Honam Mathematical J. **34** (2012), No. 4, pp. 571–584
http://dx.doi.org/10.5831/HMJ.2012.34.4.571

ZERO-DIVISOR GRAPHS OF MULTIPLICATION MODULES

Sang Cheol Lee and Rezvan Varmazyar

Abstract. In this study, we investigate the concept of zero-divisor graphs of multiplication modules over commutative rings as a natural generalization of zero-divisor graphs of commutative rings. In particular, we study the zero-divisor graphs of the module $\mathbb{Z}_n$ over the ring $\mathbb{Z}$ of integers, where n is a positive integer greater than 1.

1. Introduction

Barnard first introduced the notion of multiplication modules in 1981 [6], and then E-Bast and Smith found various properties of multiplication modules to hold in 1988 [8]. On the other hand, Beck first introduced the notion of a zero-divisor graph of a ring in 1988 [7] from the view of colorings. Since then, others, such as in [2]-[4] have studied and modified these graphs, whose vertices are the zero-divisors of R, and found various properties to hold. Multiplication modules are natural generalizations of commutative rings, and hence it is natural for us to generalize zero-divisor graphs of commutative rings to those of multiplication modules.

Throughout this paper, R will denote a commutative ring with identity and M will denote a nonzero unitary R-module. For a subset S of M, we denote the set of all nonzero elements of S by S^* as usual.

We will consider the product $N * K$ of submodules N and K of a multiplication module M over a commutative ring R. Denote

$$\{x \in M \mid Rx * Ry = 0 \text{ for some nonzero element } y \text{ of } M\}$$

by $Z(_RM)$. We define $Z(R) = Z(_RR)$. We associate a zero-divisor graph $\Gamma(_RM)$ to a multiplication module M over a commutative ring R with vertices being elements of $Z(_RM)^*$. The two distinct vertices x, y are

Received October 19, 2012. Accepted November 18, 2012.
2010 Mathematics Subject Classification. 13C12, 13A15, 05C25.
Key words and phrases. Zero-divisor, Multiplication module.

adjacent if and only if $Rx * Ry = 0$ in M. This definition is extended from that of [2]. We define $\Gamma(R) = \Gamma({}_RR)$. It is known that $\Gamma(R)$ is connected with $diam\Gamma(R) \leq 3$ and $girth\Gamma(R) \leq 4$ (if $\Gamma(R)$ contains a cycle).

Section 2 deals with the zero-divisors of multiplication modules.

In Section 3, we compare the graph $\Gamma_0({}_RM)$ and the zero-divisor graph $\Gamma({}_RM)$ of a multiplication R-module M and primarily deal with the basic properties of $\Gamma_0({}_RM)$.

If we know how to draw the zero-divisor graph $\Gamma({}_RM)$ of a multiplication module over a commutative ring R, then it is easy to draw the graph $\Gamma_0({}_RM)$. Therefore, to clarify and simplify our discussion, we mainly deal with the zero-divisor graph $\Gamma({}_RM)$ in Section 4. We show in Theorem 4.3 that the diameter of $\Gamma({}_RM))$ is bounded above for every multiplication module M. Moreover, when M_1 and M_2 are finitely generated multiplication R-modules satisfying certain condition on R, we calculate the diameter of $\Gamma({}_R(M_1 \oplus M_2))$ in Theorem 4.8.

Let n be a positive integer greater than 1. Section 5 deals with the graph of $\mathbb{Z}_n$ as a $\mathbb{Z}$-module. To do so, we decompose the positive integer n into prime numbers, say

$$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r},$$

where $p_1, p_2, \cdots, p_r$ are all distinct prime numbers and $e_1, e_2, \cdots, e_r$ are all positive ingers. If $e_1, e_2, \cdots, e_r$ are all equal to 1, then the positive integer n is called be *square-free*. Now, assume that n is not square-free. Then by Theorem 5.1, the zero-divisor graph $\Gamma({}_{\mathbb{Z}}\mathbb{Z}_n)$ is not simple, which means that it has loops. We calculate the number of its loops. We consider the relationship between the proper subgroups of the group $(\mathbb{Z}_n, +)$ and the zero-divisor graph $\Gamma({}_{\mathbb{Z}}\mathbb{Z}_n)$.

2. Zero-Divisors of Multiplication Modules

In this section we define zero-divisors of multiplication modules. The notion of *a zero-divisor of a multiplication module* is different from that of *a zero-divisor on a module.*

An R-module M is called a *multiplication module* provided that for each submodule N of M there exists an ideal I of R such that $N = IM$. We say that I is a *presentation ideal* of N. Let N and K be submodules of a multiplication R-module M. Then there exist ideals I and J of R such that $N = IM$ and $K = JM$. The *product* of N and K, denoted by $N * K$, is defined to be $(IJ)M$. By [1], the product of N and K is independent of presentation ideals of N and K.

Definition 2.1. Let M be a multiplication R-module. An element x of M is called a *zero-divisor element* of M if there exists a nonzero element y of M such that $Rx * Ry = 0$ in M.

Remark 2.2. Let M be a multiplication R-module. The cyclic submodule Rx of M should not try to be identified with x by defining the equivalence relation $\sim$ on M like this: $x \sim y$ if and only if $Rx = Ry$, where x, y in M. If we identified Rx with x, we would have self-contradictory statements. See Example 5.2.

Let M be a multiplication module. The zero element of M is a zero-divisor because M is nonzero and the zero submodule of M can be presented by the zero ideal of R. Let $Z({}_RM)$ denote the set of all zero-divisors of M.

Let M be an R-module. Recall that an element a of R is called a *zero divisor* on M if there exists a nonzero element m in M such that $am = 0$ in M. Hence the zero-divisor graph $\Gamma({}_RM)$ is the empty graph if and only if M is a torsion-free module over an integral domain. The module theoretic results on zero-divisors on M can be seen in [9, Section 2-2]. However, from now on, we do not think of *zero-divisors on modules*, but rather *zero-divisors of multiplication modules.*

3. The Comparison of the Graphs $\Gamma_0({}_RM)$ and the Zero-Divisor Graphs $\Gamma({}_RM)$ of Multiplication R-Modules

We may consider a multiplication module M as a graph $\Gamma_0({}_RM)$ whose vertices are elements of M such that two different elements x, y of M are adjacent if and only if $Rx * Ry = 0$ in M. First of all, $\Gamma({}_RM)$ is a subgraph of $\Gamma_0({}_RM)$.

Lemma 3.1. *Let M be a multiplication R-module. Then in $\Gamma_0({}_RM)$, the zero element of M is adjacent to every element of $M\backslash\{0\}$, but every element of $M\backslash Z({}_RM)$ is adjacent only to the zero element of M.*

Proof. For any element x of $M\backslash\{0\}$, $R0 * Rx = 0$. However, for any two distinct elements x, y of $M\backslash Z({}_RM)$, $Rx * Ry \neq 0$. Hence the result follows. $\square$

Example 3.2. Every ring is a multiplication module over itself. Figure 1 is the graph $\Gamma_0({}_{\mathbb{Z}}\mathbb{Z}_8)$ of the $\mathbb{Z}$-module $\mathbb{Z}_8$ and in particular this is an example of Lemma 3.1 since $Z({}_{\mathbb{Z}}\mathbb{Z}_8) = \{0, 2, 4, 6\}$. Figure 2 is the zero-divisor graph $\Gamma({}_{\mathbb{Z}}\mathbb{Z}_8)$ of the $\mathbb{Z}$-module $\mathbb{Z}_8$ since $Z({}_{\mathbb{Z}}\mathbb{Z}_8)^* = \{2, 4, 6\}$.

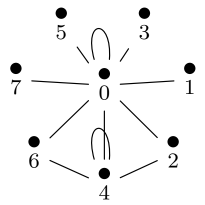


Figure 1. $\mathbf{\Gamma_0}({}_{\mathbb{Z}}\mathbb{Z}_\mathbf{8})$

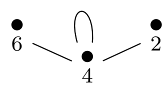


Figure 2. $\mathbf{\Gamma}({}_{\mathbb{Z}}\mathbb{Z}_\mathbf{8})$

Now, we can see that $\Gamma({}_RM)$ better illustrates the structure of $Z({}_RM)^*$. Hence in section 4 we consider and investigate the zero-divisor graphs $\Gamma({}_RM)$ of multiplication R-modules M. For each multiplication R-module M, $\Gamma_0({}_RM)$ has the fundamental property as follows. Here we denote the cardinality of a set S by $|S|$ as usual.

Theorem 3.3. *Let M be a multiplication R-module with $|M| \geq 3$. Let x, y and z be distinct vertices of $\Gamma_0({}_RM)$ such that x is adjacent to y and y is not adjacent to z. Then there exists a nonzero element m in $Ry * Rz$ such that $Rx * Rm = 0$.*

Proof. Since y is not adjacent to z, we see that $Ry*Rz \neq 0$. However, since x is adjacent to y, it follows from the independent property of presentation ideals of the zero submodule and Rz that

$$Rx * (Ry * Rz) = (Rx * Ry) * Rz = 0.$$

Take a nonzero element m in $Ry * Rz$. Then $Rm \subseteq Ry * Rz$ and so

$$Rx * Rm \subseteq Rx * (Ry * Rz) = 0.$$

Hence $Rx * Rm = 0$, as required. □

We adopt the same notations as in [2] to restate them. Let Γ be a graph. The number of edges in a path between two distinct vertices in Γ is called the *length* of the path. For distinct vertices x and y of Γ, let $d(x, y)$ be the length of the shortest path from x to y ($d(x, y) = \infty$ if there is no such path). Even though for certain distinct two vertices x and y in Γ we have a path of length n between x and y, we can not say that $d(x, y) = n$. Of course, if for certain distinct two vertices x and y

in Γ we have a path of length n between x and y, then $d(x, y) \le n$. The *diameter* of Γ is

$$diam(\Gamma) = sup\,\{\, d(x, y) \,|\, x \text{ and } y \text{ are distinct vertices of } \Gamma\}.$$

($diam(\Gamma) = -\infty$ if $\Gamma = \emptyset$).

Recall that a graph is *connected* if there is a path between any two distinct vertices. Let M be a multiplication R-module. For any two distinct vertices x, y of $\Gamma_0({}_RM)$, x is adjacent to the vertex 0 and the vertex 0 is adjacent to y and so there is a path between x and y. Hence $\Gamma_0({}_RM)$ is a connected graph with $diam(\Gamma_0({}_RM)) \le 2$.

4. The Zero-Divisor Graphs of Multiplication Modules

If we know how to draw the zero-divisor graph $\Gamma({}_RM)$ of a multiplication module over a commutative ring R, then it is easy to draw the graph $\Gamma_0({}_RM)$. Hence to clarify and simplify our discussion, we mainly deal with the zero-divisor graph $\Gamma({}_RM)$ in this section. Compare the following result with Theorem 3.3.

Lemma 4.1. *Let M be a multiplication R-module. Let x, y and z be elements of M such that $Rx * Ry = 0$ and $Ry * Rz \neq 0$. Then the following statements are true.*

1. *For every element m in $(Ry * Rz)^*$, $Rx * Rm = 0$.*
2. *If $x \neq 0$, then $(Ry * Rz)^* \subseteq Z({}_RM)^*$.*

Proof. (1) Let $m \in (Ry * Rz)^*$. Then

$$Rx * Rm \subseteq Rx * (Ry * Rz) = (Rx * Ry) * Rz = 0 * Rz = 0,$$

and so $Rx * Rm = 0$.

(2) Let $m \in (Ry * Rz)^*$. Then by (1), $Rx * Rm = 0$. If $x \neq 0$, then $m \in Z({}_RM)^*$. Hence the proof is completed. □

If M is a multiplication module, then in $\Gamma({}_RM)$, we can construct an intermediate vertex between two given distinct vertices under certain conditions. This construction is given below.

Lemma 4.2. *Let M be a multiplication module. Let x, x_1, y_1 and y be vertices of $\Gamma({}_RM)$ such that $x \neq x_1$, $y \neq y_1$, and $x_1 \neq y_1$. Assume that x is not adjacent to y and x_1 is not adjacent to y_1. If x is adjacent to x_1 and y is adjacent to y_1, then $(Rx_1 * Ry_1)^* \subseteq Z({}_RM)^*$ and there exists an element z in $(Rx_1 * Ry_1)^*$ such that x is adjacent to z and z is adjacent to y.*

Proof. Since x is adjacent to x_1, x_1 is not adjacent to y_1, and $x \neq 0$, it follows from Lemma 4.1(2) that $(Rx_1 * Ry_1)^* \subseteq Z({}_RM)^*$.

Now take an element z in $(Rx_1 * Ry_1)^*$. Since x is adjacent to x_1 and x_1 is not adjacent to y_1, it follows from Lemma 4.1(1) that $Rx * Rz = 0$. Also, since y is adjacent to y_1 and x_1 is not adjacent to y_1, it follows from Lemma 4.1(1) again that $Ry * Rz = 0$. Hence x is adjacent to z and z is adjacent to y. □

Hence in $\Gamma({}_RM)$, we have connected the following two paths with only one edge

to construct the following path of length 2 such that $d(x, y) \leq 2$.

Let M be a multiplication R-module. Let n be a nonnegative integer. For a submodule N of M, the *n-th power* of the submodule N is defined to be

$$N^n = \begin{cases} M & \text{if } n = 0 \\ \underbrace{N * N * \cdots * N}_{n \text{ times}} & \text{if } n \geq 1 \end{cases}$$

The following result is a generalization of [2, Theorem 2.3].

Theorem 4.3. *Let M be a multiplication module. The zero-divisor graph $\Gamma({}_RM)$ is connected and the following statements are true.*

1. *If $Z({}_RM)^* = \emptyset$, then $diam(\Gamma({}_RM)) = -\infty$.*
2. *If $Z({}_RM)^*$ has only one element, then $diam(\Gamma({}_RM)) = 0$.*
3. *If $|\, Z({}_RM)^* \,| \geq 2$, then $1 \leq diam(\Gamma({}_RM)) \leq 3$.*

Proof. (1) This follows from the definition.

(2) Assume that $Z({}_RM)^*$ has only one element, say x_0. Then $Rx_0 * Rx_0 = 0$. Hence $\Gamma({}_RM)$ has a loop on vertex x_0 and $diam(\Gamma({}_RM)) = 0$.

(3) Assume that $|\, Z({}_RM)^* \,| \geq 2$. If, for any two distinct vertices of $\Gamma({}_RM)$, x is adjacent to y, then $\Gamma({}_RM)$ is complete and $diam(\Gamma({}_RM)) = 1$.

Assume that x and y are distinct vertices of $\Gamma({}_RM)$ with $Rx * Ry \neq 0$. Since $x, y \in Z({}_RM)^*$, there exist nonzero elements x_1, y_1 of M such that $Rx * Rx_1 = 0$ and $Ry * Ry_1 = 0$. Further, $x_1, y_1 \in Z({}_RM)^*$.

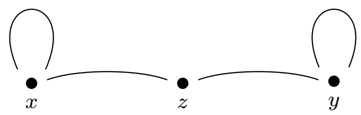
(i) Assume that $x = x_1$ and $y = y_1$. Take $z \in (Rx * Ry)^*$. Suppose that $z = x$. Then $Rx = Rz \subseteq Rx * Ry$ and so

$$Rx \subseteq Rx * Ry \subseteq (Rx * Ry) * Ry = Rx * (Ry)^2 = Rx * 0 = 0.$$

Hence $x = 0$. This contradicts to the fact that $x \in Z({}_RM)^*$. Thus $z \neq x$. By a similar proof, we can show that $z \neq y$. Moreover,

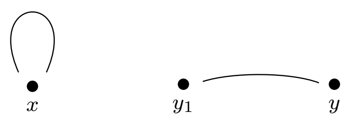
$$Rx * Rz \subseteq Rx * (Rx * Ry) = (Rx)^2 * Ry = 0 * Ry = 0$$

and thus $Rx * Rz = 0$. By a similar proof, we can show that $Rz * Ry = 0$. Therefore

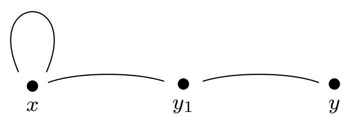


is a path of length 2 between x and y, and $d(x, y) = 2$.

(ii) Assume that $x = x_1$ and $y \neq y_1$. Consider the following auxiliary figure.

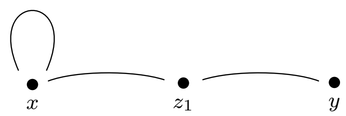


Since x is not adjacent to y, we have $x_1 \neq y_1$. If $Rx_1 * Ry_1 = 0$, then



is a path of length 2 between x and y, and $d(x, y) = 2$. Assume that $Rx_1 * Ry_1 \neq 0$. Since $y \neq 0$, it follows from Lemma 4.1 that there exists z_1 in $(Rx_1 * Ry_1)^* \subseteq Z({}_RM)^*$ such that $Rz_1 * Ry = 0$.

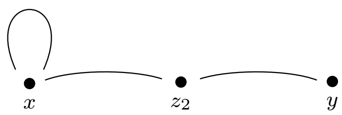
If $Rx * Rz_1 = 0$, then



is a path of length 2 between x and y, and $d(x, y) = 2$. If $Rx * Rz_1 \neq 0$, it follows from Lemma 4.1 again that there exists z_2 in $(Rx * Rz_1)^* \subseteq Z({}_RM)^*$ such that $Rz_2 * Ry = 0$. Moreover,

$$Rx * Rz_2 \subseteq Rx * (Rx * Rz_1) = (Rx)^2 * Rz_1 = 0 * Rz_1 = 0$$

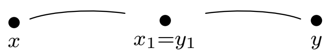
and so $Rx * Rz_2 = 0$. Hence



is a path of length 2 between x and y, and $d(x,\, y) = 2$.

(iii) A similar argument holds if $x \neq x_1$ and $y = y_1$.

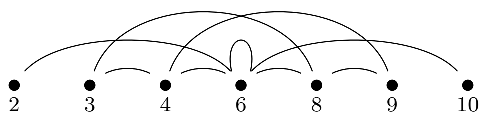
(iv) Assume that $x \neq x_1$ and $y \neq y_1$. If $x_1 = y_1$, then



is a path of length 2 between x and y, and $d(x,\, y) = 2$. Assume that $x_1 \neq y_1$. If x_1 is adjacent to y_1, then

is a path of length 3 between x and y. If x is adjacent to y_1 or x_1 is adjacent to y, then $d(x,\, y) = 2$; otherwise $d(x,\, y) = 3$. Assume that x_1 is not adjacent to y_1. Then by Lemma 4.2, there is a path between x and y with $d(x,\, y) = 2$. Therefore the zero-divisor graph $\Gamma({}_RM)$ is connected and $diam(\Gamma({}_RM)) = 1,\ 2,$ or 3. □

For each positive integer n, the ring $\mathbb{Z}_n$ is a multiplication $\mathbb{Z}_n$-module. Hence by Theorem 4.3, $diam(\Gamma(\mathbb{Z}_n)) \leq 3$. For example, the diameters of $\Gamma(\mathbb{Z}_2)$, $\Gamma(\mathbb{Z}_4)$, $\Gamma(\mathbb{Z}_9)$, $\Gamma(\mathbb{Z}_6)$, and $\Gamma(\mathbb{Z}_{12})$ are $-\infty$, 0, 1, 2 and 3, respectively. To check that $diam(\Gamma(\mathbb{Z}_{12})) = 3$, we give the zero-divisor graph $\Gamma(\mathbb{Z}_{12})$ below.



Corollary 4.4. *Let M be a multiplication module. If $|\, Z({}_RM)^* \,| \geq 2$ and the zero-divisor graph $\Gamma({}_RM)$ is not complete, then for any two distinct vertices x, y of $\Gamma({}_RM)$, $d(x,\, y) = 2$ or 3.*

Let N be a submodule of a multiplication module M. N is a *nilpotent submodule* of M if $N^n = 0$ for some positive integer n. If N is a nilpotent submodule of M, then every submodule of N is also nilpotent.

Definition 4.5. Let M be a multiplication R-module. An element x of M is called a *nilpotent element* of M if the cyclic submodule Rx of M is nilpotent.

For a multiplication module M, let $N(_RM)$ denote the set of all nilpotent elements of M. Then clearly, the zero element of M is nilpotent. $N(_RM)$ is a submodule of M. Now, assume that M is a distributive finitely generated module over a Noetherian ring R. Then $N(_RM)$ is a finitely generated submodule of M. It follows from [6, Proposition 7] that $N(_RM)$ is a multiplication module. Hence by the last paragraph of section 3, $diam(\Gamma_0(N(_RM))) \leq 2$, and by Theorem 4.3, $diam(\Gamma(N(_RM))) \leq 3$.

Proposition 4.6. *Let M be a nonzero multiplication R-module. Then the following statements are true.*

1. $N(_RM)^* \subseteq Z(_RM)^*$.
2. *Assume that x and y are any two distinct vertices of $\Gamma(_RM)$ such that $Rx * Ry \neq 0$ and $x \in N(_RM)^*$. Then $d(x, y) = 2$.*

Proof. (1) Let $x \in N(_RM)^*$. There exists a positive integer n such that $(Rx)^n = 0$. By the well-ordering property of integers, there exists the least positive integer s such that $(Rx)^s = 0$. Then $(Rx)^{s-1} \neq 0$. Take a nonzero element y in $(Rx)^{s-1}$. Then

$$Rx * Ry \subseteq Rx * (Rx)^{s-1} = (Rx)^s = 0$$

and so $Rx*Ry = 0$. Hence $x \in Z(_RM)^*$. Therefore $N(_RM)^* \subseteq Z(_RM)^*$.

(2) Let x, y be any two distinct vertices of $\Gamma(_RM)$. Assume that $Rx * Ry \neq 0$. Since $y \in Z(_RM)^*$, there exists $y_1 \in M^*$ such that $Ry*Ry_1 = 0$. If $Rx*Ry_1 = 0$, then $d(x, y) = 2$. Assume that $Rx*Ry_1 \neq 0$. Let $x \in N(_RM)^*$. Consider the set $S = \{ n \in \mathbb{N} \mid (Rx)^n * Ry_1 = 0 \}$. Then S is a nonempty subset of $\mathbb{N}$. By the well-ordering property of integers, S has the least element, say t. Then $(Rx)^t * Ry_1 = 0$, but $(Rx)^{t-1} * Ry_1 \neq 0$. Take $z \in (Rx)^{t-1} * Ry_1$. Then $Rx * Rz = 0$ and $Rz * Ry = 0$. Hence $d(x, y) = 2$. Therefore the proof is completed. □

Lemma 4.7. *Let M be the direct sum of two R-modules M_1 and M_2. If M, M_1 and M_2 are multiplication R-modules, then for any two elements (x_1, x_2) and (y_1, y_2) of M,*

$$R(x_1, x_2) * R(y_1, y_2) = (Rx_1 * Ry_1) \oplus (Rx_2 * Ry_2).$$

Proof. If M is a multiplication module, then there exist ideals I and J of R such that $R(x_1, x_2) = IM$ and $R(y_1, y_2) = JM$. Then

$$\begin{aligned} Rx_1 \oplus Rx_2 &= R(x_1, x_2) = IM = IM_1 \oplus IM_2, \\ Ry_1 \oplus Ry_2 &= R(y_1, y_2) = JM = JM_1 \oplus JM_2. \end{aligned}$$

So, $Rx_1 = IM_1$, $Rx_2 = IM_2$, $Ry_1 = JM_1$, and $Ry_2 = JM_2$. Hence

$$\begin{aligned} R(x_1,\, x_2) * R(y_1,\, y_2) &= (IJ)M \\ &= (IJ)M_1 \oplus (IJ)M_2 \\ &= (Rx_1 * Ry_1) \oplus (Rx_2 * Ry_2). \end{aligned}$$

Hence the proof is completed. □

For a multiplication module M, let

$$\mathcal{P}(_RM) = \{\, Rx * Ry \mid x,\, y \in M \,\}.$$

Theorem 4.8. *Let M_1, M_2 be finitely generated multiplication R-modules such that $(0 :_R M_1) + (0 :_R M_2) = R$. Then the following statements are true.*

1. *If $\mathcal{P}(M_1) = \{\,0\,\}$ and $\mathcal{P}(M_2) = \{\,0\,\}$, then $\Gamma(M_1 \oplus M_2)$ is complete.*
2. *$max\{\, diam(\Gamma(M_1)),\, diam(\Gamma(M_2))\,\} \le diam(\Gamma(M_1 \oplus M_2)) \le 3$*

Proof. Let $M = M_1 \oplus M_2$. Then by [8, Corollary 2.3], M is a multiplication module.

(1) Let $(x_1,\, x_2)$ and (y_1, y_2) be any two distinct elements of $Z(_RM)^*$. Then by Lemma 4.7 and by our hypothesis,

$$R(x_1,\, x_2) * R(y_1,\, y_2) = (Rx_1 * Ry_1) \oplus (Rx_2 * Ry_2) = 0 * 0 = 0.$$

Hence $\Gamma(_RM)$ is complete.

(2) Assume first that there is an edge between x_1 and x_2 in $\Gamma(M_1)$ and an edge between y_1 and y_2 in $\Gamma(M_2)$. Then $(x_1,\, y_1)$ and $(x_2,\, y_2)$ are two distinct vertices of $\Gamma(_RM)$. Moreover,

$$R(x_1,\, y_1) * R(x_2,\, y_2) = (Rx_1 * Rx_2) \oplus (Ry_1 * Ry_2) = 0 * 0 = 0.$$

Hence there is an edge between $(x_1,\, y_1)$ and $(x_2,\, y_2)$ in $\Gamma(_RM)$.

Now, let $m = diam(\Gamma(M_1))$ and $n = diam(\Gamma(M_2))$. Then by Theorem 4.3, we see that $m,\, n \in \{0,\, 1,\, 2,\, 3\}$. We may assume that $m \le n$. Note that there are vertices $x,\, x'$ in $\Gamma(M_1)$ and vertices $y,\, y'$ in $\Gamma(M_2)$ such that $d(x,\, x') = m$ and $d(y,\, y') = n$. Then there exists a path

$$x = x_0 - x_1 - x_2 - \cdots - x_{m-1} - x_m = x'$$

in $\Gamma(M_1)$ and a path

$$y = y_0 - y_1 - y_2 - \cdots - y_{n-1} - y_n = y'$$

in $\Gamma(M_2)$. Then using the previous statement it is easy to check that

$$(x_0,\, y_0) - (x_1,\, y_1) - (x_2,\, y_2) - \cdots - (x_m,\, y_m) - (0, y_{m+1}) - \cdots - (0, y_n).$$

is a path between (x_0, y_0) and $(0, y_n)$. If necessary, let $x_i = 0$ for each $i \in \{m+1, \cdots, n\}$. Then

$$(x_0, y_0)-(x_1, y_1)-(x_2, y_2)-\cdots-(x_m, y_m)-(x_{m+1}, y_{m+1})-\cdots-(x_n, y_n).$$

is a path between (x_0, y_0) and (x_n, y_n). Hence $d((x_0, y_0), (x_n, y_n)) \leq n$. We show that $d((x_0, y_0), (x_n, y_n)) = n$. To do this, suppose that $d((x_0, y_0), (x_n, y_n)) < n$. Then there are nonconsecutive integers s and t in $\{0, 1, 2, \cdots, n\}$ such that two vertices $(x_s, y_s)$and (x_t, y_t) of $\Gamma(_RM)$ can be drawn with an edge. Hence $R(x_s, y_s) * R(x_t, y_t) = 0$. In particular, $Ry_s * Ry_t = 0$. Thus $d(y_0, y_n) < n$, and so $n = d(y, y') = d(y_0, y_n) < n$. This contradiction shows that $d((x_0, y_0), (x_n, y_n)) = n$. From this, we can get that $diam(\Gamma(_RM)) \geq n$. Therefore, by Theorem 4.3,

$$max\{\, diam(\Gamma(M_1)), diam(\Gamma(M_2))\,\} \leq diam(\Gamma(_RM)) \leq 3,$$

as required. □

While discussing, Professor Maimani asked us whether there are M_1, M_2 satisfying the equation $dim(M_1 \oplus M_2) = 3$ in Theorem 4.8. We give an example of this below.

Example 4.9. Let $M_1 = \mathbb{Z}_{12}$, $M_2 = \mathbb{Z}_5$. Then $(9, 4) - (4, 0) - (6, 0) - (2, 3)$ is a shortest path (of length 3) between $(9, 4)$ and $(2, 3)$. Therefore, $dim(M_1 \oplus M_2) = 3$.

5. the zero-divisor graphs of $\mathbb{Z}_n$

Let p be a prime number. Then $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \oplus \mathbb{Z}_p$ are non-isomorphic rings. The former is a multiplication module over the ring $\mathbb{Z}$ of integers. However, the latter is not a multiplication module over the ring $\mathbb{Z}$ of integers. On the other hand, if p and q are distinct prime numbers, then $\mathbb{Z}_{pq}$ and $\mathbb{Z}_p \oplus \mathbb{Z}_q$ are isomorphic rings and they are both multiplication modules over the ring $\mathbb{Z}$ of integers. Hence, throughout this section, we consider and investigate the $\mathbb{Z}$-module $\mathbb{Z}_n$, where $n \geq 2$. First of all, $\Gamma(\mathbb{Z}_n) = \Gamma(_{\mathbb{Z}}\mathbb{Z}_n)$.

A graph Γ is said to be *simple* if Γ has no loop. For a multiplication R-module M, the zero-divisor graph $\Gamma(_RM)$ is not necessarily simple. For example, for a ring $\mathbb{Z}_6$, $\Gamma(\mathbb{Z}_6)$ is simple. However, for a ring $\mathbb{Z}_8$, $\Gamma(\mathbb{Z}_8)$ is not simple because it has a loop on vertex 4.

Compare the following result with [5, Corollary 4.6].

Theorem 5.1. *Assume that n is a positive integer greater than 1 and n is not a prime number. Then the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ is simple if and only if n is square-free.*

Proof. Assume that the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ is simple. We show that n is square-free. Suppose to the contrary that n is not square-free. Then there exist positive integers u and v such that $n = u^2v$. Let $h = uv$. Then $h \in \mathbb{Z}_n$. In $\mathbb{Z}_n$, $h^2 = 0$. Hence $\Gamma(\mathbb{Z}_n)$ has a loop on vertex h. This contradiction shows that n is square-free.

Now, assume that n is square-free. Then n can be factored as follows:

$$n = p_1 p_2 \cdots p_r,$$

where $p_1, p_2, \cdots, p_r$ are distinct prime numbers. Then $\Gamma(\mathbb{Z}_n)$ is simple. For otherwise, it has a loop on a vertex, say x. Then $x^2 = 0$ in $\mathbb{Z}_n$. So, $n \mid x^2$ in $\mathbb{Z}$. This implies that each p_i is a divisor of x^2 and hence a divisor of x. Since $p_1, p_2, \cdots, p_r$ are distinct, we can see that their product $p_1 p_2 \cdots p_r$ is a divisor of x. Hence $x = 0$ in $\mathbb{Z}_n$, so that $0 = x \in Z(\mathbb{Z}_n)^*$. This is a contradiction. $\square$

For example, the zero-divisor graph $\Gamma(\mathbb{Z}_n)$, where n is of the form $n = 2^s$, $s \geq 2$, is not simple. There are at least $2^k - 1$ loops in the the zero-divisor graph $\Gamma(\mathbb{Z}_{2^{2k}})$, where $k \geq 1$, since there is a loop on each of its vertices $m \cdot 2^k$, $1 \leq m \leq 2^k - 1$. Also, there are at least $2^k - 1$ loops in the zero-divisor graph $\Gamma(\mathbb{Z}_{2^{2k+1}})$, where $k \geq 1$, since there is a loop on each of its vertices $m \cdot 2^{k+1}$, $1 \leq m \leq 2^k - 1$.

Now let us see what happens if we define the equivalence relation $\sim$ on M as in Remark 2.2.

Example 5.2. Consider the ring $\mathbb{Z}_6$. The zero-divisor graph of the ring $\mathbb{Z}_6$ is $2 - 3 - 4$. According to our construction, the zero-divisor graph of the module $\mathbb{Z}_6$ over itself is $2 - 3 - 4$. There will be no problem with our construction. However, according to the construction defining the equivalence relation on M as in Remark 2.2, the zero-divisor graph of the module $\mathbb{Z}_6$ over itself is $[2](= [4]) - [3]$. The graphs are different. In other words, if we identify Rx with x, where $x \in M$, we never reach our goal saying that the zero-divisor graph of a ring R is identical to the zero-divisor graph of the multiplication module R over itself.

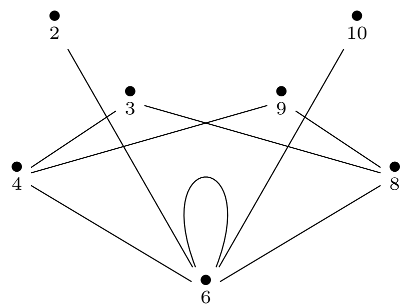
Lemma 5.3. *The class of all the proper subgroups of the group $(\mathbb{Z}_n, +)$ is equal to $\{\mathbb{Z}_n x \mid x \in Z(\mathbb{Z}_n)\}$, where $n \geq 2$.*

Consider the ring $\mathbb{Z}_n$, where $n \geq 2$. The ring can be viewed as a module over the ring $\mathbb{Z}$ of integers since the group $(\mathbb{Z}_n,\ +)$ is Abelian. Hence Lemma 5.3 can be recast as follows: the class $\mathcal{P}$ of all the proper submodules of the $\mathbb{Z}$-module $\mathbb{Z}_n$ is equal to $\{\mathbb{Z}_n x \mid x \in Z({}_{\mathbb{Z}}\mathbb{Z}_n)\}$. Hence $\mathcal{P}$ can be obtained from the set $Z({}_{\mathbb{Z}}\mathbb{Z}_n)$ of vertices of the the zero-divisor graph $\Gamma({}_{\mathbb{Z}}\mathbb{Z}_n)$.

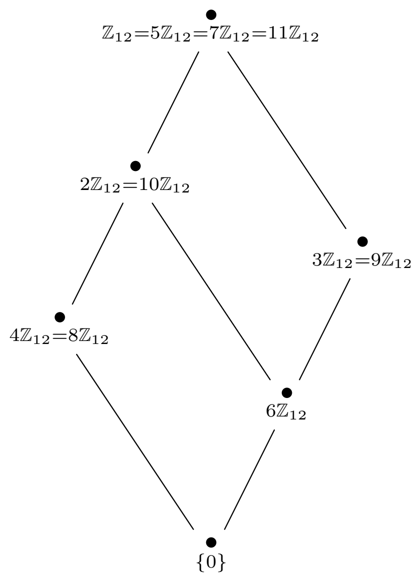
Corollary 5.4. *The lattice diagram of the group $(Z_n,\ +)$ is obtained from the zero-divisor graph of the $\mathbb{Z}$-module $\mathbb{Z}_n$, where $n \geq 2$.*

This corollary suggests us that we can draw $\Gamma({}_{\mathbb{Z}}\mathbb{Z}_n)$ by the lattice diagram of the group $(\mathbb{Z}_n,\ +)$ so that we can get the graph. The example of this is given below. Compare Example 5.5 with [10, Example 1.11].

Example 5.5.



$\mathbf{\Gamma}({}_{\mathbb{Z}}\mathbb{Z}_{\mathbf{12}})$



The lattice diagram of the group $(\mathbb{Z}_{\mathbf{12}},\ +)$

From the right side diagram, first delete the points $\mathbb{Z}_{12}$ and $\{0\}$ and all of the lines. And then introduce the rectangular coordinate system by taking the point $6\mathbb{Z}_{12}$ as the original point of the system. Now rotate all the remaining points about the y-axis through $180°$ and get all the points on the left side graph which are the non-zero zero-divisors of the $\mathbb{Z}$-module $\mathbb{Z}_{12}$. Finally, draw the lines between x and y if $\mathbb{Z}x * \mathbb{Z}y = 0$, where x, y in $\mathbb{Z}_{12}$. The resulting graph $\Gamma({}_{\mathbb{Z}}\mathbb{Z}_{12})$ is essentially the same as in the graph just prior to Corollary 4.4.

Acknowledgement. The authors appreciate the referee's valuable suggestions and we would like to thank Dr. Nichole Hansen for his feedback.

References

[1] R. Ameri, On the prime submodules of multiplication modules, *International Journal of Mathematics and Mathematical Sciences* **27**(2003), 1715–1724.

[2] D. F. Anderson and P. S. Livingston, The Zero-Divisor Graph of a Commutative Ring, *J. Algebra* **217**(1999), 434–447.

[3] S. Akbari and A. Mohammadian, On the zero-divisor graph of a commutative ring, *J. Algebra* **274** (2004), 847–855.

[4] S. Akbari, H. R. Maimani, and S. Yassemi, Zer-divisor graph is planar or a complete r-partite graph, *J. Algebra* **270** (2003) 169–180.

[5] Michael Axtell, Joe Stickers, and Wallace Trampbach, Zero-divisor ideals and realizable zero-divisor graphs, *Involve a journal of mathematics* **2**(1)(2009), 17–27.

[6] A. Barnard, Multiplication Modules, *J. Algebra* **71**(1981), 174–178.

[7] I. Beck, Coloring of Commutative Rings, *J. Algebra* **116**(1988), 208–226.

[8] Z. A. El-Bast and P. F. Smith, Multiplication Modules, *Comm. in Algebra* **16** (1988), 755–779.

[9] Irving Kaplansky, *Commutative Rings*, The University of Chicago Press, 1974.

[10] Sandra Spiroff and Cameron Wickham, A Zero-Divisor Graph Determined by Equivalence Classes of Zero Divisors, *Comm. in Algebra* **39**(7) (2011), 2338–2348.

Sang Cheol Lee
Department of Mathematics Education, and Institute of Pure and Applied Mathematics, Chonbuk National University,
Jeonju, Jeonbuk 561-756, Korea.
E-mail: scl@jbnu.ac.kr

Rezvan Varmazyar
Department of Mathematics, Islamic Azad University, Khoy Branch,
Khoy 58168-44799, Iran.
E-mail: varmazyar@iaukhoy.ac.ir