(2)$\Longrightarrow$(3) Let $x, y \in R - S(I)$, and suppose that $x+y \notin S(I)$. By assumption, since $(x + y) - y = x \notin S(I)$, we have $x + 2y = (x + y) + y \in S(I)$. Let $x + y$ and $x + 2y$ belong to $S(I)$. Then $y \in S(I)$ a contradiction. In particular, if $x \in R - S(I)$, then either $2x \in S(I)$ or $3x \in S(I)$.

(3)$\Longrightarrow$(1) Let $x, y \in R-S(I)$ be distinct elements of R such that $x+y \notin S(I)$. By assumption, since $S(I)$ is an ideal of R and $x+2y \in S(I)$, we get $2y \notin S(I)$. Thus $3y \in S(I)$ by hypothesis. Since $x + y \notin S(I)$ and $3y \in S(I)$, we conclude that $x \neq 2y$, and hence $x - 2y - y$ is a path from x to y in $\overline{S}(\Gamma_I(R))$. Thus $\overline{S}(\Gamma_I(R))$ is connected. □

Theorem 3.5. *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R, and let $|S(I)| = \alpha$ and $|R/S(I)| = \beta$ (we allow α and β to be infinite, then we have $\beta - 1 = (\beta - 1)/2 = \beta$).*

(1) *If $2 \in S(I)$, then $\overline{S}(\Gamma_I(R))$ is the union of $\beta - 1$ disjoint K^{α}'s.*

(2) *If $2 \notin S(I)$, then $\overline{S}(\Gamma_I(R))$ is the union of $(\beta - 1)/2$ disjoint $K^{\alpha,\alpha}$'s.*

Proof. (1) Suppose that $2 \in S(I)$, and let $x \in R - S(I)$. Note that each coset $x + S(I)$ is a complete subgraph of $\overline{S}(\Gamma_I(R))$ since $(x + x_1) + (x + x_2) = 2x+x_1+x_2 \in S(I)$ for all $x_1, x_2 \in S(I)$. We must have that distinct cosets form disjoint subgraphs of $\overline{S}(\Gamma_I(R))$ since if $x + x_1$ and $y + x_2$ are adjacent for some $x, y \in R - S(I)$ and $x_1, x_2 \in S(I)$, then $x+y = (x+x_1)+(y+x_2)-(x_1+x_2) \in S(I)$, and hence $x - y = (x + y) - 2y \in S(I)$ since $S(I)$ is an ideal R and $2 \in S(I)$. But then $x + S(I) = y + S(I)$. Thus $\overline{S}(\Gamma_I(R))$ is the union of $\beta - 1$ disjoint (induced) subgraphs $x + S(I)$, each of which is a K^{α}, where $\alpha = |S(I)| = |x + S(I)|$.

(2) Let $x \in R-S(I)$ and $2 \notin S(I)$. Then no two distinct elements in $x+S(I)$ are adjacent; otherwise if $(x + x_1) + (x + x_2) \in S(I)$ for $x_1, x_2 \in S(I)$ implies that $2x \in S(I)$, and hence $2 \in S(I)$, a contradiction.

On the other hand, the two cosets $x + S(I)$ and $-x + S(I)$ are disjoint, and each element of $x + S(I)$ is adjacent to each element of $-x + S(I)$. Thus $(x+S(I))\cup(-x+S(I))$ is a complete bipartite (induced) subgraph of $\overline{S}(\Gamma_I(R))$;

furthermore, if $y+x_1$ adjacent to $x+x_2$ for some $y \in R-S(I)$ and $x_1, x_2 \in S(I)$, then $x+y \in S(I)$, and hence $y+S(I) = -x+S(I)$. Thus $\overline{S}(\Gamma_I(R))$ is the union of $(\beta-1)/2$ disjoint (induced) subgraphs $(x+S(I)) \cup (-x+S(I))$, each of which is a $K^{\alpha,\alpha}$, where $\alpha = |S(I)| = |x+S(I)|$. □

Remark 3.6. If $S(I)$ is an ideal of R, according to Note 2.1, $Z(R/I) = S(I)/I$. Let $|Z(R/I)| = \alpha'$ and $|R/I/Z(R/I)| = \beta'$. With the above notation, it is easy to check that $\alpha = \alpha'|I|$ and $\beta = \beta'$. $2+I \in Z(R/I)$ if and only if $2 \in S(I)$. Let $2 \in S(I)$. By part (1) of the above theorem and [2, Theorem 2.2(1)], $\overline{S}(\Gamma_I(R))$ is the union of $\beta-1$ disjoint K^{α},s and $Reg(\Gamma(R/I))$ is the union of $\beta-1$ disjoint $K^{\alpha/|I|}$,s. Let $2 \notin S(I)$. By part (2) of the above theorem and [2, Theorem 2.2(2)], $\overline{S}(\Gamma_I(R))$ is the union of $(\beta-1)/2$ disjoint $K^{\alpha,\alpha}$,s and $Reg(\Gamma(R/I))$ is the union of $(\beta-1)/2$ disjoint $K^{\alpha/|I|,\alpha/|I|}$,$s$. It follows from Remark 2.5, $S(\Gamma_I(R))$ contains $|I|$ disjoint subgraphs isomorphic to $Z(\Gamma(R/I))$ and $\overline{S}(\Gamma_I(R))$ contains $|I|$ disjoint subgraphs isomorphic to $Reg(\Gamma(R/I))$.

Example 3.7. Let $n \geq 2$ be an integer. Then $Z(Z_n)$ is an ideal Z_n if and only if $n = p^k$ for some prime p and integer $k \geq 1$ (see, [2, Example 2.7]). Let $\langle n \rangle = n\mathbb{Z}$. Since $Z(\mathbb{Z}/\langle n \rangle) = \{a + \langle n \rangle : a \in S(\langle n \rangle)\}$; hence $S(\langle n \rangle)$ is an ideal $\mathbb{Z}$ if and only if $n = p^k$ for some prime p and integer $k \geq 1$. Let $n = p^k$ for some prime p and integer $k \geq 1$. It is easy to check that $S(\langle p^k \rangle) = \langle p \rangle$, that is $\langle p^k \rangle$ is a $p\mathbb{Z}$-primal ideal $\mathbb{Z}$. If $p = 2$, then $\overline{S}(\Gamma_{\langle p^k \rangle}(\mathbb{Z}))$ is the complete subgraph K^{α},s such that $|\langle p \rangle| = \alpha$. If $p > 2$, then $\overline{S}(\Gamma_{\langle p^k \rangle}(\mathbb{Z}))$ is the union of $p-1/2$ disjoint $K^{\alpha,\alpha}$,s.

Note 3.8. Note that if $S(I) = \{0\}$, then R is an integral domain, and $2 \in S(I)$ if and only if char $R = 2$.

Theorem 3.9. *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R. Then*

(1) $\overline{S}(\Gamma_I(R))$ *is complete if and only if* $R/S(I) \cong \mathbb{Z}_2$ *or* $R \cong \mathbb{Z}_3$.

(2) $\overline{S}(\Gamma_I(R))$ *is connected if and only if* $R/S(I) \cong \mathbb{Z}_2$ *or* $R/S(I) \cong \mathbb{Z}_3$.

(3) $\overline{S}(\Gamma_I(R))$ (*and hence* $T(\Gamma_I(R))$ *and* $S(\Gamma_I(R))$) *is totally disconnected if and only if* $I = \{0\}$ *and R is an integral domain, with* char $R = 2$.

Proof. Let $|S(I)| = \alpha$ and $|R/S(I)| = \beta$.

(1) $\overline{S}(\Gamma_I(R))$ is complete if and only if $\overline{S}(\Gamma_I(R))$ is a single K^{α} or $K^{1,1}$ by Theorem 3.5.

Let $\overline{S}(\Gamma_I(R))$ be a complete subgraph of $T(\Gamma_I(R))$. If $2 \in S(I)$, then $\beta - 1 = 1$. Thus $R/S(I) \cong \mathbb{Z}_2$. If $2 \notin S(I)$, then $\alpha = 1$ and $(\beta-1)/2 = 1$. Thus $S(I) = \{0\} = I$ and $\beta = 3$; hence $R \cong \mathbb{Z}_3$.

Conversely, if $R/S(I) \cong \mathbb{Z}_2$, then we show that $2 \in S(I)$. $R/S(I) = \{S(I), x+S(I)\}$ where $x \notin S(I)$. Thus $x+S(I) = -x+S(I)$ gives $2x \in S(I)$; hence $2 \in S(I)$. So, $\overline{S}(\Gamma_I(R))$ is a single K^{α}. Next, suppose that $R \cong \mathbb{Z}_3$, then $I = \{0\}$ is only proper ideal of R, since $T(\Gamma_0(R)) = T(\Gamma(R))$, as required.

(2) By Theorem 3.5, $\overline{S}(\Gamma_I(R))$ is a connected subgraph $T(\Gamma_I(R))$ if and only if $\overline{S}(\Gamma_I(R))$ is a single K^{α} or $K^{\alpha,\alpha}$. Let $\overline{S}(\Gamma_I(R))$ be a connected subgraph of $T(\Gamma_I(R))$. If $2 \in S(I)$, then $\beta - 1 = 1$. Thus $R/S(I) \cong \mathbb{Z}_2$. If $2 \notin S(I)$, then $\beta - 1/2 = 1$ gives $\beta = 3$; hence $R/S(I) \cong \mathbb{Z}_3$.

Conversely, by part (1), it suffices to show that $\overline{S}(\Gamma_I(R))$ is connected when $R/S(I) \cong \mathbb{Z}_3$. We claim that $2 \notin S(I)$. Suppose not. Then $R/S(I) = \{S(I), x + S(I), y + S(I)\}$ where $x, y \notin S(I)$. Since $R/S(I)$ is a cyclic group with order of 3, we have $(x + S(I)) + (x + S(I)) = y + S(I)$; hence $y \in S(I)$ $(2x \in S(I))$, a contradiction. Thus $2 \notin S(I)$ and by Theorem 3.5(2), $\overline{S}(\Gamma_I(R))$ is a single $K^{\alpha,\alpha}$ and the proof is complete.

(3) $\overline{S}(\Gamma_I(R))$ is totally disconnected if and only if it is a disjoint union of K^1's. Hence by Theorem 3.5, $2 \in S(I)$ and $|S(I)| = 1$. So R must be an integral domain with char $R = 2$. □

Remark 3.10. Let $S(I)$ be an ideal. Then $R/I/Z(R/I) = R/I/S(I)/I \cong R/S(I)$; hence $R/I/Z(R/I) \cong \mathbb{Z}_n$ if and only if $R/S(I) \cong \mathbb{Z}_n$ such that $n \geq 2$ is an integer. So the above theorem in conjunction with [2, Theorem 2.4] is the other proof of Proposition 2.7.

At the end of this section, we give further explicit descriptions of the diameter and girth of $\overline{S}(\Gamma_I(R))$.

Proposition 3.11. *Let R be a commutative ring with proper ideal I such that $S(I)$ is an ideal of R. Then*

(1) $\operatorname{diam}(\overline{S}(\Gamma_I(R))) = 0, 1, 2,$ *or* ∞. *In particular,* $\operatorname{diam}(\overline{S}(\Gamma_I(R))) \leq 2$ *if $\overline{S}(\Gamma_I(R))$ is connected.*

(2) $\operatorname{gr}(\overline{S}(\Gamma_I(R))) = 3, 4$ *or* ∞. *In particular,* $\operatorname{gr}(\overline{S}(\Gamma_I(R))) \leq 4$ *if* $\overline{S}(\Gamma_I(R)))$ *contains a cycle.*

Proof. (1) Suppose that $\overline{S}(\Gamma_I(R))$ is connected. Then $\overline{S}(\Gamma_I(R))$ is a singleton, a complete graph, or a complete bipartite graph by Theorem 3.5. Thus $\operatorname{diam}(\overline{S}(\Gamma_I(R))) \leq 2$.

(2) Let $\overline{S}(\Gamma_I(R))$ contains a cycle. Since $\overline{S}(\Gamma_I(R)))$ is a disjoint union of either complete or complete bipartite graphs by Theorem 3.5, it must contain either a 3-cycle or a 4-cycle. Thus $\operatorname{gr}(\overline{S}(\Gamma_I(R))) \leq 4$. □

Theorem 3.12. *Let R be a commutative ring with the proper ideal I such that $S(I)$ is an ideal of R.*

(1) $\operatorname{diam}(\overline{S}(\Gamma_I(R))) = 0$ *if and only if* $R \cong \mathbb{Z}_2$.

(2) $\operatorname{diam}(\overline{S}(\Gamma_I(R))) = 1$ *if and only if either* $R/S(I) \cong \mathbb{Z}_2$ *and* $|S(I)| \geq 2$ *or* $R \cong \mathbb{Z}_3$.

(3) $\operatorname{diam}(\overline{S}(\Gamma_I(R))) = 2$ *if and only if* $R/S(I) \cong \mathbb{Z}_3$ *and* $|S(I)| \geq 2$.

(4) *Otherwise,* $\operatorname{diam}(\overline{S}(\Gamma_I(R))) = \infty$.

Proof. These results all follow from Theorem 3.5, Theorem 3.9 and Proposition 3.11. □

Corollary 3.13. *Let $S(I)$ be an ideal of R and $I \neq 0$. Then we have the following results*:

(1) *If* $\operatorname{diam}(Reg(\Gamma(R/I))) = 0$, *then* $\operatorname{diam}(\overline{S}(\Gamma_I(R))) = 1$ *and* $I = S(I)$.

(2) *Let* $\operatorname{diam}(Reg(\Gamma(R/I))) = 1$. *Then* $\operatorname{diam}(\overline{S}(\Gamma_I(R))) = 1$ *if* $I \subsetneq S(I)$ *and* $\operatorname{diam}(\overline{S}(\Gamma_I(R))) = 2$ *if* $I = S(I)$.

(3) *If* $\operatorname{diam}(Reg(\Gamma(R/I))) = 2$, *then* $\operatorname{diam}(\overline{S}(\Gamma_I(R))) = 2$.

(4) $\operatorname{diam}(\overline{S}(\Gamma_I(R))) = \infty$ *if and only if* $\operatorname{diam}(Reg(\Gamma(R/I))) = \infty$.

Proof. These results all follow directly from Remark 3.10, Theorem 3.12 and [2, Theorem 2.6(1)]. Note that for (4), $\operatorname{diam}(\overline{S}(\Gamma_I(R))) = \infty$ if and only if $2 \in S(I)$ and $|R/S(I)| = \beta \geq 3$, or $2 \notin S(I)$ and $|R/S(I)| = \beta \geq 5$. So, by Note 2.1 and [2, Theorem 2.2], the proof is complete. □

Corollary 3.14. *Let $S(I)$ be an ideal of R and $I \subsetneq S(I)$. If* $\operatorname{diam}(\overline{S}(\Gamma_I(R))) = k$ *such that* $0 \leq k \leq 2$ *is an integer, then* $\operatorname{diam}(Reg(\Gamma(R/I))) = k$.

Proof. The result follows by Remark 3.10, Theorem 3.12 and [2, Theorem 2.6(1)]. □

Theorem 3.15. *Suppose that $S(I)$ is an ideal of R. Then*

(1) (a) $\operatorname{gr}(\overline{S}(\Gamma_I(R))) = 3$ *if and only if* $2 \in S(I)$ *and* $|S(I)| \geq 3$.
(b) $\operatorname{gr}(\overline{S}(\Gamma_I(R))) = 4$ *if and only if* $2 \notin S(I)$ *and* $|S(I)| \geq 2$.
(c) *Otherwise*, $\operatorname{gr}(\overline{S}(\Gamma_I(R))) = \infty$.

(2) (a) $\operatorname{gr}(T(\Gamma_I(R))) = 3$ *if and only if* $|S(I)| \geq 3$.
(b) $\operatorname{gr}(T(\Gamma_I(R))) = 4$ *if and only if* $2 \notin S(I)$ *and* $|S(I)| = 2$.
(c) *Otherwise*, $\operatorname{gr}(T(\Gamma_I(R))) = \infty$.

Proof. According to Theorem 3.1, Theorem 3.5, these results follow. □

Corollary 3.16. *Let $S(I)$ be an ideal of R. Then*

(1) (a) *If* $\operatorname{gr}(Reg(\Gamma(R/I))) = k$ *such that* $3 \leq k \leq 4$ *is an integer, then* $\operatorname{gr}(\overline{S}(\Gamma_I(R))) = k$.
(b) *If* $\{0\} \neq I \subsetneq S(I)$ *and* $\operatorname{gr}(Reg(\Gamma(R/I))) = \infty$, *then* $\operatorname{gr}(\overline{S}(\Gamma_I(R))) = 3$.

(2) (a) *If* $\operatorname{gr}(\overline{S}(\Gamma_I(R))) = 3$, *then if* $|Z(R/I)| \leq 2$, $\operatorname{gr}(\overline{S}(\Gamma_I(R))) = \infty$. *If* $|Z(R/I)| > 2$, *then* $\operatorname{gr}(\overline{S}(\Gamma_I(R))) = 3$.
(b) *If* $\operatorname{gr}(\overline{S}(\Gamma_I(R))) = 4$, *then* $\operatorname{gr}(Reg(\Gamma(R/I))) = 4$, *if* $I \subsetneq S(I)$; *otherwise* $\operatorname{gr}(Reg(\Gamma(R/I))) = \infty$.
(c) *If* $\operatorname{gr}(\overline{S}(\Gamma_I(R))) = \infty$, *then* $\operatorname{gr}(Reg(\Gamma(R/I))) = \infty$.

Proof. These results all follow directly from Note 2.1, Remark 3.10, and Theorem 3.15 and [2, Theorem 2.6(2)]. □

4. The case when $S(I)$ is not an ideal R

Given a proper ideal I of R, in this section we study the remaining case when $S(I)$ is not an ideal of R (i.e., I is not primal ideal of R). Since $S(I)$ is always closed under product by elements of R; hence there are distinct $x, y \in S(I)^*$ such that $x+y \in R-S(I)$, so $|S(I)| \geq 3$; in this case, $S(\Gamma_I(R))$ and $\overline{S}(\Gamma_I(R))$ are never disjoint subgraphs. Also, we determine when $T(\Gamma_I(R))$ is connected and compute $\operatorname{diam}(T(\Gamma_I(R)))$.

Theorem 4.1. *Suppose that $S(I)$ is not an ideal of R.*
(1) *$S(\Gamma_I(R))$ is connected with $\operatorname{diam}(S(\Gamma_I(R))) = 2$.*
(2) *Some vertex of $S(\Gamma_I(R))$ is adjacent to a vertex of $\overline{S}(\Gamma_I(S))$.*
In particular, the subgraphs $S(\Gamma_I(R))$ and $\overline{S}(\Gamma_I(S))$ are not disjoint.
(3) *If $\overline{S}(\Gamma_I(S))$ is connected, then $T(\Gamma_I(S))$ is connected.*

Proof. (1) Let $x \in S(I)^*$. Then x is adjacent to 0. Thus $x - 0 - y$ is a path in $S(\Gamma_I(R))$ of length two between any two distinct $x, y \in S(I)^*$. Moreover, there exist nonadjacent $x, y \in S(I)^*$ since $S(I)$ is not an ideal of R; thus $\operatorname{diam}(S(\Gamma_I(R))) = 2$.

(2) By assumption, there exist distinct $x, y \in S(I)^*$ such that $x + y \notin S(I)^*$; so $x+y \in R-S(I)$. Then $-x \in S(I)$ and $x+y \in R-S(I)$ are adjacent vertices in $T(\Gamma_I(R))$ since $-x + (x + y) = y \in S(I)$. The "in particular" statement is clear.

(3) By part (1) above, it suffices to show that there is a path from x to y in $T(\Gamma_I(R))$ for any $x \in S(I)$ and $y \in R - S(I)$. By part (2) above, there exist adjacent vertices u and v in $S(\Gamma_I(R))$ and $\overline{S}(\Gamma_I(R))$, respectively. Since $S(\Gamma_I(R))$ is connected, there is a path from x to u in $S(\Gamma_I(R))$; and since $\overline{S}(\Gamma_I(R))$ is connected, there is a path from v to y in $\overline{S}(\Gamma_I(R))$. Then there is a path from x to y in $T(\Gamma_I(R))$ since u and v are adjacent in $T(\Gamma_I(R))$. It follows that, $T(\Gamma_I(R))$ is connected. □

The Jacobson radical $Jac(R)$ of R is defined to be the intersection of all the maximal ideal of R, [4, Proposition 1.9]. Consider the following lemma.

Lemma 4.2. *Suppose that $S(I)$ is not an ideal of R. Then $T(\Gamma_I(R))$ is connected if and only if $R = \langle a_1, \ldots, a_k \rangle$ for some $a_1, \ldots, a_k \in S(I)$. In particular, if R/I is a finite ring and $I \subseteq Jac(R)$, then $T(\Gamma_I(R))$ is connected.*

Proof. Suppose $T(\Gamma_I(R))$ is connected. Hence there is a path $0-x_1-\cdots-x_n-1$ from 0 to 1 in $T(\Gamma_I(R))$. Now $x_1, x_1 + x_2, \ldots, x_n + 1 \in S(I)$. Hence $1 \in \langle x_1, x_1 + x_2, \ldots, x_{n-1} + x_n, x_n + 1 \rangle \subseteq \langle S(I) \rangle$; thus $R = \langle S(I) \rangle$. Conversely, suppose that $R = \langle S(I) \rangle$. We show that for each $0 \neq x \in R$, there exists a path in $T(\Gamma_I(R))$ from 0 to x. By assumption, there are elements $z_1, \ldots, z_n \in S(I)$ such that $x = z_1 + \cdots + z_n$. Set $w_0 = 0$ and $w_k = (-1)^{n+k}(z_1 + \cdots + z_k)$ for each integer k with $1 \leq k \leq n$. Then $w_k + w_{k+1} = (-1)^{n+k+1} z_{k+1} \in S(I)$ for each integer k with $0 \leq k \leq n-1$; and thus $0 - w_1 - w_2 - \cdots - w_{n-1} - w_n = x$ is a path from 0 to x in $T(\Gamma_I(R))$ of length at most n. Now let $0 \neq u, v \in R$. Then by the preceding argument, there are paths from u to 0 and 0 to v in $T(\Gamma_I(R))$; hence there is a path from u to v in $T(\Gamma_I(R))$. Thus, $T(\Gamma_I(R))$ is connected. □

In the light of Lemma 4.2, we have the following results.

Theorem 4.3. *Suppose that $S(I)$ is not an ideal of R and $R = \langle S(I)\rangle$. Let $n \geq 2$ be the least integer such that $R = \langle x_1, \dots, x_n\rangle$ for some $x_1, \dots, x_n \in S(I)$ (that is, $T(\Gamma_I(R))$ is connected). Then* $\mathrm{diam}(T(\Gamma_I(R))) = n$. *In particular, if R/I is a finite ring and $I \subseteq Jac(R)$, then* $\mathrm{diam}(T(\Gamma_I(R))) = 2$.

Proof. First, we investigate any path from 0 to 1 in $T(\Gamma_I(R))$ has length $\geq n$. Suppose that $0 - x_1 - x_2 - \cdots - x_{m-1} - 1$ is a path from 0 to 1 in $T(\Gamma_I(R))$ of length m. Thus $x_1, x_1 + x_2, \dots, x_{m-2} + x_{m-1}, x_{m-1} + 1 \in S(I)$, and hence $1 \in (x_1, x_1 + x_2, \dots, x_{m-2} + x_{m-1}, x_{m-1} + 1) \subseteq (S(I))$. Thus $m \geq n$.

Now, let x and y be distinct elements in R. We show that there is a path from x to y in $T(\Gamma_I(R))$ with length $\leq n$. Let $1 = b_1 + \cdots + b_n$ for some $b_1, \dots, b_n \in S(I)$, and let $z = y + (-1)^{n+1}x$. Define $w_0 = x$ and $w_k = (-1)^{n+k}z(b_1 + \cdots + b_k) + (-1)^k x$ for each integer k with $1 \leq k \leq n$. Then $w_k + w_{k+1} = (-1)^{n+k+1}zb_{k+1} \in S(I)$ for each integer k with $0 \leq k \leq n-1$ and $w_n = z + (-1)^n x = y$. Thus $x - w_1 - \cdots - w_{n-1} - y$ is a path from x to y in $T(\Gamma_I(R))$ with length at most n. Specially, we conclude that a shortest path between 0 and 1 in $T(\Gamma_I(R))$ has length n; hence $\mathrm{diam}(T(\Gamma_I(R))) = n$. For the "in particular" statement, note that $Z(R/I)$ is not an ideal of R. So, $x + y + I \in Reg(R/I)$ for some $x, y \in S(I)$. Since every regular element of a finite commutative ring is a unit and $I \subseteq Jac(R)$; hence $x + y$ is a unit. Now, we have $R = \langle x, y\rangle$, and thus $\mathrm{diam}(T(\Gamma_I(R))) = 2$. □

Clearly, if $R = \langle a_1, \dots, a_k\rangle$ for some $a_1, \dots, a_k \in S(I)$, then $R/I = \langle a_1 + I, \dots, a_k + I\rangle$; hence $\mathrm{diam}(T(\Gamma(R/I))) \leq \mathrm{diam}(T(\Gamma_I(R)))$ (see [2, Theorem 3.4]). Note that since, $k \geq 2$ be the least integer such that $R = \langle a_1, \dots, a_k\rangle$; hence $\mathrm{diam}(T(\Gamma(R/I))) \geq \mathrm{diam}(T(\Gamma_I(R))) - 1$.

Example 4.4. Let $n \geq 2$ be an integer, and let $n \neq p^k$ for every prime p and integer $k \geq 1$. Then $S(\langle n\rangle)$ is not an ideal of $\mathbb{Z}$ (see, Example 3.7). It is easy to check that there are distinct primes p and q, and integers $r, s \notin \langle n\rangle$ such that $pr \in \langle n\rangle$ and $qs \in \langle n\rangle$. So $\mathbb{Z} = \langle p, q\rangle$; that $p, q \in S(\langle n\rangle)$. By Theorem 4.3, $\mathrm{diam}(T(\Gamma_{\langle n\rangle}(\mathbb{Z}))) = 2$.

Theorem 4.5. *Suppose that $S(I)$ is not an ideal of R. If $T(\Gamma_I(R))$ is connected, then*

(1) $\mathrm{diam}(T(\Gamma_I(R))) = d(0, 1)$.

(2) *If* $\mathrm{diam}(T(\Gamma_I(R))) = n$, *then* $\mathrm{diam}(\overline{S}(\Gamma_I(R))) \geq n - 2$.

Proof. (1) This follows from the proof of Theorem 4.3.

(2) By part (1) above, $\mathrm{diam}(T(\Gamma_I(R))) = d(0, 1) = n$. Let $0 - x_1 - \cdots - x_{n-1} - 1$ be a shortest path from 0 to 1 in $T(\Gamma_I(R))$. Clearly, $x_1 \in S(I)$. If $x_i \in S(I)$ for some integer i with $2 \leq i \leq n-1$, then we can construct the path $0 - x_i - \cdots - x_{n-1} - 1$ from 0 to 1 in $T(\Gamma_I(R))$ which has length less than n, which is a contradiction. Thus $x_i \in R - S(I)$ for each integer i with

$2 \leq i \leq n-1$. Therefore, $x_2 - x_3 - \cdots - x_{n-1} - 1$ is a shortest path from x_2 to 1 in $\overline{S}(\Gamma_I(R))$, and it has length $n-2$. Thus $\text{diam}(\overline{S}(\Gamma_I(R))) \geq n-2$. □

Corollary 4.6. *Let $\{R_\alpha\}_{\alpha\in\Lambda}$ be a family of commutative rings with $|\Lambda| \geq 2$, and let $R = \prod_{\alpha\in\Lambda} R_\alpha$. Suppose $I = \prod_{\alpha\in\Lambda} I_\alpha$; such that for every $\alpha \in \Lambda$, I_α is a proper ideal of R_α. Then $T(\Gamma_I(R))$ is connected with* $\text{diam}(T(\Gamma_I(R))) = 2$.

Proof. It is easy to check that $e = (1, 0, 0, \ldots)$ and $1_R - e \in S(I)$. It follows that $R = \langle e, 1_R - e\rangle$; so by Theorem 4.3, the claim is true. □

Remark 4.7. Let R and U be commutative rings, I and J be proper ideals of R and U, respectively. It is clear to check that $R\times U - S(I\times J) = (R-S(I))\times(U-S(J))$. So for distinct $(x, y), (z, w) \in R\times U - S(I\times J)$, $(x, y)-(-x, -w)-(z, w)$ is a path of length at most two in $\overline{S}(\Gamma_{I\times J}(R\times U))$. Thus $\overline{S}(\Gamma_{I\times J}(R\times U))$ is connected with $\text{diam}(\overline{S}(\Gamma_{I\times J}(R\times U))) \leq 2$. By Theorem 4.1(2), it follows that $T(\Gamma_{I\times J}(R\times U))$ is connected (see Corollary 4.6).

Theorem 4.8. *Let $S(I)$ does not an ideal of R. Then $T(\Gamma_{S^{-1}I}(S^{-1}R))$; where $S = R - S(I)$, is connected with* $\text{diam}(T(\Gamma_{S^{-1}I}(S^{-1}R))) = 2$. *In particular, if R/I is a finite ring and $I \subseteq Jac(R)$, then* $\text{diam}(T(\Gamma_{S^{-1}I}(S^{-1}R))) = 2$.

Proof. Since $S(I)$ is not an ideal of R, there are $x_1, x_2 \in S(I)$ such that $s = x_1 + x_2 \in R - S(I)$. Thus $x_1/s + x_2/s = 1$ in $S^{-1}R$. It is easy to check that $S(S^{-1}I)$ is not an ideal of $S^{-1}R$ and $x_1/s, x_2/s \in S(S^{-1}I)$. Thus $S^{-1}R = \langle x_1/s, x_2/s\rangle$. The "in particular" statement is clear since every $s \in S$ is unite ($s + I \in Reg(R/I)$; hence $s + I$ is unite). It follows that $S^{-1}R = R$. □

Theorem 4.9. *Let $I \trianglelefteq R$, and P_1 and P_2 be prime ideals of R, containing I. Suppose $xy \in I$ for some $x \in P_1 \setminus P_2$ and $y \in P_2 \setminus P_1$. Then* $\text{diam}(T(\Gamma_{S^{-1}I}(R_S))) = 2$ *where $S = R \setminus P_1 \cup P_2$.*

Proof. For all $s \in S$, we have sx and $sy \notin I$; since $s, x \notin P_2$ and $s, y \notin P_1$. Thus x/s and y/s are nonzero elements of $S(S^{-1}I)$ ($(x/s)(y/1) \in S^{-1}I$ and $y/1 \notin S^{-1}I$). Let $s = x + y \in S$, hence $S^{-1}R = \langle x/s, y/s\rangle$. Thus $T(\Gamma_{S^{-1}I}(S^{-1}R))$ is connected with $\text{diam}(T(\Gamma_{S^{-1}I}(S^{-1}R))) = 2$ by Theorem 4.3. □

The following theorem give $\text{gr}(S(\Gamma_I(R)))$, $\text{gr}(\overline{S}(\Gamma_I(R)))$, and $\text{gr}(T(\Gamma_I(R)))$ when $S(I)$ is not an ideal of R.

Theorem 4.10. *Let R be a commutative ring with the proper ideal I such that $S(I)$ is not an ideal of R. Then*

(1) *If $I \neq \{0\}$,* $\text{gr}(S(\Gamma_I(R))) = 3$. *Otherwise* $\text{gr}(S(\Gamma_I(R))) = 3$ *or* ∞. *Moreover, if* $\text{gr}(S(\Gamma_I(R))) = \infty$, *then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$; so, $S(\Gamma_I(R)$ is a $K^{1,2}$ star graph with center* 0.

(2) $\text{gr}(T(\Gamma_I(R))) = 3$ *if and only if* $\text{gr}(S(\Gamma_I(R))) = 3$.

(3) *The (induced) subgraph of $S(\Gamma_I(R))$ with vertices $\sqrt{I}$ is complete; hence* $\text{gr}(S(\Gamma_I(R))) = 3$ *when* $|\sqrt{I}| \geq 3$.

(4) *If* $\text{gr}(T(\Gamma_I(R))) = 4$, *then* $\text{gr}(S(\Gamma_I(R))) = \infty$.

(5) *If* $2 \in I$, *then* $\mathrm{gr}(\overline{S}(\Gamma_I(R))) = 3$ *or* ∞.
(6) *If* $2 \notin I$, *then* $\mathrm{gr}(\overline{S}(\Gamma_I(R))) = 3, 4$ *or* ∞.

Proof. (1) Let $0 \neq x \in I$ and $y \in S(I) \setminus I$. Since $I + S(I) \subseteq S(I)$, $0 - x - y - 0$ is a 3-cycle in $S(\Gamma_I(R))$. If $I = \{0\}$, it follows from [2, Theorem 3.4(1)]. Note that if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $I = \{0\}$ is the only proper ideal of R, that $S(I)$ is not an ideal of R.

(2) It suffices to show that $\mathrm{gr}(S(\Gamma_I(R))) = 3$ when $\mathrm{gr}(T(\Gamma_I(R))) = 3$. If $2x \neq 0$ for some $x \in S(I)^*$, then $0 - x - (-x) - 0$ is a 3-cycle in $S(I)$. Thus we may assume that $2x = 0$ for all $x \in S(I)$. Since $S(I)$ is not an ideal; so there are $x \in S(I)$ such that $x \notin I$. $2x = 0 \in I$; hence $2 \in S(I)$. Let $a - b - c - a$ be a 3-cycle in $S(\Gamma_I(R))$. So $a + b, b + c, c + a \in S(I)$. If $2a = 0$, then $0 - a + b - a + c - 0$ is a 3-cycle in $S(\Gamma_I(R))$. So without loss of generality we can assume that $2a, 2b$ and $2c$ are non-zero. If $2a \neq 2b$, then $0 - 2a - 2b - 0$ is a 3-cycle in $S(\Gamma_I(R))$. Without loss of generality we can assume that $2a = 2b = 2c$. So, $2(a - b) = 2(b - c) = 0 \in I$. If $2 \notin I$, then $a - b$ and $b - c \in S(I)$; hence $0 - (a - b) - (b - c) - 0$ is a 3-cycle in $S(\Gamma_I(R))$ (if $a - b = b - c$, then $a + c = 2b = 2a$, a contradiction). Let $2 \in I$. Since $b + c \in S(I)$; hence $(b + c)r \in I$ such that $r \notin I$; thus $(2a + b + c)r \in I$. Now $0 - a + b - a + c - 0$ is a 3-cycle in $S(\Gamma_I(R))$ (if $a + b = 0$, then we have a 3-cycle $0 - a + c - b + c - 0$). Thus in all cases we get a 3-cycle in $S(\Gamma_I(R))$.

(3) It follows from $\sqrt{I} \subseteq S(I)$ is an ideal.

(4) It is clear by parts $1, 2$.

(5) Let $2 \in I$ and $\overline{S}(\Gamma_I(R))$ contains a cycle C. Hence there is a path $x - y - z$ in $\overline{S}(\Gamma_I(R))$. Without loss of generality we may assume that $x \neq 1$, $y \neq 1$. Clearly, $x + y, y + z \in S(I)$. Suppose that R contains a $a \in \sqrt{I} \setminus I$. If $a = ax = ay$, then $x + 1$, $y + 1 \in S(I)$, and thus $1 - x - y - 1$ is a 3-cycle in $\overline{S}(\Gamma_I(R))$. If either $ax \neq a$ or $ay \neq a$, then either $1 - (a + 1) - (ax + 1) - 1$ or $1 - (a + 1) - (ay + 1) - 1$ is a 3-cycle in $\overline{S}(\Gamma_I(R))$ $(a + I \in Jac(R/I))$. Let $\sqrt{I} = I$. If $I = \{0\}$ (hence $2 = 0$), then $x^2 \neq y^2$; since $x^2 + y^2 = (x + y)^2 \neq 0$. Hence $x^2 - xy - y^2 - x^2$ is a 3-cycle in $\overline{S}(\Gamma_I(R)) = Reg(R)$. Finally, let $I \neq \{0\}$. Suppose $0 \neq b \in I$. If $x + z \in S(I)$, then $x - y - z - x$ is a 3-cycle in $\overline{S}(\Gamma_I(R))$. Let $x + z \notin S(I)$. It follows that $y - x$ or $z - y \notin I$ $(2x \in I)$. Without loss of generality we can assume that $y - x \notin I$; hence $b + x - x - y - b + x$ is a 3-cycle in $\overline{S}(\Gamma_I(R))$. So, as required.

(6) Suppose that $\overline{S}(\Gamma_I(R))$ contains a cycle. So there is a path $x - y - z$ in $\overline{S}(\Gamma_I(R))$. We may assume that $x + z \notin S(I)$. It is clear that either $x + y \neq 0$ or $y + z \neq 0$ (otherwise $x = z$, a contradiction). Without loss of generality we can assume that $x + y \neq 0$. Then $x - y - (-y) - (-x) - x$ is a 4-cycle (if $x = -x$ gives $2x = 0 \in I$, then $x \in S(I)$, a contradiction). So, the proof is complete. □

References

[1] S. Akbari, D. Kiani, F. Mohammadi, and S. Moradi, *The total graph and regular graph of a commutative ring*, J. Pure Appl. Algebra **213** (2009), no. 12, 2224–2228.

[2] D. F. Anderson and A. Badawi, *The total graph of a commutative ring*, J. Algebra **320** (2008), no. 7, 2706–2719.

[3] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), no. 2, 434–447.

[4] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Readig, MA, 1969.

[5] B. Bollobás, *Graph Theory: An Introduction Course*, Springer-Verlag, New York, 1979.

[6] A. Yousefian Darani, *Primal and weakly primal submodules and related results*, Ph. D. thesis, University of Guilan, Iran, 2008.

[7] L. Fuchs, *On primal ideals*, Proc. Amer. Math. Soc. **1** (1950), 1–6.

[8] S. P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra **31** (2003), no. 9, 4425–4443.

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