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FORMULAS DEDUCIBLE FROM A GENERALIZATION OF GOTTLIEB POLYNOMIALS IN SEVERAL VARIABLES

Junesang Choi

Abstract. Gottlieb polynomials were introduced and investigated in 1938, and then have been cited in several articles. Very recently Khan and Akhlaq introduced and investigated Gottlieb polynomials in two and three variables to give their generating functions. Subsequently, Khan and Asif investigated the generating functions for the q-analogue of Gottlieb polynomials. In this sequel, by modifying Khan and Akhlaq's method, Choi presented a generalization of the Gottlieb polynomials in m variables to present two generating functions of the generalized Gottlieb polynomials $\varphi_n^m(\cdot)$. Here, we show that many formulas regarding the Gottlieb polynomials in m variables and their reducible cases can easily be obtained by using one of two generating functions for Choi's generalization of the Gottlieb polynomials in m variables expressed in terms of well-developed Lauricella series $F_D^{(m)}[\cdot]$.

1. Introduction and Preliminaries

Generating functions play an important role in the investigation of various useful properties of the sequences which they generate. They are used in finding certain properties and formulas for numbers and polynomials in a wide variety of research subjects, indeed, in modern combinatorics. For a systematic introduction to, and several interesting (and useful) applications of the various methods of obtaining linear, bilinear, bilateral or mixed multilateral generating functions for a fairly wide variety of sequences of special functions (and polynomials) in one, two and more variables, among much abundant literature, we refer to the

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extensive work by Srivastava and Manocha [13]. While concerning some orthogonal polynomials on a finite or enumerable set of points, Gottlieb [7] developed the following interesting polynomials (see also [3]; [8]; [9]; [11, p. 303]; [13, pp. 185–186]):

$$
\begin{aligned}
\varphi_n(x;\lambda) &:= e^{-n\lambda} \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} \left(1-e^{\lambda}\right)^k \\
&= e^{-n\lambda}\, {}_2F_1\left(-n,\, -x\,;\, 1\,; 1-e^{\lambda}\right),
\end{aligned} \tag{1.1}
$$

where ${}_2F_1$ denotes Gauss's hypergeometric series whose natural generalization of an arbitrary number of p numerator and q denominator parameters ($p,\, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{N}$ the set of positive integers) is called and denoted by the generalized hypergeometric series ${}_pF_q$ defined by

$$
\begin{aligned}
{}_pF_q\left[\begin{matrix}\alpha_1,\, \ldots,\, \alpha_p\,; \\ \beta_1,\, \ldots,\, \beta_q\,;\end{matrix}\; z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\
&= {}_pF_q(\alpha_1,\, \ldots,\, \alpha_p;\, \beta_1,\, \ldots,\, \beta_q;\, z).
\end{aligned} \tag{1.2}
$$

Here $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by

$$
\begin{aligned}
(\lambda)_n &:= \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases} \\
&= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-)
\end{aligned} \tag{1.3}
$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers and $\Gamma(\lambda)$ is the familiar Gamma function.

Gottlieb [7] presented many interesting identities for his polynomials $\varphi_n(x;\lambda)$, which is denoted by $l_n(x)$ in [7], including the following two generating functions (see also [8]; [9]; [11, p. 303]; [13, pp. 185–186]):

$$
\sum_{n=0}^{\infty} \varphi_n(x;\lambda)\, t^n = (1-t)^x \left(1-t\, e^{-\lambda}\right)^{-x-1} \quad (|t|<1); \tag{1.4}
$$

$$
\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \varphi_n(x;\lambda)\, t^n = \left(1-t\, e^{-\lambda}\right)^{-\mu} {}_2F_1\left[\begin{matrix}\mu,\, -x\,; \\ 1\,;\end{matrix}\; \frac{\left(1-e^{-\lambda}\right) t}{1-t\, e^{-\lambda}}\right]. \tag{1.5}
$$

Recently Khan and Akhlaq [8] introduced and investigated Gottlieb polynomials in two and three variables to give their generating functions. Subsequently, Khan and Asif [9] investigated the generating functions for the q-analogue of Gottlieb polynomials (see also [4, 5]). In this sequel, by modifying Khan and Akhlaq's method [8], Choi presented a generalization of the Gottlieb polynomials in m variables to present two generating functions of the generalized Gottlieb polynomials $\varphi_n^m(\cdot)$. Here, as noted in [3], we show that many formulas regarding the Gottlieb polynomials in m variables and their reducible cases can easily be obtained by using one of Choi's generating functions for a generalization of the Gottlieb polynomials in m variables expressed in terms of well-developed Lauricella series $F_D^{(m)}[\cdot]$.

2. Generalized Gottlieb polynomials and their generating functions

Here, we just recall the definition of a several variable analogue of the Gottlieb polynomials $\varphi_n(x;\lambda)$ and one of their generating functions in [3].

Definition. An extension of the Gottlieb polynomials $\varphi_n(x;\lambda)$ in m variables is defined by

$$\begin{aligned}
&\varphi_n^m(x_1,\, x_2,\, \ldots,\, x_m;\lambda_1,\, \lambda_2,\, \ldots,\, \lambda_m)\\
&= \exp(-n\,\sigma_m) \sum_{r_1=0}^{n} \sum_{r_2=0}^{n-r_1} \sum_{r_3=0}^{n-r_1-r_2} \cdots \sum_{r_m=0}^{n-r_1-r_2-\cdots-r_{m-1}}\\
&\cdot \frac{(-n)_{\delta_m} \cdot \prod_{j=1}^m (-x_j)_{r_j} \cdot \prod_{j=1}^m \left(1-e^{\lambda_j}\right)^{r_j}}{\prod_{j=1}^m r_j! \cdot \delta_m!} \qquad (n,\, m \in \mathbb{N}),
\end{aligned} \tag{2.1}$$

where, for convenience,

$$\sigma_m := \sum_{j=1}^m \lambda_j \quad \text{and} \quad \delta_m := \sum_{j=1}^m r_j. \tag{2.2}$$

It is noted that the special case $m=1$ of (2.1) reduces immediately to the second one of the Gottlieb polynomials $\varphi_n(x;\lambda)$ in (1.1) and the cases of (2.1) when $m=2$ and $m=3$ correspond with those in [8, 9].

The following generating function for $\varphi_n^m(x_1, x_2, \ldots, x_m; \lambda_1, \lambda_2, \ldots, \lambda_m)$ holds true:

(2.3)
$$\sum_{n=0}^{\infty} (\mu)_n \, \varphi_n^m \left(x_1, x_2, \ldots, x_m; \lambda_1, \lambda_2, \ldots, \lambda_m\right) \frac{t^n}{n!} = \left(1 - t\, e^{-\sigma_m}\right)^{-\mu}$$
$$\cdot F_D^{(m)} \left[\mu, -x_1, \ldots, -x_m; 1; \frac{t\left(e^{\lambda_1} - 1\right) e^{-\sigma_m}}{1 - t\, e^{-\sigma_m}}, \ldots, \frac{t\left(e^{\lambda_m} - 1\right) e^{-\sigma_m}}{1 - t\, e^{-\sigma_m}}\right],$$

where $F_D^{(m)}[\cdot]$ denotes one of the Lauricella series in m variables (see [12, p. 33, Eq. (4)]) defined by

$$\begin{aligned} & F_D^{(m)} \left[a, b_1, \ldots, b_m; c; x_1, \ldots, x_m\right] \\ & = \sum_{r_1=0, \ldots, r_m=0}^{\infty} \frac{(a)_{\delta_m} (b_1)_{r_1} \cdots (b_m)_{r_m}}{(c)_{\delta_m}} \frac{x_1^{r_1}}{r_1!} \cdots \frac{x_m^{r_m}}{r_m!} \\ & \qquad (\max\{|x_1|, \ldots, |x_m|\} < 1), \end{aligned} \tag{2.4}$$

and σ_m, δ_m are given in (2.2).

3. Generalized generating functions $\varphi_n^m(x_1, x_2, \ldots, x_m; \lambda_1, \lambda_2, \ldots, \lambda_m)$

In view of (2.3), we begin by recalling two known integral representations for the m variables Lauricella series $F_D^{(m)}[\cdot]$ among its several other properties (see [2, pp. 114–120]). A definite integral expression of $F_D^{(m)}[\cdot]$ is given (see [2, p. 115, Eq. (7)]):

$$\begin{aligned} & F_D^{(m)} \left[a, b_1, \ldots, b_m; c; x_1, \ldots, x_m\right] \\ & = \frac{\Gamma(c)}{\Gamma(b_1) \cdots \Gamma(b_m)\, \Gamma(c - b_1 - \cdots - b_m)} \\ & \quad \cdot \int \cdots \int u_1^{b_1 - 1} \cdots u_m^{b_m - 1} \left(1 - u_1 - \cdots - u_m\right)^{c - b_1 - \cdots - b_m - 1} \\ & \qquad \cdot \left(1 - u_1 x_1 - \cdots - u_m x_m\right)^{-a} du_1 \ldots du_m \end{aligned} \tag{3.1}$$

$$(u_1 \geqq 0, \ldots, u_m \geqq 0, \ u_1 + u_2 + \cdots + u_m \leqq 1).$$

The function $F_D^{(m)}[\cdot]$ can also be represented by a simple integral (see [2, p. 116, Eq. (8)]):

(3.2)
$$\begin{aligned}
&F_D^{(m)}\left[a,\, b_1,\, \ldots,\, b_m;\, c\,; x_1,\, \ldots,\, x_m\right] \\
&= \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 u^{a-1}\,(1-u)^{c-a-1}\,(1-u\,x_1)^{-b_1}\cdots(1-u\,x_m)^{-b_m}\,du.
\end{aligned}$$

By making in the integral (3.2) the following $2m+1$ replacements:

$$u = 1-v,\ u = \frac{v}{(1-x_1)+v\,x_1},\ \cdots,\ u = \frac{v}{(1-x_m)+v\,x_m},$$
$$u = \frac{1-v}{1-v\,x_1},\ \cdots,\ u = \frac{1-v}{1-v\,x_m},$$

we obtain $2m+1$ transformation formulas for the function $F_D^{(m)}[\cdot]$ (see [2, p. 116]):

(3.3)
$$\begin{aligned}
&F_D^{(m)}\left[a,\, b_1,\, \ldots,\, b_m;\, c\,; x_1,\, \ldots,\, x_m\right] \\
&= (1-x_1)^{-b_1}\cdots(1-x_m)^{-b_m}\; F_D^{(m)}\left[c-a,\, b_1,\, \ldots,\, b_m;\, c\,; \frac{x_1}{x_1-1},\, \ldots,\, \frac{x_m}{x_m-1}\right]
\end{aligned}$$

(3.4)
$$\begin{aligned}
&= (1-x_1)^{-a} \\
&\quad \cdot F_D^{(m)}\left[a,\, c-b_1-\cdots-b_m,\, b_2,\, \ldots,\, b_m;\, c\,; \frac{x_1}{x_1-1},\, \frac{x_1-x_2}{x_1-1},\, \ldots,\, \frac{x_1-x_m}{x_1-1}\right] \\
&\qquad \cdots\cdots
\end{aligned}$$

(3.5)
$$\begin{aligned}
&= (1-x_1)^{c-a-b_1}\,(1-x_2)^{-b_2}\cdots(1-x_m)^{-b_m} \\
&\quad \cdot F_D^{(m)}\left[c-a,\, c-b_1-\cdots-b_m,\, b_2,\, \ldots,\, b_m;\, c\,; x_1,\, \frac{x_1-x_2}{1-x_2},\, \ldots,\, \frac{x_1-x_m}{1-x_m}\right] \\
&\qquad \cdots\cdots
\end{aligned}$$

From the integral representation (3.2), diverse reduction formulas for $F_D^{(m)}$ can be deduced: For example,

$$F_D^{(m)}\left[a,\, b_1,\, \ldots,\, b_m;\, c\,; x,\, \ldots,\, x\right] = {}_2F_1(a,\, b_1+\cdots+b_m;\, c\,;\, x), \tag{3.6}$$

where, in particular, upon using Gauss's summation formula:

$$\begin{aligned}
{}_2F_1\,(a,\, b;\, c;\, 1) &= \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \\
&\quad (\Re(c-a-b) > 0;\ c \in \mathbb{C}\setminus\mathbb{Z}_0^-)
\end{aligned} \tag{3.7}$$

$\mathbb{C}$ being the set of complex numbers and $\mathbb{Z}_0^-$ the set of nonpositive integers, we get

(3.8)
$$F_D^{(m)}\left[a,\, b_1,\, \ldots,\, b_m;\, c\,;1,\, \ldots,\, 1\right] = \frac{\Gamma(c)\,\Gamma\left(c-a-b_1-\cdots-b_m\right)}{\Gamma(c-a)\,\Gamma\left(c-b_1-\cdots-b_m\right)}.$$

By applying the transformation formulas (3.3)–(3.5) for $F_D^{(m)}$ to the right-hand side of (2.3), we can obtain a variety of generating functions for $\varphi_n^m\left(x_1,\, x_2,\, \ldots,\, x_m;\lambda_1,\, \lambda_2,\, \ldots,\, \lambda_m\right)$. For example,

(3.9)
$$\begin{aligned}&\sum_{n=0}^{\infty}(\mu)_n\,\varphi_n^m\left(x_1,\, x_2,\, \ldots,\, x_m;\lambda_1,\, \lambda_2,\, \ldots,\, \lambda_m\right)\frac{t^n}{n!}\\&= \left(1-t\,e^{-\sigma_m}\right)^{-\mu}\prod_{j=1}^{m}\left(\frac{1-t\,e^{\lambda_j-\sigma_m}}{1-t\,e^{-\sigma_m}}\right)^{x_j}\\&\quad\cdot F_D^{(m)}\left[1-\mu,\, -x_1,\, \ldots,\, -x_m;\, 1\,;\frac{t\left(1-e^{\lambda_1}\right)\,e^{-\sigma_m}}{1-t\,e^{\lambda_1-\sigma_m}},\, \ldots,\, \frac{t\left(1-e^{\lambda_m}\right)\,e^{-\sigma_m}}{1-t\,e^{\lambda_m-\sigma_m}}\right].\end{aligned}$$

4. Generating functions $\varphi_n^3(x_1,\, x_2,\, x_3;\lambda_1,\, \lambda_2,\lambda_3)$

For special cases of the $2m+1$ transformation formulas for $F_D^{(m)}$ in Section 3, we give here 7 transformation formulas for $F_D^{(3)}$:

(4.1)
$$\begin{aligned}&F_D^{(3)}\left[a,\, b_1,\, b_2,\, b_3;\, c\,;x_1,\, x_2,\, x_3\right]\\&= (1-x_1)^{-b_1}\,(1-x_2)^{-b_2}\,(1-x_3)^{-b_3}\\&\quad\cdot F_D^{(3)}\left[c-a,\, b_1,\, b_2,\, b_3;\, c\,;\frac{x_1}{x_1-1},\, \frac{x_2}{x_2-1},\, \frac{x_3}{x_3-1}\right]\end{aligned}$$

(4.2)
$$= (1-x_1)^{-a}\, F_D^{(3)}\left[a,\, c-b_1-b_2-b_3,\, b_2,\, b_3;\, c\,;\frac{x_1}{x_1-1},\, \frac{x_1-x_2}{x_1-1},\, \frac{x_1-x_3}{x_1-1}\right]$$

(4.3)
$$= (1-x_2)^{-a}\, F_D^{(3)}\left[a,,\, b_1,\, c-b_1-b_2-b_3,\, b_3;\, c\,;\frac{x_2-x_1}{x_2-1},\, \frac{x_2}{x_2-1},\, \frac{x_2-x_3}{x_2-1}\right]$$

(4.4)
$$= (1-x_3)^{-a}\, F_D^{(3)}\left[a,\, b_1,\, b_2,\, c-b_1-b_2-b_3;\, c\,;\frac{x_3-x_1}{x_3-1},\, \frac{x_3-x_2}{x_3-1},\, \frac{x_3}{x_3-1}\right]$$

$$
\begin{aligned}
&= (1-x_1)^{c-a-b_1}\,(1-x_2)^{-b_2}\,(1-x_3)^{-b_3} \\
&\cdot F_D^{(3)}\left[c-a,\, c-b_1-b_2-b_3,\, b_2,\, b_3;\, c\,; x_1,\, \frac{x_1-x_2}{1-x_2},\, \frac{x_1-x_3}{1-x_3}\right]
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
&= (1-x_1)^{-b_1}\,(1-x_2)^{c-a-b_2}\,(1-x_3)^{-b_3} \\
&\cdot F_D^{(3)}\left[c-a,\, b_1,\, c-b_1-b_2-b_3,\, b_3;\, c\,; \frac{x_2-x_1}{1-x_1},\, x_2,\, \frac{x_2-x_3}{1-x_3}\right]
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
&= (1-x_1)^{-b_1}\,(1-x_2)^{-b_2}\,(1-x_3)^{c-a-b_3} \\
&\cdot F_D^{(3)}\left[c-a,\, b_1,\, b_2,\, c-b_1-b_2-b_3;\, c\,; \frac{x_3-x_1}{1-x_1},\, \frac{x_3-x_2}{1-x_2},\, x_3\right].
\end{aligned} \tag{4.7}
$$

In view of (2.3), applying these 7 transformation formulas (4.1)–(4.7) for $F_D^{(3)}$, we can get 7 generating functions for φ_n^3. For example,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} (\mu)_n\, \varphi_n^3\,(x_1,\, x_2,\, x_3; \lambda_1,\, \lambda_2,\, \lambda_3)\, \frac{t^n}{n!} \\
&= \left(1-t\,e^{-\sigma_3}\right)^{-\mu} \prod_{j=1}^{3} \left(\frac{1-t\,e^{\lambda_j-\sigma_3}}{1-t\,e^{-\sigma_3}}\right)^{x_j} \\
&\quad \cdot F_D^{(3)}\left[1-\mu,\, -x_1,\, -x_2,\, -x_3;\, 1\,; \frac{t\left(1-e^{\lambda_1}\right)\, e^{-\sigma_3}}{1-t\,e^{\lambda_1-\sigma_3}},\right. \\
&\qquad\qquad \left. \frac{t\left(1-e^{\lambda_2}\right)\, e^{-\sigma_3}}{1-t\,e^{\lambda_2-\sigma_3}},\, \frac{t\left(1-e^{\lambda_3}\right)\, e^{-\sigma_3}}{1-t\,e^{\lambda_3-\sigma_3}}\right],
\end{aligned} \tag{4.8}
$$

which is a special case of (3.9) when $m = 3$.

5. Generating functions $\varphi_n^2(x_1,\, x_2; \lambda_1,\, \lambda_2)$

We begin by recalling one of the four Appell series F_j $(j = 1,\, 2,\, 3,\, 4)$ (certain hypergeometric series in two variables) (see [1, p. 296, Eq. (1)]

and see also [12, p. 22, Eq. (2)]:

$$
\begin{aligned}
F_1\,[a,\,b_1,\,b_2;c;\,x_1,\,x_2] &= \sum_{r_1,\,r_2=0}^{\infty} \frac{(a)_{r_1+r_2}\,(b_1)_{r_1}\,(b_2)_{r_2}}{(c)_{r_1+r_2}}\,\frac{x_1^{r_1}}{r_1!}\,\frac{x_2^{r_2}}{r_2!} \\
&= \sum_{r=0}^{\infty} \frac{(a)_r\,(b_1)_r}{(c)_r}\,{}_2F_1\left[\begin{matrix} a+r,\,b_2\,; \\ c+r\,; \end{matrix}\; x_2\right] \frac{x_1^r}{r!}
\end{aligned} \tag{5.1}
$$

$$(\max\{|x_1|\,,\,|x_2|\}<1)\,.$$

We find from (2.4) and (5.1) that

$$F_D^{(2)}\,[a,\,b_1,\,b_2;c;\,x_1,\,x_2] = F_1\,[a,\,b_1,\,b_2;c;\,x_1,\,x_2]\,. \tag{5.2}$$

Setting $m=2$ in Eq. (2.3) and considering (5.2), we get a generating function for $\varphi_n^2(x_1,\,x_2;\lambda_1,\,\lambda_2)$:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} (\mu)_n\,\varphi_n^2\,(x_1,\,x_2;\lambda_1,\,\lambda_2)\,\frac{t^n}{n!} = \left(1-t\,e^{-\sigma_2}\right)^{-\mu} \\
&\quad\cdot F_1\left[\mu,\,-x_1,\,-x_2;\,1\,;\frac{t\left(e^{\lambda_1}-1\right)\,e^{-\sigma_2}}{1-t\,e^{-\sigma_2}},\,\frac{t\left(e^{\lambda_2}-1\right)\,e^{-\sigma_2}}{1-t\,e^{-\sigma_2}}\right].
\end{aligned} \tag{5.3}
$$

Recall a known reduction formula for F_1 (see [6, p. 238, Eq. (1)]):

(5.4)

$$F_1\,[a,\,b_1,\,b_2;b_1+b_2;\,x_1,\,x_2] = (1-x_2)^{-a}\;{}_2F_1\left[\begin{matrix} a,\,b_1\,; \\ b_1+b_2\,; \end{matrix}\;\frac{x_1-x_2}{1-x_2}\right].$$

Further choosing $x_2=-x_1-1$ in (5.3) and using (5.4), we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty} (\mu)_n\,\varphi_n^2\,(x_1,\,-x_1-1;\lambda_1,\,\lambda_2)\,\frac{t^n}{n!} \\
&\quad= \left(1-t\,e^{-\lambda_1}\right)^{-\mu}\;{}_2F_1\left[\begin{matrix} \mu,\,-x_1\,; \\ 1\,; \end{matrix}\;\frac{t\left(e^{-\lambda_2}-e^{-\lambda_1}\right)}{1-t\,e^{-\lambda_1}}\right].
\end{aligned} \tag{5.5}
$$

Expanding the right-hand side of (5.5) in powers of t and then equating the coefficients of t^n on each side, we have

(5.6)

$$\varphi_n^2\,(x_1,\,-x_1-1;\lambda_1,\,\lambda_2) = e^{-\lambda_1 n}\,{}_2F_1\left(-x_1,\,-n\,;1\,;\,1-e^{\lambda_1-\lambda_2}\right)\,(n\in\mathbb{N}_0)\,.$$

Expanding the right-hand side of (5.3) by using the last equality in (5.1) in powers of t and substituting the resulting series for the right-hand side of (5.3), and comparing the coefficients of t^n on both sides of the finally obtained equation, we get

$$\begin{aligned}\varphi_n^2(x_1, x_2; \lambda_1, \lambda_2) =& e^{-\sigma_2 n} \sum_{m=0}^{n} \frac{(-n)_m (-x_2)_m \left(1-e^{\lambda_2}\right)^m}{m!\, m!} \\ &\cdot {}_2F_1\left[\begin{matrix}-x_1, -m; \\ 1+x_2-m;\end{matrix} \frac{1}{e^{\lambda_2}-1}\right] \quad (n \in \mathbb{N}_0)\,.\end{aligned} \tag{5.7}$$

By using Gauss's summation formula (3.7) and a known identity, $\mathbb{Z}$ being the set of integers,

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n} \quad (\alpha \in \mathbb{C} \setminus \mathbb{Z})\,, \tag{5.8}$$

we obtain an interesting special case of (5.7) when $\lambda_2 = \ln 2$:

$$\varphi_n^2(x_1, x_2; \lambda_1, \ln 2) = \frac{e^{-\lambda_1 n}}{2^n}\, {}_2F_1\left[\begin{matrix}-n, -x_1-x_2; \\ 1;\end{matrix} -1\right] \quad (n \in \mathbb{N}_0)\,. \tag{5.9}$$

It is noted that the known reducible cases of special values of the variables for F_1 are far less numerous. Appell and Kampé de Fériet's monograph [2] gives only (see [6, p. 239, Eq. (10) and Eq. (11)]):

$$F_1[a, b_1, b_2; c; x, 1] = \frac{\Gamma(c)\,\Gamma(c-a-b_2)}{\Gamma(c-a)\,\Gamma(c-b_2)}\, {}_2F_1(a, b_1; c-b_2; x) \tag{5.10}$$

and

$$F_1[a, b_1, b_2; c; x, x] = {}_2F_1(a, b_1+b_2; c; x)\,. \tag{5.11}$$

Setting $\lambda_1 = \lambda_2 = \lambda$ in (5.3) and using (5.11), we get

$$\begin{aligned}&\sum_{n=0}^{\infty} (\mu)_n\, \varphi_n^2(x_1, x_2; \lambda, \lambda)\, \frac{t^n}{n!} \\ &\quad = \left(1-t\,e^{-2\lambda}\right)^{-\mu} {}_2F_1\left[\begin{matrix}\mu, -x_1-x_2; \\ 1;\end{matrix} \frac{t\left(e^{\lambda}-1\right) e^{-2\lambda}}{1-t\,e^{-2\lambda}}\right].\end{aligned} \tag{5.12}$$

Expanding the right-hand side of (5.12) in powers of t and equating the coefficients of t^n on both sides of the resulting series, we obtain

$$\varphi_n^2(x_1, x_2; \lambda, \lambda) = e^{-2\lambda n}\, {}_2F_1\left[\begin{matrix}-n, -x_1-x_2; \\ 1;\end{matrix} 1-e^{\lambda}\right] \quad (n \in \mathbb{N}_0)\,. \tag{5.13}$$

With the considerable number of hypergeometric series of the second order in two variables, the complete set of transformations would run into the hundreds, and only a few examples are recalled here (see [6, pp. 239–240, Equations (1)–(5)]):

$$
\begin{aligned}
&F_1\,[a,\, b_1,\, b_2; c;\, x,\, y] \\
&= (1-x)^{-b_1}\,(1-y)^{-b_2}\, F_1\left[c-a,\, b_1,\, b_2; c;\, \frac{x}{x-1},\, \frac{y}{y-1}\right] \qquad (5.14)
\end{aligned}
$$

$$
= (1-x)^{-a}\, F_1\left[a,\, c-b_1-b_2,\, b_2;\, c;\, \frac{x}{x-1},\, \frac{y-x}{1-x}\right] \qquad (5.15)
$$

$$
= (1-y)^{-a}\, F_1\left[a,\, b_1,\, c-b_1-b_2;\, c;\, \frac{y-x}{y-1},\, \frac{y}{y-1}\right] \qquad (5.16)
$$

$$
= (1-x)^{c-a-b_1}\,(1-y)^{-b_2}\, F_1\left[c-a,\, c-b_1-b_2,\, b_2;\, c;\, x,\, \frac{x-y}{1-y}\right] \qquad (5.17)
$$

$$
= (1-x)^{-b_1}\,(1-y)^{c-a-b_2}\, F_1\left[c-a,\, b_1,\, c-b_1-b_2\,;\, c;\, \frac{x-y}{x-1},\, y\right]. \qquad (5.18)
$$

Applying the transformation formulas (5.14)–(5.18) for F_1 to the right-hand side of (5.3), we get the following generating functions for $\varphi_n^2\,(x_1,\, x_2; \lambda_1,\, \lambda_2)$:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} (\mu)_n\, \varphi_n^2\,(x_1,\, x_2; \lambda_1,\, \lambda_2)\, \frac{t^n}{n!} \\
&= \left(1 - t\,e^{-\sigma_2}\right)^{-\mu-x_1-x_2} \left(1 - t\,e^{-\lambda_2}\right)^{x_1} \left(1 - t\,e^{-\lambda_1}\right)^{x_2} \\
&\quad \cdot F_1\left[1-\mu,\, -x_1,\, -x_2;\, 1\,;\, \frac{\left(1-e^{\lambda_1}\right)\, e^{-\sigma_2}\, t}{1-e^{-\lambda_2}\, t},\, \frac{\left(1-e^{\lambda_2}\right)\, e^{-\sigma_2}\, t}{1-e^{-\lambda_1}\, t}\right] \qquad (5.19)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1 - t\,e^{-\lambda_2}\right)^{-\mu} \\
&\quad \cdot F_1\left[\mu,\, 1+x_1+x_2,\, -x_2;\, 1\,;\, \frac{\left(1-e^{\lambda_1}\right)\, e^{-\sigma_2}\, t}{1-e^{-\lambda_2}\, t},\, \frac{\left(e^{-\lambda_1}-e^{-\lambda_2}\right)\, t}{1-e^{-\lambda_2}\, t}\right] \qquad (5.20)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1 - t\,e^{-\lambda_1}\right)^{-\mu} \\
&\quad \cdot F_1\left[\mu,\, -x_1,\, 1+x_1+x_2;\, 1\,;\, \frac{\left(e^{-\lambda_2}-e^{-\lambda_1}\right)\, t}{1-e^{-\lambda_1}\, t},\, \frac{\left(1-e^{\lambda_2}\right)\, e^{-\sigma_2}\, t}{1-e^{-\lambda_1}\, t}\right] \qquad (5.21)
\end{aligned}
$$

(5.22)

$$= \left(1 - t\,e^{-\sigma_2}\right)^{-1-x_1-x_2} \left(1 - t\,e^{-\lambda_2}\right)^{1-\mu+x_1} \left(1 - t\,e^{-\lambda_1}\right)^{x_2}$$

$$\cdot\, F_1 \left[1-\mu,\, 1+x_1+x_2,\, -x_2;\, 1\,;\, \frac{\left(e^{\lambda_1}-1\right)\, e^{-\sigma_2}\, t}{1-e^{-\sigma_2}\, t},\, \frac{\left(e^{-\lambda_2}-e^{-\lambda_1}\right)\, t}{1-e^{-\lambda_1}\, t}\right]$$

(5.23)

$$= \left(1 - t\,e^{-\sigma_2}\right)^{-1-x_1-x_2} \left(1 - t\,e^{-\lambda_1}\right)^{1-\mu+x_2} \left(1 - t\,e^{-\lambda_2}\right)^{x_1}$$

$$\cdot\, F_1 \left[1-\mu,\, -x_1,\, 1+x_1+x_2;\, 1\,;\, \frac{\left(e^{-\lambda_1}-e^{-\lambda_2}\right)\, t}{1-e^{-\lambda_2}\, t},\, \frac{\left(e^{\lambda_2}-1\right)\, e^{-\sigma_2}\, t}{1-e^{-\sigma_2}\, t}\right].$$

We can obtain explicit formulas for $\varphi_n^2\,(x_1,\, x_2; \lambda_1,\, \lambda_2)$ by using the transformation formulas (5.19)–(5.23). For example, expanding the right-hand side of (5.19) in powers of t and substituting the expanded series for the right-hand side of (5.19), and equating the coefficients of t^n on both sides of the finally obtained identity, we have

(5.24)

$$\frac{(\mu)_n}{n!}\, \varphi_n^2\,(x_1,\, x_2; \lambda_1,\, \lambda_2)$$

$$= \sum_{r=0}^{n} \sum_{m=0}^{r} \sum_{k=0}^{m} \frac{(1-\mu)_{r-k}\, (-x_1)_{r-m}\, (-x_2)_m\, (\mu+x_1+x_2)_{n-r}}{(n-r)!\, (r-m)!\, (m-k)!\, (r-k)!\, k!}$$

$$\cdot \left(1-e^{\lambda_1}\right)^{r-m} \left(1-e^{\lambda_2}\right)^{m-k} e^{\lambda_2\, k}\, e^{-\sigma_2\, n}$$

$$\cdot\, {}_2F_1 \left[\begin{matrix} -k,\, -x_1+r-m\,; \\ 1+x_2-m\,; \end{matrix}\; e^{\lambda_1-\lambda_2}\right] \quad (n \in \mathbb{N}_0)\,.$$

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Junesang Choi
Department of Mathematics, Dongguk University,
Gyeongju 780-714, Korea.
E-mail: junesang@mail.dongguk.ac.kr