Honam Mathematical J. **34** (2012), No. 4, pp. 585–596
http://dx.doi.org/10.5831/HMJ.2012.34.4.585

APPLICATIONS OF COUPLED $\mathcal{N}$-STRUCTURES IN BH-ALGEBRAS

Min Jeong Seo and Sun Shin Ahn*

Abstract. The notions of a $\mathcal{N}$-subalgebra, a (strong) $\mathcal{N}$-ideal of BH-algebras are introduced, and related properties are investigated. Characterizations of a coupled $\mathcal{N}$-subalgebra and a coupled (strong) $\mathcal{N}$-ideals of BH-algebras are given. Relations among a coupled $\mathcal{N}$-subalgebra, a coupled $\mathcal{N}$-ideal and a coupled strong $\mathcal{N}$ of BH-algebras are discussed.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([2,3]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. BCK-algebras have some connections with other areas: D. Mundici [9] proved MV-algebras are categorically equivalent to bounded commutative algebra, and J. Meng [10] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. Y. B. Jun, E. H. Roh, and H. S. Kim [5] introduced the notion of a BH-algebra, which is a generalization of BCK/BCI-algebras. They defined the notions of ideal, maximal ideal and translation ideal and investigated some properties. E. H. Roh and S. Y. Kim [9] estimated the number of BH^*-subalgebras of order i in a transitive BH^*-algebras by using Hao's method. In [1], S. S. Ahn and J. H. Lee introduced the notion of strong ideals in BH-algebra and investigate some properties of it. They also defined the notion of a rough sets in BH-algebras. Using a strong ideal in BH-algebras, they obtained some relations between strong ideals and upper(lower) rough strong ideals in BH-algebras. Jun et.al([4]) introduced the notion of

Received October 23, 2012. Accepted December 4, 2012.
2010 Mathematics Subject Classification. 06D72, 06F35, 03G25.
Key words and phrases. Coupled $\mathcal{N}$-structure, Coupled $\mathcal{N}$-subalgerba, Coupled $\mathcal{N}$-ideal, Coupled strong $\mathcal{N}$-ideal.
*Corresponding author.

coupled $\mathcal{N}$-structures and its application in BCK/BCI-algebras was discussed.

In this paper, we introduce the notions of a coupled $\mathcal{N}$-subalgebra, a coupled (strong) $\mathcal{N}$-ideals of BH-algebras are introduced, and related properties are investigated. Characterizations of a coupled $\mathcal{N}$-subalgebra and a coupled (strong) $\mathcal{N}$-ideals of BH-algebras are given. Relations among a coupled $\mathcal{N}$-subalgebra, a coupled $\mathcal{N}$-ideal and a coupled strong $\mathcal{N}$-ideal of BH-algebras are discussed.

2. Preliminaries

By a *BH-algebra*([5]), we mean an algebra $(X;*,0)$ of type (2,0) satisfying the following conditions:

(I) $x*x=0$,
(II) $x*0=x$,
(III) $x*y=0$ and $y*x=0$ imply $x=y$, for all $x,y\in X$.

For brevity, we also call X a BH-algebra. In X we can define a binary operation " $\leq$ " by $x\leq y$ if and only if $x*y=0$. A non-empty subset S of a BH-algebra X is called a *subalgebra* of X if, for any $x,y\in S$, $x*y\in S$, i.e., S is a closed under binary operation.

Definition 2.1.([5]) A non-empty subset A of a BH-algebra X is called an *ideal* of X if it satisfies:

(I1) $0\in A$,
(I2) $x*y\in A$ and $y\in A$ imply $x\in A$, $\forall x,y\in X$.

An ideal A of a BH-algebra X is said to be a *translation ideal* of X if it satisfies:

(I3) $x*y\in A$ and $y*x\in A$ imply $(x*z)*(y*z)\in A$ and $(z*x)*(z*y)\in A$, $\forall x,y,z\in X$.

Definition 2.2.([9]) A BH-algebra X is called a *BH^*-algebra* if it satisfies the identity $(x*y)*x=0$ for all $x,y\in X$.

Example 2.3.([5]) Let $X:=\{0,a,b,c\}$ be a BH-algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	1	0	0
1	1	0	0	0
2	2	2	0	3
3	3	3	3	0

Then $A := \{0, 1\}$ is a translation ideal of X.

Definition 2.4.([1]) A non-empty subset A of a BH-algebra X is called a *strong ideal* of X if it satisfies (I1) and

(I4) $(x * y) * z, y \in A$ imply $x * z \in A$.

Lemma 2.5.([1]) *In a BH-algebra, any strong ideal is an ideal.*

Lemma 2.6.([1]) *In a BH^*-algebra X, any ideal is a subalgebra.*

Corollary 2.7.([1]) *Any strong ideal of BH^*-algebra is a subalgebra.*

3. Coupled $\mathcal{N}$-structures applied to subalgebras and ideals in BH-algebras

Definition 3.1.([4]) A *coupled $\mathcal{N}$-structure* $\mathcal{C}$ in a nonempty set X is an object of the form

$$\mathcal{C} = \{\langle x; f_{\mathcal{C}}, g_{\mathcal{C}}\rangle : x \in X\}$$

where $f_{\mathcal{C}}$ and $g_{\mathcal{C}}$ are $\mathcal{N}$-functions on X such that $-1 \leq f_{\mathcal{C}}(x)+g_{\mathcal{C}}(x) \leq 0$ for all $x \in X$.

A coupled $\mathcal{N}$-structure $\mathcal{C} = \{\langle x; f_{\mathcal{C}}, g_{\mathcal{C}}\rangle : x \in X\}$ in X can be identified to an ordered pair $(f_{\mathcal{C}}, g_{\mathcal{C}})$ in $\mathcal{F}(X, [-1, 0]) \times \mathcal{F}(X, [-1, 0])$. For the sake of simplicity, we shall use the notation $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ instead of $\mathcal{C} = \{\langle x; f_{\mathcal{C}}, g_{\mathcal{C}}\rangle : x \in X\}$.

For a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in X and $t, s \in [-1, 0]$ with $t + s \geq -1$, the set

$$\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\} = \{x \in X \mid f_{\mathcal{C}}(x) \leq t,\ g_{\mathcal{C}}(x) \geq s\}$$

is called an $\mathcal{N}(t, s)$-level set of $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$. An $\mathcal{N}(t, t)$-level set of $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is called an $\mathcal{N}$-level set of $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$.

Definition 3.2.([4]) A coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a BH-algebra X is called a *coupled $\mathcal{N}$-subalgebra* of X if it satisfies:

(3.1) $f_{\mathcal{C}}(x * y) \leq \bigvee \{f_{\mathcal{C}}(x), f_{\mathcal{C}}(y)\}$ and $g_{\mathcal{C}}(x * y) \geq \bigwedge \{g_{\mathcal{C}}(x), g_{\mathcal{C}}(y)\}$

for all $x, y \in X$.

Theorem 3.3.([4]) *A coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a BH-algebra X is a coupled $\mathcal{N}$-subalgebra of X if and only if the nonempty $\mathcal{N}(t, s)$-level set $\mathcal{N}\{(f_{\mathcal{C}}, g_{\mathcal{C}}); (t, s)\}$ is a subalgebra of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.*

Lemma 3.4.([4]) *Every coupled $\mathcal{N}$-subalgebra $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ of a BH-algebra X satisfies $f_{\mathcal{C}}(0) \leq f_{\mathcal{C}}(x)$ and $g_{\mathcal{C}}(0) \geq g_{\mathcal{C}}(x)$ for all $x \in X$.*

Proposition 3.5. *If every $\mathcal{N}$-subalgebra $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ of a BH-algebra X satisfies the inequalities $f_{\mathcal{C}}(x * y) \leq f_{\mathcal{C}}(y)$ and $g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(y)$ for any $x, y \in X$, then $f_{\mathcal{C}}$ and $g_{\mathcal{C}}$ are constant functions.*

Proof. Let $x \in X$. Using (II) and assumption, we have $f_{\mathcal{C}}(x) = f_{\mathcal{C}}(x * 0) \leq f_{\mathcal{C}}(0)$ and $g_{\mathcal{C}}(x) = g_{\mathcal{C}}(x * 0) \geq g_{\mathcal{C}}(0)$. It follows from Lemma 3.4 that $f_{\mathcal{C}}(x) = f_{\mathcal{C}}(0)$ and $g_{\mathcal{C}}(x) = g_{\mathcal{C}}(0)$. Hence $f_{\mathcal{C}}$ and $g_{\mathcal{C}}$ are constant functions. $\square$

Definition 3.6.([4]) A coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a BH-algebra X is called a *coupled $\mathcal{N}$-ideal* of X if it satisfies:

(c81) $f_{\mathcal{C}}(0) \leq f_{\mathcal{C}}(x)$ and $g_{\mathcal{C}}(0) \geq g_{\mathcal{C}}(x)$,
(c82) $f_{\mathcal{C}}(x) \leq \bigvee \{f_{\mathcal{C}}(x * y), f_{\mathcal{C}}(y)\}$ and $g_{\mathcal{C}}(x) \geq \bigwedge \{g_{\mathcal{C}}(x * y), g_{\mathcal{C}}(y)\}$,

for all $x, y \in X$.

Example 3.7. (1) Let $X = \{0, 1, 2, 3, 4\}$ be a BH-algebra([1]), which is not a BCK/BCI-algebra, with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in X defined by

$$\mathcal{C} = \big\{\langle 0; -0.8, -0.2\rangle, \quad \langle 1; -0.6, -0.2\rangle, \langle 2; -0.5, -0.2\rangle, \langle 3; -0.5, -0.2\rangle, \langle 4; -0.1, -0.6\rangle\big\}.$$

Then $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled $\mathcal{N}$-subalgebra, but not a coupled $\mathcal{N}$-ideal of X since

$$f_{\mathcal{C}}(4) = -0.1 \not\leq -0.5 = \bigvee \{f_{\mathcal{C}}(4 * 3), f_{\mathcal{C}}(3)\}$$

and/or

$$g_{\mathcal{C}}(4) = -0.6 \not\geq -0.2 = \bigwedge \{g_{\mathcal{C}}(4 * 3), g_{\mathcal{C}}(3)\}.$$

(2) Let $X = \{0, a, b, c\}$ be a set with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then $(X;*,0)$ is a BH-algebra, which is not a BCK/BCI-algebra. Let $\mathcal{C}=(f_{\mathcal{C}},g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in X defined by

$\mathcal{C}=\big\{\langle 0;-0.8,-0.2\rangle,\langle a;-0.4,-0.5\rangle,\langle b;-0.4,-0.5\rangle,\langle c;-0.2,-0.6\rangle\big\}.$

It is easy to check that $\mathcal{C}=(f_{\mathcal{C}},g_{\mathcal{C}})$ is both a coupled $\mathcal{N}$-subalgebra and a coupled $\mathcal{N}$-ideal of X.

(3) Let $X=\{0,1,2,3\}$ be a BH-algebra([5]), which is not a BCK/BCI-algebra, with the following Cayley table:

$*$	0	1	2	3
0	0	1	3	0
1	1	0	2	0
2	2	2	0	3
3	3	3	3	0

Let $\mathcal{C}=(f_{\mathcal{C}},g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in X defined by

$\mathcal{C}=\big\{\langle 0;-0.7,-0.2\rangle,\langle 1;-0.5,-0.4\rangle,\langle 2;-0.5,-0.4\rangle,\langle 3;-0.3,-0.6\rangle\big\}.$

Then $\mathcal{C}=(f_{\mathcal{C}},g_{\mathcal{C}})$ is a coupled $\mathcal{N}$-ideal of X, but not a coupled $\mathcal{N}$-subalgebra of X, since

$$f_{\mathcal{C}}(0*2)=f_{\mathcal{C}}(3)=-0.3\not\leq -0.5=\bigvee\{f_{\mathcal{C}}(0),f_{\mathcal{C}}(2)\}$$

and/or

$$g_{\mathcal{C}}(0*2)=g_{\mathcal{C}}(3)=-0.6\not\geq -0.4=\bigwedge\{g_{\mathcal{C}}(0),g_{\mathcal{C}}(2)\}.$$

Proposition 3.8.([4]) *Every coupled $\mathcal{N}$-ideal of a BH-algebra X satisfies the following assertions:*

(i) $(\forall x,y,z\in X)(x*y\leq z\ \Rightarrow f_{\mathcal{C}}(x)\leq\bigvee\{f_{\mathcal{C}}(y),f_{\mathcal{C}}(z)\},\ g_{\mathcal{C}}(x)\geq\bigwedge\{g_{\mathcal{C}}(y),g_{\mathcal{C}}(z)\})$.

(ii) $(\forall x,y\in X)(x\leq y\Rightarrow f_{\mathcal{C}}(x)\leq f_{\mathcal{C}}(y),g_{\mathcal{C}}(x)\geq g_{\mathcal{C}}(y))$.

Theorem 3.9.([4]) *For a coupled $\mathcal{N}$-structure $\mathcal{C}=(f_{\mathcal{C}},g_{\mathcal{C}})$ in a BH-algebra X, the following are equivalent:*

(1) *$\mathcal{C}=(f_{\mathcal{C}},g_{\mathcal{C}})$ is a coupled $\mathcal{N}$-ideal of X.*

(2) *The nonempty $\mathcal{N}(t,s)$-level set $\mathcal{N}\{(f_{\mathcal{C}},g_{\mathcal{C}});(t,s)\}$ is an ideal of X for all $t,s\in[-1,0]$ with $t+s\geq -1$.*

Definition 3.10. A coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ in a BH-algebra X is called a *coupled strong $\mathcal{N}$-ideal* of X if it satisfies (c81) and

(c83) $f_{\mathcal{C}}(x*z) \leq \bigvee \{f_{\mathcal{C}}((x*y)*z), f_{\mathcal{C}}(y)\}$ and $g_{\mathcal{C}}(x*z) \geq \bigwedge \{g_{\mathcal{C}}((x*y)*z), g_{\mathcal{C}}(y)\}$

for all $x, y \in X$.

Example 3.11. (1) Consider a BH-algebra $X = \{0,1,2,3,4,5\}$ and a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ as in Example 3.7(1). Then $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled $\mathcal{N}$-subalgebra of X, but not a coupled $\mathcal{N}$-ideal of X (see Example 3.7(1)). Also it is not not a coupled strong $\mathcal{N}$-ideal of X since

$$f_{\mathcal{C}}(4*2) = -0.1 \not\leq \ -0.6 = \bigvee \{f_{\mathcal{C}}((4*1)*2), f_{\mathcal{C}}(1)\}$$

and/or

$$g_{\mathcal{C}}(4*2) = -0.6 \not\geq -0.2 = \bigwedge \{g_{\mathcal{C}}((4*1)*2), g_{\mathcal{C}}(1)\}.$$

(2) Let $X = \{0,1,2,3\}$ be a BH-algebra as in Example 3.7(1). Let $\mathcal{D} = (f_{\mathcal{D}}, g_{\mathcal{D}})$ be a coupled $\mathcal{N}$-structure in X defined by

$$\mathcal{D} = \big\{\langle 0; -0.7, -0.1\rangle, \quad \langle 1; -0.6, -0.2\rangle, \langle 2; -0.3, -0.5\rangle, \\ \langle 3; -0.3, -0.5\rangle, \langle 4; -0.3, -0.5\rangle\big\}.$$

It is easy to show that $\mathcal{D} = (f_{\mathcal{D}}, g_{\mathcal{D}})$ is both a coupled $\mathcal{N}$-subalgebra and a coupled $\mathcal{N}$-ideal of X, but not a coupled strong $\mathcal{N}$-ideal of X, since

$$f_{\mathcal{D}}(4*2) = f_{\mathcal{D}}(4) = -0.3 \not\leq \ -0.6 = \bigvee \{f_{\mathcal{D}}((4*1)*2), f_{\mathcal{D}}(1)\}$$

and/or

$$g_{\mathcal{D}}(4*2) = g_{\mathcal{D}}(4) = -0.5 \not\geq \ -0.2 = \bigwedge \{g_{\mathcal{D}}((4*1)*2), g_{\mathcal{D}}(1)\}.$$

(3) Let $X := \{0,1,2,3,4,5\}$ be a BH-algebra ([1]), which is not a BCK/BCI-algebra, with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	0	1
2	2	2	0	0	0	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in X defined by

$$\mathcal{C} = \{ \quad \langle 0; -0.8, -0.2\rangle, \quad \langle 1; -0.7, -0.3\rangle, \langle 2; -0.7, -0.3\rangle, \\ \langle 3; -0.7, -0.3\rangle, \quad \langle 4; -0.7, -0.3\rangle, \langle 5; -0.2, -0.5\rangle\big\}.$$

It is easy to check that $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ is both a coupled $\mathcal{N}$-ideal of X and a coupled strong $\mathcal{N}$-ideal of X.

Theorem 3.12. *For a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ in a BH-algebra X, the following are equivalent:*

(1) *$\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ is a coupled strong $\mathcal{N}$-ideal of X.*
(2) *The nonempty $\mathcal{N}(t,s)$-level set $\mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t,s)\}$ is a strong ideal of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.*

Proof. Assume that $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ is a coupled strong $\mathcal{N}$-ideal of X. Let $t, s \in [-1, 0]$ be such that $t + s \geq -1$. Obviously, $0 \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t,s)\}$. Let $x, y, z \in X$ be such that $(x * y) * z, y \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t,s))\}$. Then $f_\mathcal{C}((x*y)*z) \leq t, f_\mathcal{C}(y) \leq t$ and $g_\mathcal{C}((x*y)*z) \geq s, g_\mathcal{C}(y) \geq s$. It follows from (c83) that $f_\mathcal{C}(x*z) \leq \bigvee\{f_\mathcal{C}((x*y)*z), f_\mathcal{C}(y)\} \leq t$ and $g_\mathcal{C}(x*z) \geq \bigwedge\{g_\mathcal{C}((x*y)*z), g_\mathcal{C}(y)\} \geq s$, which imply that $x*z \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t,s)\}$. Hence the nonempty $\mathcal{N}(t,s)$-level set of $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ is a strong ideal of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$.

Conversely, suppose that the nonempty $\mathcal{N}(t,s)$-level set of $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ is a strong ideal of X for all $t, s \in [-1, 0]$ with $t + s \geq -1$. Since $0 \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t,s)\}$, the condition (c81) is valid. Assume that there exist $a, b, c \in X$ such that $f_\mathcal{C}(a*c) > \bigvee\{f_\mathcal{C}((a*b)*c), f_\mathcal{C}(b)\}$ or $g_\mathcal{C}(a*c) < \bigwedge\{g_\mathcal{C}((a*b)*c), g_\mathcal{C}(b)\}$. For the case $f_\mathcal{C}(a*c) > \bigvee\{f_\mathcal{C}((a*b)*c), f_\mathcal{C}(b)\}$ and $g_\mathcal{C}(a*c) \geq \bigwedge\{g_\mathcal{C}((a*b)*c), g_\mathcal{C}(b)\}$, there exist $s_0, t_0 \in [-1, 0)$ such that $f_\mathcal{C}(a*c) > t_0 > \bigvee\{f_\mathcal{C}((a*b)*c), f_\mathcal{C}(b)\}$ and $s_0 = \bigwedge\{g_\mathcal{C}((a*b)*c), g_\mathcal{C}(b)\}$. It follows that $(a*b)*c, b \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_0, s_0)\}$, but $a*c \notin \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_0, s_0)\}$. This is impossible. For the case $f_\mathcal{C}(a*c) \geq \bigvee\{f_\mathcal{C}((a*b)*c), f_\mathcal{C}(b)\}$ and $g_\mathcal{C}(a*c) < \bigwedge\{g_\mathcal{C}((a*b)*c), g_\mathcal{C}(b)\}$, there exist $s_0, t_0 \in [-1, 0)$ such that $t_0 = f_\mathcal{C}(a*b)$ and $g_\mathcal{C}(a*c) < s_0 < \bigwedge\{g_\mathcal{C}((a*b)*c), g_\mathcal{C}(b)\}$. Then $(a*b)*c, b \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_0, s_0)\}$, but $a*c \notin \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_0, s_0)\}$. This is a contradiction. If $f_\mathcal{C}(a*c) > \bigvee\{f_\mathcal{C}((a*b)*c), f_\mathcal{C}(b)\}$ and $g_\mathcal{C}(a*c) < \bigwedge\{g_\mathcal{C}((a*b)*c), g_\mathcal{C}(b)\}$, then $(a*b)*c, b \in \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_0, s_0)\}$, but $a*c \notin \mathcal{N}\{(f_\mathcal{C}, g_\mathcal{C}); (t_0, s_0)\}$, where $t_0 := \frac{1}{2}(f_\mathcal{C}(a*c) + \bigvee\{f_\mathcal{C}((a*b)*c), f_\mathcal{C}(b)\})$ and $s_0 := \frac{1}{2}(g_\mathcal{C}(a*c) + \bigwedge\{g_\mathcal{C}((a*b)*c), g_\mathcal{C}(b)\})$. This is a contradiction. Therefore $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ is a coupled strong $\mathcal{N}$-ideal of X. □

Proposition 3.13. *For any BH^*-algebra X, every coupled $\mathcal{N}$-ideal is a coupled $\mathcal{N}$-subalgebra of X.*

Proof. Let a coupled $\mathcal{N}$-structure $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ be a coupled $\mathcal{N}$-ideal of a BH^*-algebra X and let $x, y \in X$. Then

$$f_\mathcal{C}(x * y) \leq \bigvee \{f_\mathcal{C}((x * y) * x), f_\mathcal{C}(x)\} = \bigvee\{f_\mathcal{C}(0), f_\mathcal{C}(x)\} \leq \bigvee\{f_\mathcal{C}(x), f_\mathcal{C}(y)\}$$

and

$$g_\mathcal{C}(x * y) \geq \bigwedge \{g_\mathcal{C}((x * y) * x), g_\mathcal{C}(x)\} = \bigwedge\{g_\mathcal{C}(0), g_\mathcal{C}(x)\} \geq \bigwedge\{g_\mathcal{C}(x), g_\mathcal{C}(y)\}.$$

Hence $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ is a coupled $\mathcal{N}$-subalgebra of X. □

The converse of Theorem 3.13 may not be true in general as seen in the following example.

Example 3.14. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	3	3	0

It is easily to check that $(X; *, 0)$ is a BH^*-algebra. Let $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ be a coupled $\mathcal{N}$-structure in X defined by

$$\mathcal{C} = \big\{\langle 0; -0.7, -0.1\rangle, \langle 1; -0.7, -0.1\rangle, \langle 2; -0.3, -0.5\rangle, \langle 3; -0.6, -0.2\rangle\big\}.$$

Then $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ is a a coupled $\mathcal{N}$-subalgebra, but not a coupled $\mathcal{N}$-ideal of X, since

$$f_\mathcal{C}(2) = -0.3 \not\leq\ -0.6 = \bigvee \{f_\mathcal{C}(2 * 3), f_\mathcal{C}(3)\}$$

and/or

$$g_\mathcal{C}(2) = -0.5 \not\geq\ -0.2 = \bigwedge \{g_\mathcal{C}(2 * 3), g_\mathcal{C}(3)\}.$$

Proposition 3.15. *Every coupled strong $\mathcal{N}$-ideal $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ of a BH-algebra X is a coupled $\mathcal{N}$-ideal of X.*

Proof. Put $z := 0$ in (c83). □

Proposition 3.16. *Let $\mathcal{C} = (f_\mathcal{C}, g_\mathcal{C})$ be a coupled strong $\mathcal{N}$-ideal of a BH-algebra X. Then the following hold:*

(i) *If $x \leq y$ for any $x, y \in X$, then $f_\mathcal{C}(x) \leq f_\mathcal{C}(y)$, $g_\mathcal{C}(x) \geq g_\mathcal{C}(y)$.*
(ii) *If $f_\mathcal{C}(x * y) = f_\mathcal{C}(0)$ for any $x, y \in X$, then $f_\mathcal{C}(x) \leq f_\mathcal{C}(y)$.*
(iii) *If $g_\mathcal{C}(x * y) = g_\mathcal{C}(0)$ for any $x, y \in X$, then $g_\mathcal{C}(x) \geq g_\mathcal{C}(y)$.*

Proof. (i) It follows from Proposition 3.8 and Proposition 3.15.
(ii) For any $x, y \in X$, we have

$$\begin{aligned} f_{\mathcal{C}}(x) = f_{\mathcal{C}}(x * 0) \leq& \bigvee\{f_{\mathcal{C}}((x * y) * 0), f_{\mathcal{C}}(y * 0)\} \\ =& \bigvee\{f_{\mathcal{C}}(x * y), f_{\mathcal{C}}(y)\} \\ =& \bigvee\{f_{\mathcal{C}}(0), f_{\mathcal{C}}(y)\} \\ =& f_{\mathcal{C}}(y). \end{aligned}$$

(iii) For any $x, y \in X$, we have

$$\begin{aligned} g_{\mathcal{C}}(x) = g_{\mathcal{C}}(x * 0) \geq& \bigwedge\{g_{\mathcal{C}}((x * y) * 0), g_{\mathcal{C}}(y * 0)\} \\ =& \bigwedge\{g_{\mathcal{C}}(x * y), g_{\mathcal{C}}(y)\} \\ =& \bigwedge\{g_{\mathcal{C}}(0), g_{\mathcal{C}}(y)\} \\ =& g_{\mathcal{C}}(y). \end{aligned}$$

□

Proposition 3.17. *Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled strong $\mathcal{N}$-ideal of a BH^*-algebra X. Then the following hold:*

(i) $(\forall x, y \in X)(f_{\mathcal{C}}(x * y) \leq f_{\mathcal{C}}(x), g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(x))$.
(ii) $(\forall x, y \in X)(f_{\mathcal{C}}(x*y) \leq \bigvee\{f_{\mathcal{C}}(x), f_{\mathcal{C}}(y)\}, g_{\mathcal{C}}(x*y) \geq \bigwedge\{g_{\mathcal{C}}(x), g_{\mathcal{C}}(y)\})$.
(iii) $(\forall x, y, z \in X)(f_{\mathcal{C}}(x * (y * z)) \leq \bigvee\{f_{\mathcal{C}}(x), f_{\mathcal{C}}(y), f_{\mathcal{C}}(z)\}, g_{\mathcal{C}}(x * (y * z)) \geq \bigwedge\{g_{\mathcal{C}}(x), g_{\mathcal{C}}(y), g_{\mathcal{C}}(z)\})$.

Proof. (i) Since X is a BH^*-algebra, we have $(x * y) * x = 0$ for any $x, y \in X$. Hence $x * y \leq x$ for any $x, y \in X$. Using Proposition 3.16(i), we have $f_{\mathcal{C}}(x * y) \leq f_{\mathcal{C}}(x)$ and $g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(x)$ for any $x, y \in X$.
(ii) It is easily verified from Proposition 3.13 and Proposition 3.15.
(iii) For any $x, y, z \in X$, using (ii) we have

$$\begin{aligned} f_{\mathcal{C}}(x * (y * z)) \leq& \bigvee\{f_{\mathcal{C}}(x), f_{\mathcal{C}}(y * z)\} \\ \leq& \bigvee\{f_{\mathcal{C}}(x), f_{\mathcal{C}}(y), f_{\mathcal{C}}(z)\} \end{aligned}$$

and

$$\begin{aligned} g_{\mathcal{C}}(x * (y * z)) \geq& \bigwedge\{g_{\mathcal{C}}(x), g_{\mathcal{C}}(y * z)\} \\ \geq& \bigwedge\{g_{\mathcal{C}}(x), g_{\mathcal{C}}(y), g_{\mathcal{C}}(z)\}. \end{aligned}$$

□

For any element a of a d-algebra X, let

$$X_a := \{x \in X \mid f_{\mathcal{C}}(x) \leq f_{\mathcal{C}}(a),\ g_{\mathcal{C}}(x) \geq g_{\mathcal{C}}(a)\}.$$

Obviously, X_a is a non-empty subset of X.

Theorem 3.18. *Let a be any element of a BH-algebra X. If $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ is a coupled (strong) $\mathcal{N}$-ideal of X, then the set X_a is a (strong) ideal of X.*

Proof. Since $f_{\mathcal{C}}(0) \leq f_{\mathcal{C}}(x)$ and $g_{\mathcal{C}}(0) \geq g_{\mathcal{C}}(x)$ for any $x \in X$, we have $0 \in X_a$. Let $x, y \in X$ be such that $x * y \in X_a$ and $y \in X_a$. Then $f_{\mathcal{C}}(x * y) \leq f_{\mathcal{C}}(a), g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(a), f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$. It follows from (c82) that $f_{\mathcal{C}}(x) \leq \bigvee\{f_{\mathcal{C}}(x * y), f_{\mathcal{C}}(y)\} \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(x) \geq \bigwedge\{g_{\mathcal{C}}(x * y), g_{\mathcal{C}}(y)\} \geq g_{\mathcal{C}}(a)$ so that $x \in X_a$. Therefore X_a is an ideal of X.

Let $x, y, z \in X$ be such that $(x*y)*z \in X_a$ and $y \in X_a$. Then $f_{\mathcal{C}}((x*y)*z) \leq f_{\mathcal{C}}(a), g_{\mathcal{C}}((x*y)*z) \geq g_{\mathcal{C}}(a), f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$. It follows from (c83) that $g_{\mathcal{C}}(x * z) \leq \bigvee\{f_{\mathcal{C}}((x * y) * z), f_{\mathcal{C}}(y)\} \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(x * z) \leq \bigwedge\{g_{\mathcal{C}}((x * y) * z), g_{\mathcal{C}}(y)\} \geq g_{\mathcal{C}}(a)$ so that $x * z \in X_a$. Therefore X_a is a strong ideal of X. □

Proposition 3.19. *Let a be any element of a BH-algebra X and let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in X. Then*

(i) *If X_a is an ideal of X, then $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ satisfies the following assertion:*

(3.2)

$$(\forall x, y, z \in X)\ \left(\begin{array}{l} f_{\mathcal{C}}(x) \geq \bigvee\{f_{\mathcal{C}}(y * z), f_{\mathcal{C}}(z)\} \ \Rightarrow\ f_{\mathcal{C}}(x) \geq f_{\mathcal{C}}(y) \\ g_{\mathcal{C}}(x) \leq \bigwedge\{g_{\mathcal{C}}(y * z), g_{\mathcal{C}}(z)\} \ \Rightarrow\ g_{\mathcal{C}}(x) \leq g_{\mathcal{C}}(y) \end{array} \right).$$

(ii) *If $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ satisfies* (3.2) *and*

$$(\forall x \in X)\,(f_{\mathcal{C}}(0) \leq f_{\mathcal{C}}(x),\ g_{\mathcal{C}}(0) \geq g_{\mathcal{C}}(x)), \tag{3.3}$$

then X_a is an ideal of X.

Proof. (i) Assume that X_a is an ideal of X for all $a \in X$. Let $x, y, z \in X$ be such that $f_{\mathcal{C}}(x) \geq \bigvee\{f_{\mathcal{C}}(y * z), f_{\mathcal{C}}(z)\}$ and $g_{\mathcal{C}}(x) \leq \bigwedge\{g_{\mathcal{C}}(y * z), g_{\mathcal{C}}(z)\}$. Then $y * z \in X_x$ and $z \in X_x$. Since X_x is an ideal of X, it follows that $y \in X_x$ so that $f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(x)$ and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(x)$.

(ii) Suppose that $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ satisfies two conditions (3.2) and (3.3). Let $x, y \in X$ be such that $x*y \in X_a$ and $y \in X_a$. Then $f_{\mathcal{C}}(x*y) \leq f_{\mathcal{C}}(a)$, $g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(a)$, $f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$. Hence $f_{\mathcal{C}}(a) \geq \bigvee\{f_{\mathcal{C}}(x * y), f_{\mathcal{C}}(y)\}$ and $g_{\mathcal{C}}(a) \leq \bigwedge\{g_{\mathcal{C}}(x * y), g_{\mathcal{C}}(y)\}$, which imply from

(3.2) that $f_{\mathcal{C}}(a) \geq f_{\mathcal{C}}(x)$ and $g_{\mathcal{C}}(a) \leq g_{\mathcal{C}}(x)$. Thus $x \in X_a$. Obviously, $0 \in X_a$. Therefore X_a is an ideal of X. □

Theorem 3.20. *Let $\mathcal{C} = (f_{\mathcal{C}}, g_{\mathcal{C}})$ be a coupled $\mathcal{N}$-structure in a BH-algebra X. Then X_a is a coupled $\mathcal{N}$-ideal of X for any $a \in X$ if and only if*

(i) $f_{\mathcal{C}}(0) \leq f_{\mathcal{C}}(a), g_{\mathcal{C}}(0) \geq g_{\mathcal{C}}(a))$.

(ii) $(\forall x, y \in X)(f_{\mathcal{C}}(x * y) \leq f_{\mathcal{C}}(a)$ *and* $f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$ *imply* $f_{\mathcal{C}}(x) \leq f_{\mathcal{C}}(a))$.

(iii) $(\forall x, y \in X)(g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(a)$ *and* $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$ *imply* $g_{\mathcal{C}}(x) \geq g_{\mathcal{C}}(a))$.

Proof. Assume that X_a is a coupled $\mathcal{N}$-ideal of X. Then $0 \in X_a$ and so $f_{\mathcal{C}}(0) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(0) \geq g_{\mathcal{C}}(a)$. Let $x, y, z \in X$ be such that $f_{\mathcal{C}}(x * y) \leq f_{\mathcal{C}}(a), g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(a), f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$, and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$. Then $x * y, y \in X_a$. Since X_a is an ideal of X, we have $x \in X_a$. Hence $f_{\mathcal{C}}(x) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(x) \geq g_{\mathcal{C}}(a)$.

Conversely, consider X_a for any $a \in X$. Obviously, $0 \in X_a$ for any $a \in X$. Assume that $x * y, y \in X_a$. Then $f_{\mathcal{C}}(x * y) \leq f_{\mathcal{C}}(a), g_{\mathcal{C}}(x * y) \geq g_{\mathcal{C}}(a), f_{\mathcal{C}}(y) \leq f_{\mathcal{C}}(a)$, and $g_{\mathcal{C}}(y) \geq g_{\mathcal{C}}(a)$. It follows from hypothesis that $f_{\mathcal{C}}(x) \leq f_{\mathcal{C}}(a)$ and $g_{\mathcal{C}}(x) \geq g_{\mathcal{C}}(a)$. Hence $x \in X_a$. Thus X_a is a coupled $\mathcal{N}$-ideal of X.

4. Acknowledgements

The authors thank the referees for their valuable suggestions.

References

[1] S. S. Ahn and J. H. Lee, *Rough strong ideals in BH-algebras*, Honam Math. J. **32**(2010), 203-215.

[2] K. Iséki, *On BCI-algebras*, Mathe. Seminar Notes **8**(1980), 125-130.

[3] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Jpn. **23**(1978), 1-26.

[4] Y. B. Jun, S. S. Ahn and D. R. Prince Williams, *Coupled $\mathcal{N}$-structures and its applications in BCK/BCI-algebras*, Iranian J. Sci. Tech. (in press).

[5] Y. B. Jun, E. H. Roh and H. S. Kim, *On BH-algebras*, Scientae Math. **1**(1998), 347-354.

[6] D. Mundici, *MV-algebras are categorically equivalent to bounded commutative BCK-algebras*, Math. Jpn. **31**(1986), 889-894.

[7] J. Meng, *Implicative commutative semigroups are equivalent to a class of BCK-algebras*, Semigroup Forum **50**(1995), 89-96.

[8] J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa, Seoul, 1994.

[9] E. H. Roh and S. Y. Kim, *On BH^*-subalgebras of transitive BH^*-algebras*, Far East J. Math. Sci. **1**(1999), 255-263.
[10] Q. Zhang, Y. B. Jun and H. Roh, *On the branch of BH-algebras*, Sci. Math. Jpn. **54**(2001), 363-367.

Min Jeong Seo
Department of Mathematics Education, Dongguk University,
Seoul 100-715, Korea.
E-mail: cury13@naver.com

Sun Shin Ahn
Department of Mathematics Education, Dongguk University,
Seoul 100-715, Korea.
E-mail: sunshine@dongguk.edu