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NEIGHBORHOOD STRUCTURES IN ORDINARY SMOOTH TOPOLOGICAL SPACES

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Abstract. We construct a new definition of a base for ordinary smooth topological spaces and introduce the concept of a neighborhood structure in ordinary smooth topological spaces. Then, we state some of their properties which are generalizations of some results in classical topological spaces.

1. Introduction

In 1985, Sostak [6] defined a fuzzy topology τ on a nonempty set X as a mapping $\tau : I^X \to I$ satisfying three axioms, where I^X denotes the set of all fuzzy sets in X. He considered the degree of openness of fuzzy sets, gave some basic rules and proved how such an extension can be done. In 1992, Chattopadhyay et al. [1] studied the fuzzy topological spaces in Sostak's sense. In the same year, Ramadan [5] introduced similar concepts under the name of smooth topological spaces working in terms of lattices L and L' instead of $I = [0, 1]$. In particular, Demirci [3] introduced the concepts of neighborhood structures in smooth topological spaces. Moreover, Ying [7] investigated fuzzifying topological spaces (called ordinary smooth topological spaces by Hur et al. [4]) considering of degree of openness of ordinary subsets. Recently, Chae et al. [2] constructed the set OST(X) of all ordinary smooth topologies on a set X and studied it in the sense of a lattice.

In this paper,we construct a new definition of a base for ordinary smooth topological spaces and introduce the concept of a neighborhood structure in ordinary smooth topological spaces. Then, we state some

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of their properties which are generalizations of some results in classical topological spaces.

2. Preliminaries

Let $2 = \{0, 1\}$ and let 2^X denote the set of all ordinary subsets of X.

Definition 2.1[4]. Let X be an nonempty set. Then a mapping $\tau : 2^X \to I$ is called an *ordinary smooth topology* (in short, *ost*) on X or a *gradation of openness of ordinary subsets* of X if τ satisfies the following axioms:

(OST$_1$) $\tau(\emptyset) = \tau(X) = 1$.

(OST$_2$) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$, $\forall A, B \in 2^X$.

(OST$_3$) $\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha)$, $\forall \{A_\alpha\} \subset 2^X$.

The pair (X, τ) is called an *ordinary smooth topological space* (in short, *osts*). We will denote the set of all osts's on X as OST(X).

Remark 2.2. Ying [7] called the mapping $\tau : 2^X \to I$ [resp. $\tau : I^X \to 2$ and $\tau : I^X \to I$] satisfying the axioms in Definition 2.1 as a *fuzzyfying topology* [resp. *fuzzy topology* and *bifuzzy topology*] on X.

Remark 2.3. If $I = 2$, then Definition 2.1 coincides with the known definition of the classical topology.

Definition 2.4. Let X be a nonempty set. Then a mapping $\mathcal{C} : 2^X \to I$ is called an *ordinary smooth cotopology* (in short, *osct*) on X or a *gradation of closedness of ordinary subsets* of X if $\mathcal{C}$ satisfies the following axioms :

(OSCT$_1$) $\mathcal{C}(\emptyset) = \mathcal{C}(X) = 1$.

(OSCT$_2$) $\mathcal{C}(A \cup B) \geq \mathcal{C}(A) \wedge \mathcal{C}(B)$, $\forall A, B \in 2^X$.

(OSCT$_3$) $\mathcal{C}(\bigcap_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mathcal{C}(A_\alpha)$, $\forall \{A_\alpha\} \subset 2^X$.

The pair $(X, \mathcal{C})$ is called an *ordinary smooth cotopological space* (in short, *oscts*). We will denote the set of all oscts's on X as OSCT(X).

Remark 2.5. If $I = 2$, then Definition 2.4 also coincides with the known definition of the classical topology.

The following is the immediate result of Definition 2.1 and 2.4.

Result 2.A[4, Proposition 2.7]. Let X be a nonempty set. We define two mappings f : OST(X) $\to$ OSCT(X) and g : OSCT(X) $\to$ OST(X) as follows, respectively :

$[f(\tau)](A) = \tau(A^c)$, $\forall \tau \in$ OST(X), $\forall A \in 2^X$

and

$[g(\mathcal{C})](A) = \mathcal{C}(A^c)$, $\forall \mathcal{C} \in$ OSCT(X), $\forall A \in 2^X$.

Then f and g are well-defined. Furthermore $g \circ f = id_{\text{OST(X)}}$ and $f \circ g = id_{\text{OSCT(X)}}$.

Remark 2.6. Let $f(\tau) = \mathcal{C}_\tau$ and $g(\mathcal{C}) = \tau_{\mathcal{C}}$. Then, Result 2.A, we can easily see that $\tau_{\mathcal{C}_\tau} = \tau$ and $\mathcal{C}_{\tau_{\mathcal{C}}} = \mathcal{C}$.

Definition 2.7[4]. Let (X, τ) be an osts and let $r \in I$. Then we define two ordinary subsets of X as follows :

$[\tau]_r = \{A \in 2^X : \tau(A) \geq r\}$

and

$[\tau]^*_r = \{A \in 2^X : \tau(A) > r\}$.

We call these the $r-$*level* set and the *strong r-level* set of τ, respectively.

It is clear that $[\tau]_0 = 2^X$, the classical discrete topology on X and $[\tau]^*_1 = \emptyset$. Also it can be easily seen that $[\tau]^*_r \subset [\tau]_r$ for each $r \in I$.

Result 2.B[4, Proposition 2.10]. Let (X, τ) be an osts and let $T(X)$ be the set of all classical topologies on X. Then :

(a) $[\tau]_r \in$ T(X), $\forall r \in I$.

(a)′ $[\tau]^*_r \in$ T(X), $\forall r \in I_1$.

(b) For any $r, s \in I$, if $r \leq s$, then $[\tau]_s \subset [\tau]_r$ and $[\tau]^*_s \subset [\tau]^*_r$.

(c) $[\tau]_r = \bigcap_{s<r} [\tau]_s$, $\forall r \in I_0$.

(c)′ $[\tau]^*_r = \bigcup_{s>r} [\tau]^*_s$, $\forall r \in I_1$, where $I_0 = (0, 1]$ and $I_1 = [0, 1)$.

3. Main Results

For a mapping $t : 2^X \to I$ and $r \in I_1$, let us define the family $[t]^*_r = \{A \in 2^X : t(A) > r\}$ which will play an important role in our study. From Result 2.B, it is clear that if $t \in$ OST(X), then $[t]^*_r \in T(X)$.

Definition 3.1. Let (X, τ) be an ordinary smooth topological space. Then a mapping $\beta : 2^X \to I$ is called an *ordinary smooth base* for τ if $[\beta]_r^*$ is a classical base for $[\tau]_r^*$.

Let (X, T) be a classical topological space and for each $p \in X$ and let $\mathcal{N}_T(p)$ denote the classical neighborhood system of p.

The following is the characterization of Definition 3.1.

Theorem 3.2. Let (X, τ) be an ordinary smooth topological space. Then a mapping $\beta : 2^X \to I$ is an ordinary smooth base for τ if and only if for each $r \in I_1$ and each $p \in X$, if $A \in \mathcal{N}_{[\tau]_r^*}(p)$, then there exists $B \in [\beta]_r^*$ such that $p \in B \subset A$.

Proof. ($\Rightarrow$) Suppose $\beta : 2^X \to I$ is an ordinary smooth base for τ. Then, by Definition 3.1, $[\beta]_r^*$ is a classical base for $[\tau]_r^*$ for each $r \in I_1$. For each $p \in X$, let $A \in \mathcal{N}_{[\tau]_r^*}(p)$. Then there exists $U \in [\tau]_r^*$ such that $p \in U \subset A$. Since $U \in [\tau]_r^*$, there exists $\beta_0 \subset [\beta]_r^*$ such that $U = \cup\beta_0$. Since $p \in U$, $p \in \cup\beta_0$. Thus there exists $B \in \beta_0$ such that $p \in B \subset U$. So there exists $B \in [\beta]_r^*$ such that $p \in B \subset A$. ($\Leftarrow$) Suppose the necessary condition holds. Assume that the mapping $\beta : 2^X \to I$ is not an ordinary smooth base for τ. Then, by Definition 3.1, there exist $r_0 \in I_1$ and $A \in [\tau]_{r_0}^*$ such that $A \neq \cup\beta'$, $\forall \beta' \subset [\beta]_{r_0}^*$. Consider the family $\beta^* = \{B \in [\beta]_{r_0}^* : B \subset A\}$ and let $G = \cup\beta^*$. Then clearly $A \neq G$. Thus there exists $p \in X$ such that $p \in A$ and $p \notin G$. Since $A \in [\tau]_{r_0}^*$ and $p \in A$, $A \in \mathcal{N}_{[\tau]_{r_0}^*}(p)$. By hypothesis, there exists $B \in [\beta]_{r_0}^*$ such that $p \in B \subset A$. By the definition of β^*, $B \in \beta^*$. So $B \subset G$. Since $p \notin G$, $p \notin B$. This is a contradiction. Hence the mapping $\beta : 2^X \to I$ is an ordinary smooth base for τ. This completes the proof. $\square$

The following is another characterization of Definition 3.1.

Theorem 3.3. Let (X, τ) be an ordinary smooth topological space. Then a mapping $\beta : 2^X \to I$ is an ordinary smooth base for τ if and only if $\tau(A) \leq \bigvee\{\beta(B) : B \in 2^X \quad \text{and} \quad \mathrm{p \in B \subset A}\}$ for each $A \in 2^X$ with $p \in A$.

Proof. ($\Rightarrow$): Suppose $\beta : 2^X \to I$ is an ordinary smooth base for τ. Let $A \in 2^X$ with $p \in A$. Then clearly $\tau(A) = 0$ or $\tau(A) \neq 0$.

Case (i) : Suppose $\tau(A) = 0$. Then the required inequality is obvious.

Case (ii) : Suppose $\tau(A) = r > 0$. Let $\epsilon > 0$ be arbitrary such that $\epsilon \leq r$. Then clearly $\tau(A) > r - \epsilon$, i.e., $A \in [\tau]_{r-\epsilon}^*$. Thus $A \in \mathcal{N}_{[\tau]_{r-\epsilon}^*}(p)$.

By Theorem 3.2, there exists $B \in [\beta]_{r-\epsilon}^*$ such that $p \in B \subset A$. So

$$\bigvee\{\beta(B) : B \in 2^X \quad \text{and} \quad \mathrm{p \in B \subset A}\} > \mathrm{r} - \epsilon$$

Since $\epsilon > 0$ is arbitrary,

$$\bigvee\{\beta(B) : B \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{B} \subset \mathrm{A}\} \geq \mathrm{r} = \tau(\mathrm{A}).$$

Hence, in any cases, the required inequality holds.

($\Leftarrow$): Suppose the necessary condition holds. Let $r \in I_1$ and let $U \in [\tau]^*_r$ such that $p \in U$ for each $p \in X$. Then, by the hypothesis,

$$\alpha < \tau(U) \leq \bigvee\{\beta(B) : B \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{B} \subset \mathrm{U}\}.$$

Thus there exists $B \in 2^X$ such that $p \in B \subset U$ and $\beta(B) > \alpha$. So $B \in [\beta]^*_\alpha$ and $p \in B \subset U$, i.e., the necessary condition of Theorem 3.2 is satisfied. Hence, by Theorem 3.2, β is an ordinary smooth base for τ. This completes the proof. □

Definition 3.4. Let (X, τ) be an ordinary smooth topological space and let $p \in X$. Then a mapping $N_p : 2^X \to I$ is called the *ordinary smooth neighborhood system of* p w.r.t. τ if $[N_p]^*_r = \mathcal{N}_{[\tau]^*_r}(p)$, for each $r \in I_1$. In this case, we will call $N_p(A)$ as the *degree of neighborhood of* A *to* p and each member of $[N_p]^*_r$ is called an *ordinary smooth neighborhood of* p.

The following is the characterization of Definition 3.4.

Theorem 3.5. Let (X, τ) be an ordinary smooth topological space and let $p \in X$ be fixed. Then a mapping $N_p : 2^X \to I$ is the ordinary smooth neighborhood system of p if and only if for each $A \in 2^X$,

$$N_p(A) = \begin{cases} \bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{A}\}, & \text{if } \mathrm{p} \in \mathrm{A}, \\ 0, & \text{if } \mathrm{p} \notin \mathrm{A}. \end{cases}$$

Proof. ($\Rightarrow$): Suppose $N_p : 2^X \to I$ is the ordinary smooth neighborhood system of p w.r.t. τ and let $A \in 2^X$. Then clearly $p \in A$ or $p \notin A$.

Case (i) Suppose $p \notin A$ and $N_p(A) > 0$. Then, from the hypothesis and Definition 3.4, there exists $U \in [\tau]^*_0$ such that $p \in U \subset A$. Thus $p \in A$. This is a contradiction. So $N_p(A) = 0$.

Case (ii) Suppose $p \in A$. Then we may have $N_p(A) = 0$ or $N_p(A) > 0$. If $N_p(A) = 0$, then it is obvious that

$$N_p(A) = 0 \leq \bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{A}\}.$$

Furthermore, assume that

$$\bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{A}\} = \beta > 0.$$

Then there exists $V \in 2^X$ such that $\tau(V) > 0$ and $p \in V \subset A$. Thus, by the hypothesis and Definition 3.4, $A \in [N_p]^*_0$, i.e., $N_p(A) > 0$. This is a contradiction. So, for the case $N_p(A) = 0$, we have

$$N_p(A) = \bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{A}\} = 0.$$

Then, suppose $N_p(A) = r > 0$. Let $\epsilon > 0$ be arbitrary such that $\epsilon \leq r$. Then $N_p(A) > r - \epsilon$, i.e., $A \in [N_p]^*_{r-\epsilon}$. Since N_p is an ordinary smooth neighborhood system of p, there exists $U \in [\tau]^*_{r-\epsilon}$ such that $p \in U \subset A$. Thus

$$\bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{A}\} > \mathrm{r} - \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{A}\} \geq \mathrm{r} = \mathrm{N_p(A)}. \tag{3.1}$$

On the other hand, let $\bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{A}\} = \mathrm{s}$. Then clearly $s > 0$. Let $\epsilon > 0$ be arbitrary such that $\epsilon \leq s$. Then there exists $V \in 2^X$ such that $\tau(V) > s - \epsilon$ and $p \in V \subset A$. Thus $V \in [\tau]^*_{s-\epsilon}$ and $p \in V \subset A$. Thus, by the hypothesis, $A \in [N_p]^*_{s-\epsilon}$. So $N_p(A) > s - \epsilon$. Since $\epsilon > 0$ is arbitrary,

$$N_p(A) \geq s = \bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{A}\}. \tag{3.2}$$

Hence, by (3.1) and (3.2).

$$N_p(A) = \bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{A}\} \quad \text{for} \quad \mathrm{N_p(A)} > 0.$$

This completes the proof of the necessity.

($\Leftarrow$): Suppose the mapping $N_p : 2^X \to I$ is given by

$$N_p(A) = \begin{cases} \bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{A}\}, & \text{if}\, \mathrm{p} \in \mathrm{A}, \\ 0, & \text{if}\, \mathrm{p} \notin \mathrm{A}, \end{cases}$$

for each $A \in 2^X$.

For each $r \in I_1$, let $U \in [N_p]^*_r$, i.e., $N_p(U) > r$. Then, from the hypothesis,

$$r < N_p(U) = \bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{U}\}$$

Thus there exists $V \in 2^X$ such that $\tau(V) > r$ and $p \in V \subset U$. So $V \in [\tau]^*_r$ and $p \in V \subset U$, i.e., $U \in \mathcal{N}_{[\tau]^*_r}(p)$. Hence $[N_p]^*_r \subset \mathcal{N}_{[\tau]^*_r}(p)$.

Now let $r \in I_1$ and let $U \in \mathcal{N}_{[\tau]^*_r}(p)$. Then there exists $B \in [\tau]^*_r$ such that $p \in B \subset U$. Thus $\tau(B) > r$ and $p \in B \subset U$. So $N_p(U) = \bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{U}\} > \mathrm{r}$, i.e., $U \in [N_p]^*_r$. Hence

$\mathcal{N}_{[\tau]_r^*}(p) \subset [N_p]_r^*$. Therefore $[N_p]_r^* = \mathcal{N}_{[\tau]_r^*}(p)$ for each $r \in I_1$, i.e., N_p is the ordinary smooth neighborhood system of p w.r.t. τ. □

The following is another characterization of Definition 3.4.

Theorem 3.6. Let (X, τ) be an ordinary smooth topological space and let $\beta : 2^X \to I$ be an ordinary smooth base for τ. Then a mapping $N_p : 2^X \to I$ is the ordinary smooth neighborhood system of p w.r.t. τ if and only if for each $A \in 2^X$,

$$N_p(A) = \begin{cases} \bigvee\{\beta(U) : U \in 2^X \quad \text{and} \quad \mathrm{p \in U \subset A}\}, & \mathrm{if\,p \in A,} \\ 0, & \mathrm{if\,p \notin A,} \end{cases}$$

where $p \in X$.

Proof. By considering Theorem 3.2 and Definition 3.4, we can easily obtain that a mapping $N_p : 2^X \to I$ is the ordinary smooth neighborhood system of p w.r.t. τ if and only if $[N_p]_r^* = \{U \in 2^X : \exists B \in [\beta]_r^*$ such that $p \in B \subset U\}$, for each $r \in I_1$. Using this equivalence, the proof is completed in a way similar to that of Theorem 3.5. □

Proposition 3.7. Let (X, τ) be an ordinary smooth topological space and let $p \in X$. If the mapping $N_p : 2^X \to I$ is the ordinary smooth neighborhood system of p w.r.t. τ, then the followings hold:

(OSN_1) If $N_p(U) > 0$, then $p \in U$, where $U \in 2^X$.

(OSN_2) $\bigvee\{N_p(U) : U \in 2^X\} = 1$

(OSN_3) $N_p(U_1 \cap U_2) \geq N_p(U_1) \wedge N_p(U_2), \quad \forall U_1, U_2 \in 2^X$.

(OSN_4) If $U_1 \subset U_2$ and $U_1, U_2 \in 2^X$, then $N_p(U_1) \leq N_p(U_2)$.

(OSN_5) $\forall U \in 2^X$, $N_p(U) \leq \bigvee\{N_p(V) \wedge (\bigwedge_{e\in V} N_e(V)) : V \in 2^X$ and $\mathrm{V \subset U}\}$.

Proof. (OSN_1), (OSN_2) and (OSN_4) follows directly from Theorem 3.5.

(OSN_3) Let $U_1, U_2 \in 2^X$.

Case (i): Suppose $N_p(U_1) = 0$ or $N_p(U_2) = 0$. Then the required inequality is obvious.

Case (ii): Suppose $N_p(U_1) = r_1 > 0$ and $N_p(U_2) = r_2 > 0$. Let $\epsilon > 0$ be arbitrary such that $\epsilon \leq r_1 \wedge r_2$. Then

$$N_p(U_1) > r_1 - \epsilon \geq 0$$

and

$$N_p(U_2) > r_2 - \epsilon \geq 0.$$

By Definition 3.4, there exist $T_1, T_2 \in 2^X$ such that

$$\tau(T_1) > r_1 - \epsilon \quad \text{and} \quad \mathrm{p \in T_1 \subset U_1}$$

and

$$\tau(T_2) > r_2 - \epsilon \quad \text{and} \quad \mathrm{p \in T_2 \subset U_2}.$$

Thus

$$\begin{aligned}\tau(T_1 \cap T_2) &\geq \tau(T_1) \wedge \tau(T_2) \quad [\mathrm{Since}\, \tau \in \mathrm{OST(X)}]\\ &> (r_1 - \epsilon) \wedge (r_2 - \epsilon)\\ &= (r_1 \wedge r_2) - \epsilon.\end{aligned}$$

and

$$p \in T_1 \cap T_2 \subset U_1 \cap U_2.$$

So, by the hypothesis,

$$N_p(U_1 \cap U_2) > (r_1 \wedge r_2) - \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$N_p(U_1 \cap U_2) \geq r_1 \wedge r_2 = N_p(U_1) \wedge N_p(U_2).$$

(OSN_5) Let $U \in 2^X$.

Case (i): Suppose $N_p(U) = 0$. Then the required inequality is obvious.

Case (ii): Suppose $N_p(U) = r > 0$. Let $\epsilon > 0$ be arbitrary such that $\epsilon \leq r$. Then $N_p(U) > r - \epsilon$. Thus, by Definition 3.4, there exists $V_0 \in 2^X$ and $\tau(V_0) > r - \epsilon$ and $p \in V_0 \subset U$. Since $V_0 \subset V_0$ and $\tau(V_0) > r - \epsilon$, $N_e(V_0) > r - \epsilon$ for each $e \in V_0$. So $\bigwedge_{e \in V_0} N_e(V_0) \geq r - \epsilon$.

On the other hand, in particular, $N_p(V_0) > r - \epsilon$. Thus

$$\begin{aligned}&\bigvee\{N_p(V) \wedge (\bigwedge_{e \in V} N_e(V)) : V \in 2^X \quad \text{and} \quad \mathrm{V \subset U}\}\\ &\geq N_p(V_0) \wedge (\bigwedge_{e \in V_0} N_e(V_0))\\ &\geq r - \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\begin{aligned}N_p(U) &= r\\ &\leq \bigvee\{N_p(V) \wedge (\bigwedge_{e \in V} N_e(V)) : V \in 2^X \quad \text{and} \quad \mathrm{V \subset U}\}.\end{aligned}$$

This completes the proof. □

Proposition 3.8. Let a mapping$N_p : 2^X \to I$ satisfy the conditions (OSN$_1$) $\sim$ (OSN$_5$), where $p \in X$. We define the mapping $\tau : 2^X \to I$ as follows: For each $U \in 2^X$,

$$\tau(U) = \begin{cases} 1, & \text{if U} = \emptyset, \\ \bigwedge_{e \in U} N_e(U), & \text{otherwise.} \end{cases}$$

Then $\tau \in$ OST(X). Moreover, the mapping N_p is exactly the ordinary smooth neighborhood system of p w.r.t. τ.

Proof. It is obvious that $\tau(\emptyset) = 1$. By (OSN$_2$), $\bigvee\{N_p(U) : U \in 2^X\} = 1$. By (OSN$_4$), $N_p(U) \leq N_p(X)$ for each $U \in 2^X$. Thus

$$\bigvee\{N_p(U) : U \in 2^X\} = N_p(X) = 1, \quad \forall p \in X.$$

So $\tau(U) = \bigwedge_{p \in X} N_p(X) = 1$. Hence τ satisfies the condition (OST$_1$).

Let $U_1, U_2 \in 2^X$. If $U_1 \cap U_2 = \emptyset$, then it is obvious that $\tau(U_1 \cap U_2) = 1 \geq \tau(U_1) \wedge \tau(U_2)$. Now suppose $U_1 \cap U_2 \neq \emptyset$. Then

$$\begin{aligned} \tau(U_1 \cap U_2) &= \bigwedge_{e \in U_1 \cap U_2} N_e(U_1 \cap U_2) \\ &\geq \bigwedge_{e \in U_1 \cap U_2} [N_e(U_1) \wedge N_e(U_2)] \quad [\text{By}\,(\text{OSN}_1)] \\ &= (\bigwedge_{e \in U_1 \cap U_2} N_e(U_1)) \wedge (\bigwedge_{e \in U_1 \cap U_2} N_e(U_2)) \\ &\geq (\bigwedge_{e \in U_1} N_e(U_1)) \wedge (\bigwedge_{e \in U_2} N_e(U_2)) \\ &= \tau(U_1) \wedge \tau(U_2). \end{aligned}$$

Thus τ satisfies the condition (OST$_2$). Let $\{U_\alpha\}_{\alpha \in \Gamma} \subset 2^X$. If $\cup_{\alpha \in \Gamma} U_\alpha = \emptyset$. Then it is obvious that

$$\tau(\cup_{\alpha \in \Gamma} U_\alpha) = 1 \geq \bigwedge_{\alpha \in \Gamma} \tau(U_\alpha).$$

Now suppose $\cup_{\alpha \in P} U_\alpha \neq \emptyset$. Then

$$\begin{aligned} N_p(\cup_{\alpha \in \Gamma} U_\alpha) &\geq N_p(U_{\alpha_0}) \quad [\text{By (OSN}_4)] \\ &\geq \bigwedge_{e \in U_{\alpha_0}} N_e(U_{\alpha_0}) = \tau(U_{\alpha_0}). \end{aligned}$$

Thus

$$\begin{aligned}\tau(\cup_{\alpha\in\Gamma}U_\alpha) &= \bigwedge_{p\in\cup_{\alpha\in\Gamma}} U_\alpha N_p(\cup_{\alpha\in\Gamma}U_\alpha)\\ &\geq \bigwedge_{\alpha\in\Gamma}\tau(U_\alpha).\end{aligned}$$

So τ satisfies the condition (OST_1). Hence $\tau \in \mathrm{OST(X)}$.

Now we show that the mapping $N_p : 2^X \to I$ satisfying the conditions (OSN_1) $\sim$ (OSN_5) is exactly the ordinary smooth neighborhood system of p w.r.t. τ.

Let a mapping $M_p : 2^X \to I$ be the ordinary smooth neighborhood system of p w.r.t. τ. Then, by Theorem 3.5 and the condition (OSN_1),

$$M_p(U) = 0 = N_p(U) \quad \text{for each}\, U \in 2^X \,\text{with}\, \mathrm{p} \notin \mathrm{U}. \tag{3.3}$$

For each $U \in 2^X$, let $p \in U$. Then

$$\begin{aligned}M_p(U) &= \bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p \in V \subset U}\} \quad \text{[By Theorem 3.5]}\\ &= \bigvee_{e\in V} N_e(V) : V \in 2^X \quad \text{and} \quad \mathrm{p \in V \subset U}\}. \quad \text{[By the definition of}\, \tau].\end{aligned}$$

It is clear that

$$\bigwedge_{e\in V} N_e(V) \leq N_p(V) \quad \mathrm{for\, p \in V,\, where\, V \in 2^X}.$$

By the condition (OSN_4)

$$\bigwedge_{e\in V} N_e(V) \leq N_p(V) \leq N_p(U) \quad \mathrm{for\, p \in V \subset U,\, where\, U, V \in 2^X}.$$

Thus

$$\begin{aligned}M_p(U) &= \bigvee\{\tau(V) : V \in 2^X \quad \text{and} \quad \mathrm{p \in V \subset U}\}\\ &= \bigvee\{\bigwedge_{e\in V} N_e(V) : V \in 2^X \quad \text{and} \quad \mathrm{p \in V \subset U}\}\\ &\leq N_p(U) \quad \mathrm{for\, p \in U \in 2^X}.\end{aligned} \tag{3.4}$$

On the other hand,

$$N_p(U) \leq \bigvee\{N_p(V) \wedge (\bigwedge_{e\in V} N_e(V)) : V \in 2^X \text{ and } \mathrm{V} \subset \mathrm{U}\}\ [\text{By}\ (\mathrm{OSN}_5)]$$
$$= [\bigvee\{N_p(V) \wedge (\bigwedge_{e\in V} N_e(V)) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{U}\}]$$
$$\vee\, [\bigvee\{N_p(V) \wedge (\bigwedge_{e\in V} N_e(V)) : V \in 2^X \quad \text{and} \quad \mathrm{p} \notin \mathrm{V} \subset \mathrm{U}\}]$$
$$= \bigvee\{N_p(V) \wedge (\bigwedge_{e\in V} N_e(V)) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{U}\}$$
$$[\text{Since}\ \mathrm{N_p(V)} = 0 \quad \text{for}\ \mathrm{p} \notin \mathrm{V}]$$
$$\leq \bigvee\{\bigwedge_{e\in V} N_e(V) : V \in 2^X \quad \text{and} \quad \mathrm{p} \in \mathrm{V} \subset \mathrm{U}\}$$
$$= M_p(U) \quad \text{for}\ \mathrm{p} \in \mathrm{U} \in 2^{\mathrm{X}}.$$

So $N_p(U) \leq M_p(U)$ for $p \in U \in 2^X$. (3.5)
Hence, by (3.3), (3.4) and (3.5), $M_p = N_p$. This completes the proof. □

References

[1] K. C. Chattopadhyay, R. N. Hazra and S. K. Samanta, Gradation of openness : fuzzy topology, Fuzzy Sets and Systems 49(1992), 237-242.
[2] M. S. Cheng, G. P. Chae, K. Hur and S. M. Kim, The lattice of ordinary smooth topologies, Honam Math. J. 33(4)(2011), 453-465.
[3] M. Demirci, Neighborhood structures of smooth topological spaces, Fuzzy Sets and Systems 92(1997), 123-128.
[4] P. K. Lim, B. K. Ryou and K. Hur, Ordinary smooth topological spaces, IJFIS. 12(1)(2012), 66-76.
[5] A. A. Ramadan, Smooth topological spaces, Fuzzy Sets and Systems 48(1992), 371-375.
[6] A. P. Sostak, On a fuzzy topological structure, Rend. Circ. Matem. Palermo Ser. II, 11(1985), 89-103.
[7] M. S. Ying, A new approach for fuzzy topology (I), Fuzzy Sets and Systems 39(1991), 303-321.

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