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EXPLICIT FORMULAS FOR THE BERGMAN KERNEL ON CERTAIN FORELLI-RUDIN CONSTRUCTION

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Abstract. In this note, we present certain circular domain, named Forelli-Rudin construction or Hua construction, which is built on Cartan domains. We compute the explicit Bergman kernel for it and get the corresponding weighted Bergman kernel on its base.

1. Introduction and main results

The Bergman kernel was introduced by S. Bergman in the 1920's [1]. In 1933, Bergman generalized this theory to the case of several complex variables [2]. It's known that every bounded domain in $\mathbb{C}^n$ admits a non trivial Bergman kernel. Let Ω be a bounded domain in $\mathbb{C}^n$ and $L^2(\Omega)$ denote the space that consists of all the square-integral functions associated to the Lebesgue measure dV. Let $L^2_h(\Omega) := L^2(\Omega) \cap \mathcal{O}(\Omega)$. The Bergman kernel $K(z, \bar{w})$ of the domain Ω is the unique sesqui-holomorphic functions satisfying the skew-symmetry property that $K(z, \bar{w}) = \overline{K(w, \bar{z})}$ and the reproducing property that $f(z) = \int_\Omega f(w)K(z, \bar{w})dV$ for any $f \in L^2_h(\Omega)$. If multiply the Lebesgue measure dV by a measurable non-negative function ρ, then the corresponding weighted Bergman kernel $K_\rho(z, \bar{w})$ for $L^2_h(\Omega, \rho)$ can be obtained according to the Riesz representation theorem, where $L^2_h(\Omega, \rho)$ consists of all square-integrable functions associated to the measure ρdV.

Stefan Bergman himself [3] computed the kernel of the domain $\{(w, z) \in \mathbb{C}^2 : |w|^{2p} + |z|^2 < 1\}$ in 1936. Using holomorphic automorphism group, Hua [13] obtained the Bergman kernels for the bounded symmetric domains in the 1950's. D'Angelo [6] calculated the Bergman kernel of the domain $\{w \in \mathbb{C}, z \in \mathbb{C}^n : |w|^{2p} + \|z\|^2 < 1\}$ as the sum of an orthonormal series in 1978 and generalized his results in 1994 while w is also a vector in $\mathbb{C}^m$ [7]. For the complex ellipsoid in $\mathbb{C}^n$, Franciscs and Hanges [12] gave the Bergman kernel in terms of hypergeometric functions. Oeljeklaus and Pflug [16] computed the Bergman

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kernel for minimal ball $\mathbb{B}_*$ in 1997. In 1998, Roos and Yin [20] introduced so called Cartan-Hartogs domains on which Bergman kernels were obtained in close form by Yin in 1999. In [8], the Bergman kernel for the symmetrized polydisc $\mathbb{G}_n$ was obtained by Edigarian and Zwonek. Recently, Jong-Do Park computed the explicit formula of the Bergman kernel for a nonhomogeneous domain $\{z \in \mathbb{C}^2 : |z_1|^4 + |z_2|^4 < 1\}$ and the kernel is not algebraic [17].

As for the weighted Bergman kernel, the important formula for the Hartogs domain

$$\Omega_\rho^m = \{(z, w) \in \Omega \times \mathbb{C}^m : |w|^2 < \rho(z)\}$$

is

$$K_{\Omega_\rho^m}((z, w), \overline{(t, s)}) = \frac{m!}{\pi^m} \sum_{k=0}^{\infty} \frac{(m+1)_k}{k!} K_{\Omega, \rho^{k+m}}(z, \bar{t}) \langle w, s \rangle^k \tag{1.1}$$

due to Ligocka [14], where $K_{\Omega,\rho^{k+m}}$ denotes weighted Bergman kernel on Ω with respect to the weight ρ^{k+m} and the Pochhammer symbol $(m + 1)_k = (m + 1)(m + 2) \cdots (m + k)$.

The formula was shown firstly by Forelli and Rudin [11] for Ω the unit disc and $\rho(z) = 1 - |z|^2$ and later generalized by Engliš to the following so-called Forelli-Rudin construction [9]

$$\tilde{\Omega}^{n_1,n_2}_{\varphi_1,\varphi_2} = \left\{ (s, t, z) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \Omega : \frac{|s|^2}{\varphi_1(z)} + \frac{|t|^2}{\varphi_2(z)} < 1 \right\},$$

where φ_1 and φ_2 are two positive functions on Ω.

In 2006, Engliš and Zhang constructed a type of Hartogs domain, named generalized Forelli-Rudin construction, whose "fiber" of the base domain Ω is, instead of the disc or the ball, an arbitrary irreducible bounded symmetric domain. They expressed the Bergman kernel in terms of weighted Bergman kernels on its base Ω [10].

Inspired by the work of above, we study the Bergman kernel for a certain Forelli-Rudin construction which we call Hua construction of the first type. The aim of the present paper is to give an explicit formula for the Bergman kernel without the use of the weighted Bergman kernel for the base domain Ω. As an application, we can also obtain the corresponding weighted Bergman kernel for Ω explicitly.

In more details, let the base domain Ω be the irreducible bounded symmetric domain of the first type $\mathfrak{R}_{\mathrm{I}}(m, n)$, and $\varphi_j(Z) = \det(I - Z\bar{Z}^t)^{q_j}$, $(j = 1, 2, \ldots, r)$ positive continuous functions on $\mathfrak{R}_{\mathrm{I}}$. We consider a certain Forelli-Rudin Construction in $\mathbb{C}^{N_1+N_2+\cdots+N_r} \times \mathfrak{R}_{\mathrm{I}}$, that is

$$\tilde{\Omega} = \widetilde{\Omega}^{N_1,\ldots,N_r}_{\varphi_1,\ldots,\varphi_r} = \left\{ w_l \in \mathbb{C}^{N_l},\ Z \in \mathfrak{R}_{\mathrm{I}}(m, n) :\ \sum_{l=1}^{r} \frac{\|w_l\|^{2p_l}}{\varphi_l(Z)} < 1 \right\},$$

where $N_1, \dots, N_r$, r are all positive integers, $p_j > 0$, $q_j > 0$ and the norm $\|w_l\|^2 = |w_{l1}|^2 + \cdots + |w_{lN_l}|^2$. Of course we can construct the similar Forelli-Rudin constructions based on the other irreducible bounded symmetric domains, we take $\mathfrak{R}_{\mathrm{I}}$ for example only.

We will use the integral formula of $\varphi_j(Z)$ over $\mathfrak{R}_{\mathrm{I}}(m,n)$ in the proof of our main theorem, which is an important result in Hua's book [13]. Therefore, we call $\widetilde{\Omega}^{N_1,\dots,N_r}_{\varphi_1,\dots,\varphi_r}$ Hua construction also. Obviously, when $r = 1$ the domain $\widetilde{\Omega}^N_\varphi$ is exact the so-called Cartan-Hartogs domain which was studied by many authors [21, 19, 18]. Moreover, when $r = m = 1$ the domain is the complex ellipsoid in complex Euclidean space and a further special case $r = m = n = N_1 = 1$ is the Thullen domain in $\mathbb{C}^2$.

Due to Boas-Fu-Straube's inflation theorem in [5] (see also [4]), it's enough to consider only the case $N_1 = N_2 = \cdots = N_r = 1$. Denote by $K_{\widetilde{\Omega}}$ the Bergman kernel on the diagonal for $\widetilde{\Omega}^{1,\dots,1}_{\varphi_1,\dots,\varphi_r}$, we have the following main results.

Theorem 1. *For any $p_j \in \mathbb{R}^+$, the Bergman kernel for $\widetilde{\Omega}^{1,\dots,1}_{\varphi_1,\dots,\varphi_r}$ can be formulated as the following infinite series*

$$K_{\widetilde{\Omega}} = C \sum_{v_1=0}^{mn} \sum_{u_1=0}^{v_1} \sum_{v_2=0}^{v_1-u_1} \sum_{u_2=0}^{v_2} \cdots \sum_{v_r=0}^{v_{r-1}-u_{r-1}} \sum_{u_r=0}^{v_r} b^{(1)}_{v_1} \cdots b^{(r)}_{v_r} \Gamma(v_r + 1 - u_r) \cdot \prod_{l=1}^{r} L_l \frac{\partial^{u_1+\cdots+u_r}}{\partial x_1^{u_1} \cdots \partial x_r^{u_r}} x_1^{u_1} \cdots x_r^{u_r} H(x) \det(I - Z\bar{Z}^t)^{-d},$$

where $b^{(j)}_{v_j} (j = 1, 2, \dots, r)$ are constant coefficients and the notations

$$C = \frac{\prod_{l=1}^r p_l}{\pi^{r+mn}}, \quad L_l = \frac{\Gamma(v_l + 1)}{\Gamma(u_l + 1)\Gamma(v_l + 1 - u_l)}, \quad d = m + n + \sum_{l=1}^{r} \frac{q_l}{p_l},$$

$$H(x) = \sum_{|j| \geqslant 0} T(j_1, \dots, j_r) x_1^{j_1} \cdots x_r^{j_r}, \quad T(j_1, \dots, j_r) = \frac{\Gamma\left(1 + \sum_{l=1}^r \frac{j_l+1}{p_l}\right)}{\prod_{l=1}^r \Gamma\left(\frac{j_l+1}{p_l}\right)}.$$

The components of the vector x in $H(x)$ are defined by $x_l = |w_l|^2 \varphi_l(Z)^{-\frac{1}{p_l}}$, $l = 1, 2, \dots, r$.

If all $p_j (j = 1, 2, \dots, r)$, the power of $|w_j|^2$, are positive integers, the Bergman kernel on the diagonal $K_{\widetilde{\Omega}}((w, Z); \overline{(w, Z)})$ can be expressed in terms of Appell's hypergeometric functions. We have:

Corollary 1. *For all $p_j \in \mathbb{Z}^+$, $j = 1, 2, \dots, r$, the Bergman kernel on the diagonal is*

$$K_{\widetilde{\Omega}} = C \sum_{v_1=0}^{mn} \sum_{u_1=0}^{v_1} \cdots \sum_{v_r=0}^{v_{r-1}-u_{r-1}} \sum_{u_r=0}^{v_r} \sum_{k_1=0}^{p_1-1} \cdots \sum_{k_r=0}^{p_r-1} b^{(1)}_{v_1} \cdots b^{(r)}_{v_r}$$

$$\cdot\ \Gamma(v_r+1-u_r)\prod_{l=1}^{r} L_l T(k_1,\dots,k_r)\ G(x)\det(I-Z\bar{Z}^t)^{-d},$$

where the function

$$G(x)=\frac{\partial^{u_1+\cdots+u_r}}{\partial x_1^{u_1}\cdots\partial x_r^{u_r}}\prod_{j=1}^{r} x_j^{u_j+k_j}F_A^{(r)}\left(1+\sum_{l=1}^{r}\frac{k_l+1}{p_l},\mathbf{1},\frac{\mathbf{k}+\mathbf{1}}{\mathbf{p}},\mathbf{x}^{\mathbf{p}}\right).$$

The notations $\mathbf{1}=(1,\dots,1)$, $\frac{\mathbf{k+1}}{\mathbf{p}}=\left(\frac{k_1+1}{p_1},\dots,\frac{k_r+1}{p_r}\right)$ *and* $F_A^{(r)}$ *is Appell's hypergeometric function with* r *variables.*

General speaking, the hypergeometric function $F_A^{(r)}$ is still a series of infinite form. Notice that whenever $(w,Z)\in\widetilde{\Omega}^{1,\dots,1}_{\varphi_1,\dots,\varphi_r}$ then $\sum_{l=1}^r |x_l|^{p_l}=\sum_{l=1}^r\frac{|w_l|^{2p_l}}{\varphi_l(Z)}$ <1, therefore $F_A^{(r)}$ converges at the point $(x_1^{p_1},\dots,x_r^{p_r})$. The presence of the hypergeometric functions in Corollary 1 provides an opportunity to approach the asymptotic behavior of the Bergman kernel near some weakly pseudoconvex boundary points. It can also be used to obtain an asymptotic expansion of the Bergman kernel, see ref. [12].

Now we consider the case that the reciprocal of p_j is an integer for $j=1,2,\dots,r-1$ and $p_r\in\mathbb{R}^+$. In this case the domain $\widetilde{\Omega}^{1,\dots,1}_{\varphi_1,\dots,\varphi_r}$ doesn't admit smooth boundary. We have the following theorem.

Theorem 2. *Let* $1/p_j\in\mathbb{Z}^+$, $j=1,2,\dots,r-1$ *and* $p_r\in\mathbb{R}^+$. *Then the Bergman kernel of the domain* $\widetilde{\Omega}^{1,\dots,1}_{\varphi_1,\dots,\varphi_r}$ *is*

$$K_{\widetilde{\Omega}}=C\cdot\sum_{v_1=0}^{mn}\sum_{u_1=0}^{v_1}\sum_{v_2=0}^{v_1-u_1}\sum_{u_2=0}^{v_2}\cdots\sum_{v_{r-1}=0}^{v_{r-2}-u_{r-2}}\sum_{u_{r-1}=0}^{v_{r-1}}\sum_{v_r=0}^{v_{r-1}-u_{r-1}+1}$$
$$\cdot\, b_{v_1}^{(1)}\cdots b_{v_{r-1}}^{(r-1)}\tilde{b}_{v_r}^{(r)}\prod_{l=1}^{r-1}L_l K(x)\det(I-Z\bar{Z}^t)^{-d},$$

where

$$K(x)=\frac{\partial^{u_1+\cdots+u_{r-1}}}{\partial x_1^{u_1}\cdots\partial x_{r-1}^{u_{r-1}}}x_1^{u_1}\cdots x_{r-1}^{u_{r-1}}F(x),$$

$$F(x)=\frac{\partial^{r-1}}{\partial x_1\cdots\partial x_{r-1}}\sum_{\alpha_1=0}^{1/p_1-1}\cdots\sum_{\alpha_{r-1}=0}^{1/p_{r-1}-1}\frac{\Gamma(v_r+1)(1-t)^{-(v_r+1)}}{\left(1-\sum_{\iota=1}^{r-1}\omega_\iota^{\alpha_\iota}x_\iota^{p_\iota}\right)^{1+1/p_r}},$$

$$t=x_r(1-\sum_{\iota=1}^{r-1}\omega_\iota^{\alpha_\iota}x_\iota^{p_\iota})^{-\frac{1}{p_r}},\ \omega_\iota=e^{2\pi p_\iota\sqrt{-1}}.$$

Applying Ligocki's formula (1.1), one can quickly obtain the weighted Bergman kernel of the bounded symmetric domain $\mathfrak{R}_{\mathrm{I}}(m,n)$ with respect to the weight $\varphi_l(Z)$. We consider only the fiber dimension 1, that is $r=1$.

Corollary 2. *Denote by* $K_{\widetilde{\Omega},\varphi}$ *the weighted Bergman kernel for* $\mathfrak{R}_{\mathrm{I}}(m,n)$ *with respect to the weight* $\varphi(Z)=\det(I-Z\bar{Z}^t)^{q/p}$. *We have*

$$K_{\mathfrak{R}_{\mathrm{I}},\varphi}(Z,\bar{Z})=\frac{1}{\pi^{mn+1}}\sum_{\nu=0}^{mn}b_\nu\Gamma(v+1)\det(I-Z\bar{Z}^t)^{-(m+n+\frac{q}{p})}.$$

2. Preliminaries

In this section, some elementary knowledge of Hua construction $\widetilde{\Omega}^{1,\dots,1}_{\varphi_1,\dots,\varphi_r}$ we introduced, including some properties and definitions, will be given.

Definition 1. Let Ω be a bounded domain in $\mathbb{C}^{M+N}$ with the center 0. If the holomorphic automorphism group $\mathrm{Aut}(\Omega)$ includes the following mapping:

$$\begin{cases} w_l^*=e^{\sqrt{-1}\theta_l}w_l,\ \theta_l\in\mathbb{R},\ l=1,2,\dots,M,\\ z_k^*=e^{\sqrt{-1}\theta}z_k,\ \theta\in\mathbb{R},\ k=1,2,\dots,N,\end{cases}$$

then we call Ω a semi-Reinhardt domain.

Note that a Reinhardt domain is semi-Reinhardt, and a semi-Reinhardt domain is circular. On the contrary, this is not true. Evidently, $\widetilde{\Omega}^{1,\dots,1}_{\varphi_1,\dots,\varphi_r}$ is a semi-Reinhardt domain, "Reinhardt" in w and "circular" in z.

We recall the definition of one of Appell's hypergeometric functions in several variables.

Definition 2. Denote that $(s)_m=\frac{\Gamma(s+m)}{\Gamma(s)}=s(s+1)\cdots(s+m-1)$. If $s=(s_1,\dots,s_\nu)$ and $m=(m_1,\dots,m_\nu)$ is a multi-index, then $(s)_m=\prod_{k=1}^{\nu}(s_k)_{m_k}$. The multi-variable hypergeometric function is defined as

$$F_A^{(\nu)}(\alpha,\beta,\gamma,x)=\sum_{|m|\geqslant 0}\frac{(\alpha)_{|m|}(\beta)_m}{(\gamma)_m m!}x^m,$$

where $x\in\mathbb{C}^\nu$, and $\alpha\in\mathbb{R},\ \beta=(\beta_1,\dots,\beta_\nu),\ \gamma=(\gamma_1,\dots,\gamma_\nu)$ are parameters.

The multi-variable series $F_A^{(\nu)}$ converges when $|x_1|+\cdots+|x_\nu|<1$ and diverges when $|x_1|+\cdots+|x_\nu|>1$. If $\nu=1$, $F_A^{(1)}(\alpha,\beta,\gamma,x)$ coincides with the classical Euler-Gauss hypergeometric function $F(\alpha,\beta,\gamma,x)$.

Analogous to the automorphism group of irreducible bounded symmetric domains, we get a subset of the holomorphic automorphism group $\mathrm{Aut}(\widetilde{\Omega}^{1,\dots,1}_{\varphi_1,\dots,\varphi_r})$. Denote by $\mathrm{Aut}(\widetilde{\Omega}_\varphi)$ for short.

Lemma 1. $\mathrm{Aut}(\widetilde{\Omega}_\varphi)$, *the holomorphic automorphism subgroup of* $\widetilde{\Omega}^{1,\dots,1}_{\varphi_1,\dots,\varphi_r}$, *consists of the following maps,*

$$\begin{cases} w_l^*=w_l\det(I-Z_0\bar{Z}_0^t)^{\frac{q_l}{2p_l}}\det(I-Z\bar{Z}_0^t)^{-\frac{q_l}{p_l}},\ 1\leqslant l\leqslant r,\\ Z^*=A(Z-Z_0)(I-\bar{Z}_0^tZ)^{-1}D^{-1},\end{cases}$$

where $Z_0\in\mathfrak{R}_{\mathrm{I}}(m,n)$, $\bar{A}^tA=(I-Z_0\bar{Z}_0^t)^{-1}$, $\bar{D}^tD=(I-\bar{Z}_0^tZ_0)^{-1}$.

Proof. It's known $Z^* = A(Z - Z_0)(I - \bar{Z}_0 Z)^{-1} D^{-1}$ is the biholomorphic transformation of $\mathfrak{R}_{\mathrm{I}}(m,n)$, which maps the point Z_0 onto 0 [15]. We have

$$I - Z^* \bar{Z^*}^t = (\bar{A}^t)^{-1}(I - Z\bar{Z}_0^t)^{-1}(I - Z\bar{Z}^t)(I - Z_0\bar{Z}^t)^{-1}A^{-1}$$

and

$$\det(I - Z^*\bar{Z^*}^t) = \det(I - Z_0\bar{Z}_0^t)|\det(I - Z\bar{Z}_0^t)|^{-2}\det(I - Z\bar{Z}^t),$$

$$w_l^* \bar{w_l^*}^t = w_l \bar{w}_l{}^t \det(I - Z_0\bar{Z}_0^t)^{\frac{q_l}{p_l}} |\det(I - Z\bar{Z}_0^t)|^{-\frac{2q_l}{p_l}}.$$

Therefore,

$$\frac{|w_l^*|^{2p_l}}{\det(I - Z^*\bar{Z^*}^t)^{q_l}} = \frac{|w_l|^{2p_l}}{\det(I - Z\bar{Z}^t)^{q_l}},$$

which complete the proof. □

The lemma tells us that for any point $(w, Z) \in \widetilde{\Omega}$, we can choose an element $F_Z \in \mathrm{Aut}(\widetilde{\Omega}_\varphi)$ such that $F_Z(w, Z) = (w^*, 0)$.

Lemma 2. *Let Ω be a semi-Reinhardt domain in $\mathbb{C}^{M+N}$. We can choose $P_{kl}^{(j)}(z)$ such that*

$$\left\{w^j P_{kl}^{(j)}(z)\right\} = \left\{w_1^{j_1} w_2^{j_2} \cdots w_M^{j_M} P_{kl}^{(j)}(z_1, \ldots, z_N)\right\}$$

is a complete orthonormal basis of $L_h^2(\Omega)$, where

$$j = (j_1, \ldots, j_M),\ j_1, j_2, \ldots, j_M,\ k = 0, 1, \ldots,$$
$$l = 1, 2, \ldots, m_k;\ m_k = (N + k - 1)![k!(N - 1)!]^{-1}$$

and $P_{kl}^{(j)}(z)$ is a homogeneous polynomial of degree k in $z_1, \ldots, z_N$. For any fixed k, j, the m_k polynomials $P_{k1}^{(j)}(z)$, $P_{k2}^{(j)}(z)$, $\ldots$, $P_{km_k}^{(j)}(z)$ are linear independent.

Proof. It is known $\{w_1^{j_1} w_2^{j_2} \cdots w_M^{j_M}\}$ forms a complete orthogonal basis of a bounded Reinhardt domain containing the origin and $P_{kl}(z)(k = 0, 1, \ldots; l = 1, 2, \ldots, m_k)$ is complete orthogonal for a bounded circular domain. Then it is standard to prove $w^j P_{kl}^{(j)}(z)$ is orthonormal by normalization.

Let $\Phi_{jkl}(w, z) = w^j P_{kl}^{(j)}(z)$. For any holomorphic function $f(w, z)$, $f(w, z)$ can be expanded with respect to w and z respectively, i.e.,

$$f(w, z) = \sum b_j(z) w^j = \sum b_{jkl} P_{kl}^{(j)}(z) w^j = \sum b_{jkl} \Phi_{jkl}(w, z).$$

Therefore it's sufficient to verify $b_{jkl} = \int_\Omega f(w, z)\overline{\Phi_{jkl}(w, z)} dV$ to prove the completeness. Take a normal exhausting sequence $\Omega_1, \Omega_2, \ldots$ that converges to Ω, where $\overline{\Omega}_j \subset \Omega_{j+1}$, and assume all Ω_j are semi-Reinhardt. The series $\sum b_{jkl}\Phi_{jkl}(w, z)$ converges uniformly on $\overline{\Omega}_j$. The conclusion then follows after taking a limit

$$\lim_{j\to\infty} \int_{\Omega_j} b_{jkl}\Phi_{jkl}(w, z)\overline{\Phi_{jkl}(w, z)} dV = \int_\Omega b_{jkl}\Phi_{jkl}(w, z)\overline{\Phi_{jkl}(w, z)} dV = b_{jkl}.$$ □

Lemma 3. *Let* $x_l = x_l(w, Z) = |w_l|^2 \left[\det(I - Z\bar{Z}^t)\right]^{-\frac{q_l}{p_l}}$. *Then* x_l *is invariant function under the action of* $\mathrm{Aut}(\widetilde{\Omega}_\varphi)$, *i.e.,* $x_l(w^*, Z^*) = x_l(w, Z)$, $l = 1, 2, \dots, r$.

Proof.

$$\begin{aligned}
x_l(w^*, Z^*) &= |w_l^*|^2[\det(I - Z^*\bar{Z^*}^t)]^{-\frac{q_l}{p_l}} \\
&= |w_l|^2 \left[\det(I - Z_0\bar{Z}_0^t)\right]^{\frac{q_l}{p_l}} |\det(I - Z\bar{Z}_0^t)|^{-\frac{2q_l}{p_l}} \\
&\quad \cdot \left[\det(I - Z_0\bar{Z}_0^t)|\det(I - Z\bar{Z}_0^t)|^{-2}\det(I - Z\bar{Z}^t)\right]^{-\frac{q_l}{p_l}} \\
&= |w_l|^2 \left[\det(I - Z\bar{Z}^t)\right]^{-\frac{q_l}{p_l}} = x_l(w, Z).
\end{aligned}$$

□

We give the following lemma we used for simplifying the proof of theorems. One can find it in the text book of several complex variables.

Lemma 4. *Let* $j_1, j_2, \dots, j_r$ *be nonnegative real numbers. Integration in polar coordinates in each variable reveals that*

$$\int_{\sum\limits_{j=1}^{r} |w_j|^{2p_j} < R} \prod_{l=1}^{r} |w_l|^{2j_l} dV_w = \frac{\pi^r}{\prod_{l=1}^r p_l} \frac{\prod\limits_{l=1}^{r} \Gamma\left(\frac{j_l+1}{p_l}\right)}{\Gamma\left(1 + \sum\limits_{l=1}^{r} \frac{j_l+1}{p_l}\right)} R^{\sum\limits_{l=1}^{r} \frac{j_l+1}{p_l}}, \tag{2.1}$$

where $dV_w = \left(\frac{\sqrt{-1}}{2}\right)^n dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_r \wedge d\bar{w}_r$.

Lemma 5. *Let* $Z \in \mathfrak{R}_{\mathrm{I}}(m, n)$, $\lambda > -1$ *and* $J_{m,n} = \int_{\mathfrak{R}_{\mathrm{I}}(m,n)} \det(I - Z\bar{Z}^t)^\lambda dV$. *Then*

$$J_{m,n} = \pi^{mn} \frac{\prod_{k_1=1}^{n} \Gamma(\lambda + k_1) \prod_{k_2=1}^{m} \Gamma(\lambda + k_2)}{\prod_{k_3=1}^{m+n} \Gamma(\lambda + k_3)}. \tag{2.2}$$

This is an important integral formula in Hua's book [13].

Lemma 6. *Suppose that* $P(x)$ *is a polynomial of degree* n *in* x,

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1.$$

Then for any $\alpha \in \mathbb{R} \setminus \{0\}$, *there are constants* b_j *such that*

$$P(x) = \sum_{j=0}^{n} b_j \frac{\Gamma(\alpha x + j + 1)}{\Gamma(\alpha x + 1)}, \tag{2.3}$$

where $b_0 = P\left(\frac{-1}{\alpha}\right)$ *and the other coefficients* b_j *can be evaluated by*

$$\Gamma(\nu + 1) b_\nu = \sum_{k=0}^{\nu} (-1)^k \binom{\nu}{k} P\left(\frac{-k-1}{\alpha}\right), \ \nu = 1, \dots, n.$$

Proof. See [22] for direct computations. □

To make the paper more readable, now we collect the following several identities used in the text omitting the proofs. One can check them by induction.

Lemma 7. *For any $s, t, r \in \mathbb{R}$ and $k,\ l \in \mathbb{Z}^+$, the following formulas are true*

$$\sum_{k=0}^{l+1} \frac{\Gamma(k+s)\Gamma(l+1-k+t)}{\Gamma(k+1)\Gamma(l+2-k)} = \frac{s+t+l}{l+1} \sum_{k=0}^{l} \frac{\Gamma(k+s)\Gamma(l-k+t)}{\Gamma(k+1)\Gamma(l+1-k)}, \tag{2.4}$$

$$\frac{\Gamma(s+t+k)}{\Gamma(s+t)} = \frac{\Gamma(k+1)}{\Gamma(s)\Gamma(t)} \sum_{p=0}^{k} \frac{\Gamma(p+s)\Gamma(t+k-p)}{\Gamma(p+1)\Gamma(k+1-p)}. \tag{2.5}$$

Next we need to know the sum of a certain Taylor series.

Lemma 8. *For $\|x\| < 1$ and $s > 0$, we have*

$$\sum_{\alpha \geq 0} \frac{\Gamma(|\alpha| + s)}{\Gamma(s) \prod \Gamma(\alpha_j + 1)} x^{2\alpha} = \frac{1}{(1 - \|x\|^2)^s}. \tag{2.6}$$

We will use the following converging power series in the proof of the main theorem. See [12] for a proof.

Lemma 9. *For $s \in \mathbb{R}^+$ and $\frac{1}{p_\iota} \in \mathbb{Z}^+$, $1 \leqslant \iota \leqslant r-1$. Let*

$$h_s(x_1, \ldots, x_{r-1}) = \sum_{j \geqslant 0} \frac{\Gamma\left(s + \sum\limits_{\iota=1}^{r-1} \frac{j_\iota + 1}{p_\iota}\right)}{\prod_{\iota=1}^{r-1} \Gamma\left(\frac{j_\iota+1}{p_\iota}\right)} x_1^{j_1} x_2^{j_2} \cdots x_{r-1}^{j_{r-1}}.$$

If $\sum_{\iota=1}^{r-1} |x_\iota|^{p_\iota} < 1$, then

$$h_s(x_1, \ldots, x_{r-1}) = \frac{\partial^{r-1}}{\partial x_1 \cdots \partial x_{r-1}} \left(\sum_{\alpha_1=0}^{\frac{1}{p_1}-1} \cdots \sum_{\alpha_{r-1}=0}^{\frac{1}{p_{r-1}}-1} \frac{\Gamma(s)}{\left(1 - \sum_{\iota=1}^{r-1} \omega_\iota^{\alpha_\iota} x_\iota^{p_\iota}\right)^s} \right),$$

where $\omega_\iota = e^{2\sqrt{-1}\pi p_\iota}$, $x_\iota^{p_\iota} = |x_\iota|^{p_\iota} e^{\sqrt{-1}\phi_\iota p_\iota}$, $-\pi \leqslant \phi_\iota = \arg x_\iota < \pi$, $1 \leqslant \iota \leqslant r-1$.

3. Proofs

Proof of Theorem 1. For any point $(w, Z_0) \in \widetilde{\Omega}^{1,\ldots,1}_{\varphi_1,\ldots,\varphi_r}$, it follows from Lemma 1, there exists an element $F \in \mathrm{Aut}(\widetilde{\Omega}_\varphi)$ such that $F(w, Z_0) = (w^*, 0)$. According to the rule of Bergman kernels transformation, we have

$$K_{\widetilde{\Omega}}(w, Z; \bar{w}, \bar{Z}) = K_{\widetilde{\Omega}}(w^*, 0; \bar{w}^*, 0) \left|\det J_F\right|^2_{Z_0 = Z}, \tag{3.1}$$

where

$$J_F|_{Z_0=Z} = \left(\frac{\partial(w^*, Z^*)}{\partial(w, Z)} \right)_{Z_0 = Z} = \begin{pmatrix} \frac{\partial w^*}{\partial w} & 0 \\ * & \frac{\partial Z^*}{\partial Z} \end{pmatrix}_{Z_0 = Z}.$$

Therefore,

$$
\begin{aligned}
|\det J_F|^2_{Z_0=Z} &= \left|\det\left(\frac{\partial w^*}{\partial w}\right)\det\left(\frac{\partial Z^*}{\partial Z}\right)\right|^2_{Z_0=Z} \\
&= \det(I - Z\bar{Z}^t)^{-(m+n+\sum\limits_{l=1}^{r}\frac{q_l}{p_l})}.
\end{aligned}
\tag{3.2}
$$

Next we need only compute $K_{\widetilde{\Omega}}(w^*, 0; \bar{w}^*, 0)$. Denote w by w^* for convenience. Now we take the complete orthonormal bases $\left\{w^j P^{(j)}_{kl}(Z)\right\}$ of $L^2_h(\widetilde{\Omega}^{1,\ldots,1}_{\varphi_1,\ldots,\varphi_r})$ due to Lemma 2 since $\widetilde{\Omega}^{1,\ldots,1}_{\varphi_1,\ldots,\varphi_r}$ is a semi-Reinhardt domain, where

$$
\begin{aligned}
&j = (j_1, \ldots, j_M),\ j_1, j_2, \ldots, j_M,\ k = 0, 1, \ldots, \\
&l = 1, 2, \ldots, m_k;\ m_k = (N + k - 1)![k!(N-1)!]^{-1}.
\end{aligned}
$$

Hence

$$
K_{\widetilde{\Omega}}(w, 0; \bar{w}, 0) = \sum_{j,k,l}\left|w^j P^{(j)}_{kl}(0)\right|^2 = \sum_{|j|\geqslant 0}\left|w^j P^{(j)}_{01}(0)\right|^2 = \sum_{|j|\geqslant 0}|a_j|^2\,|w|^{2j},
$$

where $a_j = P^{(j)}_{01}(0)$ is a constant.

We are going to determine the constant coefficient a_j. Notice that $a_j w^j = w^j P^{(j)}_{01} = \Phi_{j01}$ is an element in the set of complete orthonormal basis, then

$$
\int_{\widetilde{\Omega}}\left|a_j w^j\right|^2 dV_w dV_z = 1,
$$

which implies

$$
\begin{aligned}
|a_j|^{-2} &= \int_{\widetilde{\Omega}} |w_1|^{2j_1}\cdots|w_r|^{2j_r}\, dV_w dV_z \\
&= \int_{\mathfrak{R}_{\mathrm{I}}(m,n)} \det(I - Z\bar{Z}^t)^{\lambda_1} dV_z/\pi^{mn} C\cdot T(j_1, \ldots, j_r)
\end{aligned}
$$

according to Lemma 4, where

$$
C = \frac{\prod_{l=1}^r p_l}{\pi^{r+mn}},\ T(j_1, \ldots, j_r) = \frac{\Gamma\left(1 + \sum\limits_{l=1}^{r}\frac{j_l+1}{p_l}\right)}{\prod_{l=1}^r \Gamma\left(\frac{j_l+1}{p_l}\right)} \quad \text{and} \quad \lambda_1 = \sum_{l=1}^{r}\frac{q_l}{p_l}(j_l+1).
$$

By Lemma 5, we have

$$
|a_j|^2 = C\cdot T(j_1, \ldots, j_r)\cdot\frac{\prod_{k=1}^{m+n}\Gamma(\lambda_1 + k)}{\prod_{k=1}^{m}\Gamma(\lambda_1 + k)\prod_{k=1}^{n}\Gamma(\lambda_1 + k)}.
$$

Observe that

$$\frac{\prod_{k=1}^{m+n}\Gamma(\lambda_1+k)}{\prod_{k=1}^{m}\Gamma(\lambda_1+k)\prod_{k=1}^{n}\Gamma(\lambda_1+k)} = \frac{\prod_{k=n+1}^{m+n}\Gamma(\lambda_1+k)}{\prod_{k=1}^{m}\Gamma(\lambda_1+k)} = \prod_{k=1}^{m}\frac{\Gamma(\lambda_1+n+k)}{\Gamma(\lambda_1+k)} \tag{3.3}$$

is a polynomial of degree mn in λ_1, then by Lemma 6 the polynomial (3.3) can be rewritten as

$$f_1(\lambda_1) = \sum_{v_1=0}^{mn} b_{v_1}^{(1)} \frac{\Gamma\left(\frac{p_1}{q_1}\lambda_1+v_1+1\right)}{\Gamma\left(\frac{p_1}{q_1}\lambda_1+1\right)}, \tag{3.4}$$

where $b_0^{(1)} = f_1\left(-\frac{q_1}{p_1}\right)$ and

$$\Gamma(\nu+1)b_\nu^{(1)} = \sum_{k=0}^{\nu}(-1)^k\binom{\nu}{k} f_1\left(-\frac{(k+1)q_1}{p_1}\right),\ \nu=1,2,\ldots,mn.$$

Substitute $\sum\limits_{l=1}^{r}\frac{q_l}{p_l}(j_l+1)$ for λ_1 in the function $f_1(\lambda_1)$, we have

$$f_1(\lambda_1) = \sum_{v_1=0}^{mn} b_{v_1}^{(1)} \frac{\Gamma((j_1+1)+\lambda_2+v_1+1)}{\Gamma((j_1+1)+\lambda_2+1)},$$

where $\lambda_2 = \frac{p_1}{q_1}\sum\limits_{l=2}^{r}\frac{q_l}{p_l}(j_l+1)$.

Let

$$f_\kappa(\lambda_\kappa) = \frac{\Gamma(\lambda_\kappa+1+v_{\kappa-1}-u_{\kappa-1})}{\Gamma(\lambda_\kappa+1)},\ 2\leqslant\kappa\leqslant r, \tag{3.5}$$

$$L_l = \frac{\Gamma(v_l+1)}{\Gamma(u_l+1)\Gamma(v_l+1-u_l)},\ B_l = \frac{\Gamma(j_l+1+u_l)}{\Gamma(j_l+1)},\ 1\leqslant l\leqslant r, \tag{3.6}$$

applying Lemma 7, we can separate j_1+1 form λ_2. That is

$$f_1(\lambda_1) = \sum_{v_1=0}^{mn}\sum_{u_1=0}^{v_1} b_{v_1}^{(1)} L_1 B_1 f_2(\lambda_2),$$

where $f_2(\lambda_2)$ is a polynomial of degree v_1-u_1 in λ_2.

Again by Lemma 6 and Lemma 7, we can deal with $f_2(\lambda_2)$ using the same method and we have

$$f_2(\lambda_2) = \sum_{v_2=0}^{v_1-u_1} b_{v_2}^{(2)} \frac{\Gamma\left(\frac{p_2q_1}{q_2p_1}\lambda_2+v_2+1\right)}{\Gamma\left(\frac{p_2q_1}{q_2p_1}\lambda_2+1\right)}, \tag{3.7}$$

where $b_0^{(2)} = f_2\left(-\frac{q_2p_1}{p_2q_1}\right)$ and

$$\Gamma(\nu+1)b_\nu^{(2)} = \sum_{k=0}^{\nu}(-1)^k\binom{\nu}{k}f_2\left(-\frac{(k+1)q_2p_1}{p_2q_1}\right),\ \nu = 1,2,\ldots,v_1-u_1.$$

Then

$$f_1(\lambda_1) = \sum_{v_1=0}^{mn}\sum_{u_1=0}^{v_1}\sum_{v_2=0}^{v_1-u_1}\sum_{u_2=0}^{v_2} b_{v_1}^{(1)}b_{v_2}^{(2)}L_1L_2B_1B_2f_3(\lambda_3),$$

where $\lambda_3 = \frac{p_2}{q_2}\sum\limits_{k=3}^{r}\frac{q_k}{p_k}(j_k+1)$.

If we continue the above steps, we obtain

$$f_1(\lambda_1) = \sum_{v_1=0}^{mn}\sum_{u_1=0}^{v_1}\cdots\sum_{v_{r-1}=0}^{v_{r-2}-u_{r-2}}\sum_{u_{r-1}=0}^{v_{r-1}} b_{v_1}^{(1)}\cdots b_{v_{r-1}}^{(r-1)}\prod_{l=1}^{r-1}L_lB_lf_r(\lambda_r), \tag{3.8}$$

where

$$f_l(\lambda_l) = \sum_{v_l=0}^{v_{l-1}-u_{l-1}} b_{v_l}^{(l)}\frac{\Gamma\left(\frac{p_lq_{l-1}}{q_lp_{l-1}}\lambda_l+v_l+1\right)}{\Gamma\left(\frac{p_lq_{l-1}}{q_lp_{l-1}}\lambda_l+1\right)},$$

$$\lambda_l = \frac{p_{l-1}}{q_{l-1}}\sum_{k=l}^{r}\frac{q_k(j_k+1)}{p_k},\ 1\leqslant l\leqslant r$$

and

$$\Gamma(\nu+1)b_\nu^{(l)} = \sum_{k=0}^{\nu}(-1)^k\binom{\nu}{k}f_l\left(-\frac{(k+1)q_lp_{l-1}}{p_lq_{l-1}}\right),\ \nu = 1,\ldots,v_{l-1}-u_{l-1}.$$

Now we consider the function $f_r(\lambda_r)$. Since $f_r(\lambda_r)$ is a polynomial of degree $v_{r-1}-u_{r-1}$, it can be formulated to

$$f_r(\lambda_r) = \sum_{v_r=0}^{v_{r-1}-u_{r-1}}\sum_{u_r=0}^{v_r} b_{v_r}^{(r)}\frac{\Gamma(v_r+1)}{\Gamma(u_r+1)}\frac{\Gamma(j_r+1+u_r)}{\Gamma(j_r+1)}.$$

Let $x_l = |w_l|^2,\ l = 1,2,\ldots,r$. Then the Bergman kernel

$$K_{\widetilde{\Omega}}(w,0;\bar{w},0) = \sum_{|j|\geqslant 0}|a_j|^2|w|^{2j} = \sum_{|j|\geqslant 0}C\cdot T(j_1,\ldots,j_r)f_1(\lambda_1)x_1^{j_1}\cdots x_r^{j_r}.$$

Note that

$$f_1(\lambda_1)\prod_{l=1}^{r}x_l^{j_l} = \sum_{v_1=0}^{mn}\sum_{u_1=0}^{v_1}\cdots\sum_{v_r=0}^{v_{r-1}-u_{r-1}}\sum_{u_r=0}^{v_r} b_{v_1}^{(1)}\cdots b_{v_r}^{(r)}\Gamma(v_r+1-u_r)\prod_{l=1}^{r}L_lB_lx_l^{j_l}$$

and

$$B_lx_l^{j_l} = \frac{\Gamma(j_l+1+u_l)}{\Gamma(j_l+1)}x_l^{j_l} = \frac{\partial^{u_l}}{\partial x_l^{u_l}}x_l^{j_l+u_l},$$

we have

$$K_{\widetilde{\Omega}}(w,0;\bar{w},0) = C \cdot \sum_{v_1=0}^{mn} \sum_{u_1=0}^{v_1} \sum_{v_2=0}^{v_1-u_1} \sum_{u_2=0}^{v_2} \cdots \sum_{v_r=0}^{v_{r-1}-u_{r-1}} \sum_{u_r=0}^{v_r} b_{v_1}^{(1)} \cdots b_{v_r}^{(r)}$$
$$\cdot \Gamma(v_r+1-u_r) \prod_{l=1}^{r} L_l \frac{\partial^{u_1+\cdots+u_r}}{\partial x_1^{u_1} \cdots \partial x_r^{u_r}} x_1^{u_1} \cdots x_r^{u_r} H(x),$$

where $H(x) = \sum_{|j|\geqslant 0} T(j_1,\ldots,j_r) x_1^{j_1} \cdots x_r^{j_r}$.

Replace w by w^* and this completes the proof of Theorem 1 since we have formula (3.1) already. □

Proof of Corollary 1. Let $p_1,\ldots,p_r \in \mathbb{Z}^+$. Recall that

$$T(j_1,\ldots,j_r) = \frac{\Gamma\left(1+\sum_{l=1}^{r} \frac{j_l+1}{p_l}\right)}{\prod_{l=1}^{r} \Gamma\left(\frac{j_l+1}{p_l}\right)},$$

then

$$H(x) = \sum_{j_1=0}^{\infty} \cdots \sum_{j_r=0}^{\infty} \frac{\Gamma\left(1+\sum_{l=1}^{r} \frac{j_l+1}{p_l}\right)}{\prod_{l=1}^{r} \Gamma\left(\frac{j_l+1}{p_l}\right)} x_1^{j_1} \cdots x_r^{j_r}. \tag{3.9}$$

Let $j_l = p_l k_l^* + k_l$, $k_l \in \mathbb{Z}$ and $0 \leqslant k_l \leqslant p_l - 1$, the formula (3.9) is

$$\begin{aligned} &H(x) \\ &= \sum_{k_1=0}^{p_1-1} \cdots \sum_{k_r=0}^{p_r-1} \sum_{k_1^*=0}^{\infty} \cdots \sum_{k_r^*=0}^{\infty} \frac{\Gamma\left(1+\sum_{l=1}^{r} k_j^* + \sum_{l=1}^{r} \frac{k_l+1}{p_l}\right)}{\prod_{l=1}^{r} \Gamma\left(k_l^* + \frac{k_l+1}{p_l}\right)} x_1^{p_1k_1^*+k_1} \cdots x_r^{p_rk_r^*+k_r} \\ &= \sum_{k_1=0}^{p_1-1} \cdots \sum_{k_r=0}^{p_r-1} x_1^{k_1} \cdots x_r^{k_r} T(k_1,\ldots,k_r) F_A^{(r)}\left(1+\sum_{l=1}^{r} \frac{k_l+1}{p_l}, \mathbf{1}, \frac{\mathbf{k}+\mathbf{1}}{\mathbf{p}}, \mathbf{x}^{\mathbf{p}}\right). \end{aligned}$$

Here the notations $\mathbf{1} = (1,\ldots,1)$, $\frac{\mathbf{k}+\mathbf{1}}{\mathbf{p}} = (\frac{k_1+1}{p_1},\ldots,\frac{k_r+1}{p_r})$, and $F_A^{(r)}$ is Appell's hypergeometric function. This completes the proof of Corollary 1. □

Proof of Theorem 2. Let $\frac{1}{p_1},\ldots,\frac{1}{p_{r-1}} \in \mathbb{Z}^+$ and $p_r \in \mathbb{R}^+$. Start from formula (3.8), the Bergman kernel

$$\begin{aligned} &K_{\widetilde{\Omega}}(w,0;\bar{w},0) \\ &= \sum_{|j|\geqslant 0} C \cdot T(j_1,\ldots,j_r) f_1(\lambda_1) x_1^{j_1} \cdots x_r^{j_r} \\ &= C \sum_{v_1=0}^{mn} \sum_{u_1=0}^{v_1} \cdots \sum_{v_{r-1}=0}^{v_{r-2}-u_{r-2}} \sum_{u_{r-1}=0}^{v_{r-1}} b_{v_1}^{(1)} \cdots b_{v_{r-1}}^{(r-1)} \prod_{l=1}^{r-1} L_l \frac{\partial^{u_1+\cdots+u_{r-1}}}{\partial x_1^{u_1} \cdots \partial x_{r-1}^{u_{r-1}}} \end{aligned}$$

$$\cdot\, x_1^{u_1} \cdots x_{r-1}^{u_{r-1}} \sum_{j_r=0}^{\infty} \frac{f_r(\lambda_r) x_r^{j_r}}{\Gamma\left(\frac{j_r+1}{p_r}\right)} h_s\left(x_1, \dots, x_{r-1}\right).$$

Here we used Lemma 9,

$$h_s(x_1, \dots, x_{r-1}) = \frac{\partial^{r-1}}{\partial x_1 \cdots \partial x_{r-1}} \left(\sum_{\alpha_1=0}^{\frac{1}{p_1}-1} \cdots \sum_{\alpha_{r-1}=0}^{\frac{1}{p_{r-1}}-1} \frac{\Gamma(s)}{\left(1 - \sum\limits_{l=1}^{r-1} \omega_l^{\alpha_l} x_l^{p_l}\right)^s} \right),$$

where $s = 1 + \frac{j_r+1}{p_l}$, $\omega_l = e^{2\sqrt{-1}\pi p_l}$, $x_l^{p_l} = |x_l|^{p_l} e^{\sqrt{-1}\phi_l p_l}$, $-\pi \leqslant \phi_l = \arg x_l < \pi$, $1 \leqslant l \leqslant r-1$.

Now we concentrate on the only infinite series, with respect to the index j_r, hiding in the above Bergman kernel. Denote it by $\hat{f}_{j_r}$, that is

$$\hat{f}_{j_r}(t) = \sum_{j_r=0}^{\infty} \frac{\Gamma\left(1 + \frac{j_r+1}{p_r}\right)}{\Gamma\left(\frac{j_r+1}{p_r}\right)} f_r(\lambda_r)\, t^{j_r} = \sum_{j_r=0}^{\infty} \frac{j_r+1}{p_r} f_r(\lambda_r)\, t^{j_r},$$

where $t = x_r \left(1 - \sum_{l=1}^{r-1} \omega_l^{\alpha_l} x_l^{p_l}\right)^{-\frac{1}{p_r}}$. We define $f(j_r) \triangleq \frac{j_r+1}{p_r} f_r(\lambda_r)$. Note that $f_r(\lambda_r)$ is a polynomial of degree $v_{r-1} - u_{r-1}$ in j_r, then in terms of Lemma 6 we can write

$$f(j_r) = \sum_{v_r=0}^{v_{r-1}-u_{r-1}+1} \tilde{b}_{v_r}^{(r)} \frac{\Gamma(j_r + v_r + 1)}{\Gamma(j_r+1)},$$

where $\tilde{b}_0^{(r)} = f(-1) = 0$,

$$\Gamma(\nu+1)\tilde{b}_\nu^{(r)} = \sum_{k=0}^{\nu} (-1)^k \binom{\nu}{k} f\left(-k-1\right),\ \nu = 1, \dots, v_{r-1} - u_{r-1} + 1.$$

From Lemma 8,

$$\sum_{j_r=0}^{\infty} \frac{\Gamma(j_r + v_r + 1)}{\Gamma(j_r+1)} t^{j_r} = \frac{\Gamma(v_r+1)}{(1-t)^{v_r+1}},$$

we have

$$\hat{f}_{j_r}(t) = \sum_{v_r=0}^{v_{r-1}-u_{r-1}+1} \tilde{b}_{v_r}^{(r)} \frac{\Gamma(v_r+1)}{(1-t)^{v_r+1}}.$$

So the Bergman kernel in close form is

$$K_{\widetilde{\Omega}}(w, 0; \bar{w}, 0) = C \sum_{v_1=0}^{mn} \sum_{u_1=0}^{v_1} \cdots \sum_{v_{r-1}=0}^{v_{r-2}-u_{r-2}} \sum_{u_{r-1}=0}^{v_{r-1}} \sum_{v_r=0}^{v_{r-1}-u_{r-1}+1} b_{v_1}^{(1)} \cdots b_{v_{r-1}}^{(r-1)} \tilde{b}_{v_r}^{(r)}$$

$$\prod_{l=1}^{r-1} L_l \frac{\partial^{u_1+\cdots+u_{r-1}}}{\partial x_1^{u_1} \cdots \partial x_{r-1}^{u_{r-1}}} x_1^{u_1} \cdots x_{r-1}^{u_{r-1}} \frac{\partial^{r-1}}{\partial x_1 \cdots \partial x_{r-1}} \sum_{\alpha_1=0}^{1/p_1-1} \cdots$$

$$\cdots \sum_{\alpha_{r-1}=0}^{1/p_{r-1}-1} \frac{\Gamma(v_r+1)(1-t)^{-(v_r+1)}}{\left(1-\sum_{l=1}^{r-1}\omega_l^{\alpha_l}x_l^{p_l}\right)^{1+1/p_r}},$$

where the parameter $t = x_r(1-\sum_{l=1}^{r-1}\omega_l^{\alpha_l}x_l^{p_l})^{-\frac{1}{p_r}}$, $\omega_l = e^{2\pi p_l\sqrt{-1}}$. This completes the proof of Theorem 2. □

Proof of Corollary 2. We consider only the case $r = 1$ for simplicity. In this case, the domain is

$$\Omega_\varphi^1 = \left\{(w, Z) \in \mathbb{C} \times \mathfrak{R}_{\mathrm{I}}(m,n) : \ |w|^2 < \det(I - Z\bar{Z}^t)^{\frac{q}{p}},\ p, q \in \mathbb{R}^+\right\}.$$

We know from Theorem 2 that the Bergman kernel is

$$K_{\Omega_\varphi^1}(w, Z; \bar{w}, \bar{Z}) = \frac{1}{\pi^{mn+1}} \sum_{\nu=0}^{mn} b_\nu \Gamma(v+1)(1-t)^{-(v+1)} \det(I - Z\bar{Z}^t)^{-(m+n+\frac{q}{p})},$$

where $t = x^* = |w|^2 \det(I - Z\bar{Z}^t)^{-q/p}$. The constants $b_v(\nu = 0, 1, 2, \ldots, mn)$ are determined independently by

$$\Gamma(\nu+1)b_\nu = \sum_{k=0}^{\nu}(-1)^k \binom{\nu}{k} f\left(-\frac{(k+1)q}{p}\right),$$

where f is Hua polynomial (3.3) of degree mn.

Applying formula (1.1) and let $w = 0$, then we get the weight Bergman kernel for Cartan domain of the first type $\mathfrak{R}_{\mathrm{I}}(m,n)$ with respect to the weight $\varphi(Z) = \det(I - Z\bar{Z}^t)^{q/p}$. That is

$$K_{\mathfrak{R}_{\mathrm{I}},\varphi}(Z, \bar{Z}) = \frac{1}{\pi^{mn+1}} \sum_{\nu=0}^{mn} b_\nu \Gamma(v+1) \det(I - Z\bar{Z}^t)^{-(m+n+\frac{q}{p})}.$$

□

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