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DENSITY OF IMAGINARY CUBIC FUNCTION FIELDS HAVING INFINITE HILBERT 3-CLASS FIELD TOWER

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Abstract. In this paper we study the density of imaginary cubic function fields having the infinite Hilbert 3-class field tower.

1. Introduction and Statement of Result

Let $k = \mathbb{F}_q(T)$ be the rational function field over the finite field $\mathbb{F}_q$ of q elements, $\mathbb{A} = \mathbb{F}_q[T]$ and $\infty = (1/T)$. Assume that q is odd with $q \equiv 1 \bmod 3$. Let F be an imaginary cubic function field, i.e., F is a finite cyclic extension of k of degree 3 in which ∞ is ramified. Let $F_1^{(3)}$ be the Hilbert 3-class field of $F_0^{(3)} = F$ and inductively, $F_{n+1}^{(3)}$ be the Hilbert 3-class field of $F_n^{(3)}$ for $n \geq 0$. The sequence of fields

$$F_0^{(3)} = F \subset F_1^{(3)} \subset \cdots \subset F_n^{(3)} \subset \cdots$$

is called the *Hilbert* 3*-class field tower of* F. We say that F has infinite Hilbert 3-class field tower if $F_n^{(3)} \neq F_{n+1}^{(3)}$ for each $n \geq 0$. Let $\mathcal{C}l_F$ be the ideal class group of $\mathcal{O}_F$ and $\mathcal{O}_F^*$ be the group of units of $\mathcal{O}_F$. Let $r_3(A) = \dim_{\mathbb{F}_3}(A/A^3)$ be the 3-rank of an abelian group A. In [5], Schoof proved that the Hilbert 3-class field tower of F is infinite if $r_3(\mathcal{C}l_F) \geq 2 + 2\sqrt{r_3(\mathcal{O}_F^*) + 1}$. For more details on Hilbert 3-class field tower of F, we refer [3, 4, 5]. In [3], we studied the infiniteness of the Hilbert 3-class field tower of imaginary cubic function fields. The aim of this paper is to study the density of imaginary cubic function fields having the infinite Hilbert 3-class field tower.

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Let γ be a generator of $\mathbb{F}_q^*$. For any $0 \neq N \in \mathbb{A}$, let $\omega(N)$ be the number of distinct monic irreducible divisors of N. Write $\mathcal{P}$ for the set of all monic irreducible polynomials in $\mathbb{A}$. Assume that we have given a total ordering "$<$" on $\mathcal{P}$ such that $P < Q$ for any $P, Q \in \mathcal{P}$ with $\deg P < \deg Q$. For any positive integers n and t, let $\mathcal{P}(n,t)$ be the set of monic square free polynomials N in $\mathbb{A}$ with $\deg N = n$ and $\omega(N) = t$. If we write $N = P_1 \cdots P_t \in \mathcal{P}(n,t)$, it is always assumed that $P_1 < \cdots < P_t$. Let $\mathbb{E}_t$ be the set of all t-tuples $(1, e_2, \ldots, e_t)$ with $e_i \in \{1,2\}$ for $2 \leq i \leq t$. For any $N = P_1 \cdots P_t \in \mathcal{P}(n,t)$ and $\mathbf{e} = (1, e_2, \ldots, e_t) \in \mathbb{E}_t$, we write $N^{\mathbf{e}} = P_1 P_2^{e_2} \cdots P_t^{e_t}$. Then any imaginary cubic function field F can be written uniquely as $F = k(\sqrt[3]{\gamma^a N^{\mathbf{e}}})$, where $a \in \{0,1,2\}$, $N \in \mathcal{P}(n,t)$ and $\mathbf{e} \in \mathbb{E}_t$ with $3 \nmid \deg N^{\mathbf{e}}$.

For positive integers n and t, let $X_{t;n}$ be the set of imaginary cubic function fields $F = k(\sqrt[3]{\gamma^a N^{\mathbf{e}}})$, where $a \in \{0,1,2\}$, $N \in \mathcal{P}(n,t)$ and $\mathbf{e} \in \mathbb{E}_t$ with $3 \nmid \deg N^{\mathbf{e}}$, and $X_{t;n}^*$ be the subset of $X_{t;n}$ consisting of $F \in X_{t;n}$ having infinite Hilbert 3-class field tower. We define a density

$$\alpha_t = \lim_{n \to \infty} \frac{|X_{t;n}^*|}{|X_{t;n}|}.$$

If $t \geq 6$, then $X_{t;n}^* = X_{t;n}$, so we have $\alpha_t = 1$ (see [3]). In this paper we are interested in the densities α_4 and α_5. The main result of this paper is the following theorem.

Theorem 1.1. *We have $\alpha_4 \geq \frac{1}{27}$ and $\alpha_5 \geq \frac{29}{81}$.*

2. Preliminaries

2.1. Hilbert 3-class field tower and the invariant λ_2

Assume that q is odd with $q \equiv 1 \bmod 3$. Let F be an imaginary cubic function field. Write $\mathcal{C}l_F$ for the ideal class group of the integral closure $\mathcal{O}_F$ of $\mathbb{A}$ in F. Let $r_3(\mathcal{C}l_F) = \dim_{\mathbb{F}_3}(\mathcal{C}l_F/\mathcal{C}l_F^3)$ be the 3-rank of $\mathcal{C}l_F$. Let σ is a generator of $G = Gal(F/k)$. Then we have

$$r_3(\mathcal{C}l_F) = \lambda_1(F) + \lambda_2(F), \tag{2.1}$$

where $\lambda_i(F) = \dim_{\mathbb{F}_3}\left(\mathcal{C}l_F^{(1-\sigma)^{i-1}}/\mathcal{C}l_F^{(1-\sigma)^i}\right)$ for $i = 1, 2$.

We give a sufficient condition for an imaginary cubic function field to have an infinite Hilbert 3-class field tower.

Proposition 2.1. *Assume that q is odd with $q \equiv 1 \bmod 3$. Let $F = k(\sqrt[3]{\gamma^a N^{\mathbf{e}}})$ be an imaginary cubic function field over k with $N \in \mathcal{P}(n,t)$. If $\lambda_2(F) \geq 6 - t$, then the Hilbert 3-class field tower of F is infinite.*

Proof. By Schoof's theorem, the Hilbert 3-class field tower of F is infinite if $r_3(\mathcal{C}l_F) \geq 5$. Since F is imaginary, we have $\lambda_1(F) = t-1$. Then the result follows from (2.1). $\square$

2.2. Some asymptotic formulas

For positive integers n and t, write $\mathcal{I}(n,t)$ for the set of all $N^{\mathbf{e}}$ with $N \in \mathcal{P}(n,t), \mathbf{e} \in \mathbb{E}_t$ and $3 \nmid \deg N^{\mathbf{e}}$. Then we have

$$X_{t;n} = \left\{ k(\sqrt[3]{\gamma^a N^{\mathbf{e}}}) : a \in \{0,1,2\}, N^{\mathbf{e}} \in \mathcal{I}(n,t) \right\}.$$

Let $\mathcal{P}'(n,t)$ be the subset of $\mathcal{P}(n,t)$ consisting of $N = P_1 \cdots P_t \in \mathcal{P}(n,t)$ with $\deg P_i \neq \deg P_j$ for $1 \leq i \neq j \leq t$. Let $\mathcal{I}'(n,t)$ be the subset of $\mathcal{I}(n,t)$ consisting of $N^{\mathbf{e}} \in \mathcal{I}(n,t)$ with $N \in \mathcal{P}'(n,t)$ and

$$X'_{t;n} = \left\{ k(\sqrt[3]{\gamma^a N^{\mathbf{e}}}) \in X_{t;n} : a \in \{0,1,2\}, N^{\mathbf{e}} \in \mathcal{I}'(n,t) \right\}.$$

For a nonnegative integer s with $0 \leq s \leq t-1$, let $X_{t,s;n}$ be the subset of $X_{t;n}$ consisting of $F \in X_{t;n}$ with $\lambda_2(F) = s$ and $X'_{t,s;n} = X_{t,s;n} \cap X'_{t;n}$. It follows from [2, Lemma 3.1, (3.2), (3.7)] that as $n \to \infty$,

$$|X_{t;n}| \sim |X'_{t;n}|, \tag{2.2}$$

$$|X'_{t;n}| = 3^{1-t}(4^t - 1)\frac{q^n (\log n)^{t-1}}{(t-1)!\, n} + O\Big(\frac{q^n (\log n)^{t-2}}{n}\Big) \tag{2.3}$$

and

$$\begin{aligned} |X'_{t,s;n}| &= 3^{1-\frac{t^2+t}{2}}(4^t - 1)\nu_3(t-1, t-1-s) \\ &\quad \times \left[\frac{q^n (\log n)^{t-1}}{(t-1)!\, n} + O\Big(\frac{q^n (\log n)^{t-2}}{n}\Big) \right], \end{aligned} \tag{2.4}$$

where

$$\nu_3(n,a) = \prod_{i=1}^{[\frac{a}{2}]} \frac{3^{2i}}{(3^{2i}-1)} \prod_{i=0}^{a-1} (3^{n-i} - 1).$$

3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. Let ${X'}^*_{t;n} = {X'}_{t;n} \cap X^*_{t;n}$. By (2.2), we have

$$\alpha_t = \lim_{\substack{n \to \infty \\ 3 \nmid n}} \frac{|{X'}^*_{t;n}|}{|{X'}_{t;n}|}.$$

We first consider the case $t = 4$. For $F \in X'_{4;n}$, by Proposition 2.1, the Hilbert 3-class field tower of F is infinite if $\lambda_2(F) \geq 2$. Hence, we have

$$X'_{4,2;n} \cup X'_{4,3;n} \subset X'^{*}_{4;n}$$

and so

$$|X'^{*}_{4;n}| \geq |X'_{4,2;n}| + |X'_{4,3;n}|.$$

By (2.3) and (2.4), we have

$$\begin{aligned}
|X'_{4;n}| &= 3^{-3}(4^4-1)\frac{q^n(\log n)^3}{3!\,n} + O\Big(\frac{q^n(\log n)^2}{n}\Big),\\
|X'_{4,2;n}| &= 3^{-9}(4^4-1)(3^3-1)\left[\frac{q^n(\log n)^3}{3!\,n} + O\Big(\frac{q^n(\log n)^2}{n}\Big)\right],\\
|X'_{4,3;n}| &= 3^{-9}(4^4-1)\left[\frac{q^n(\log n)^3}{3!\,n} + O\Big(\frac{q^n(\log n)^2}{n}\Big)\right]
\end{aligned}$$

as $n \to \infty$. Hence, we have $\alpha_4 \geq \frac{3^{-9}(4^4-1)(3^3-1)+3^{-9}(4^4-1)}{3^{-3}(4^4-1)} = \frac{1}{27}$.

Now, we consider the case $t = 5$. For $F \in X'_{5;n}$, by Proposition 2.1, the Hilbert 3-class field tower of F is infinite if $\lambda_2(F) \geq 1$. Hence, we have

$$\bigcup_{s=1}^{4} X'_{5,s;n} \subset X'^{*}_{5;n}$$

and so

$$|X'^{*}_{5;n}| \geq \sum_{s=1}^{4} |X'_{5,s;n}|.$$

By (2.3) and (2.4), we have

$$\begin{aligned}
|X'_{5;n}| &= 3^{-4}(4^5-1)\frac{q^n(\log n)^4}{4!\,n} + O\Big(\frac{q^n(\log n)^3}{n}\Big),\\
|X'_{5,s;n}| &= 3^{-14}(4^5-1)\nu_3(4,4-s)\left[\frac{q^n(\log n)^4}{4!\,n} + O\Big(\frac{q^n(\log n)^3}{n}\Big)\right]
\end{aligned}$$

with

$$\nu_3(4,4-s) = \begin{cases} \frac{9}{8}(3^4-1)(3^3-1)(3^2-1) & \text{if } s=1,\\ \frac{9}{8}(3^4-1)(3^3-1) & \text{if } s=2,\\ (3^4-1) & \text{if } s=3,\\ 1 & \text{if } s=4.\end{cases}$$

Hence, we have $\alpha_5 \geq \frac{29}{81}$.

References

[1] J. Ahn and H. Jung, *Hilbert* 2*-class field towers of imaginary quadratic function fields.* J. Chungcheong Math. Soc. 23 (2010), no 4, 699–704.
[2] H. Jung, *Density of class groups of imaginary* ℓ*-cyclic function fields.* Finite fields and their applications 17 (2011), 286–293.
[3] ______, *Hilbert* 3*-class field towers of imaginary cubic function fields.* submitted.
[4] M. Rosen, *The Hilbert class field in function fields.* Exposition. Math. **5** (1987), no. 4, 365–378.
[5] R. Schoof, *Algebraic curves over* $\mathbb{F}_2$ *with many rational points.* J. Number Theory **41** (1992), no. 1, 6–14.

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