

## CONVOLUTION SUM $\sum_{k < N/3} \sigma_1(3^m k) \sigma_1(2^n(N - 3k))$

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**Abstract.** Let  $\sigma_s(N) = \sum_{d|N} d^s$ . Next, the convolution sums  $\sum_{k < N/3} \sigma_1(3^m k) \sigma_1(2^n(N - 3k))$ ,  $\sum_{k < N/3} \sigma_1(2^m k) \sigma_1(3^n(N - 3k))$ , etc., are evaluated for all  $N \in \mathbb{N}$  with  $m, n \in \mathbb{N} \cup \{0\}$ .

### 1. Introduction

In many areas of number theory, we usually accept  $\sigma_s(N) = \sum_{d|N} d^s$ . In this paper, we also use the divisor function  $\sigma_s(N)$  and specifically define

$$\sigma_{s,r}(N; m) = \sum_{\substack{d|N \\ d \equiv r \pmod{m}}} d^s.$$

Using these divisor functions, we construct some convolution sums and derive formulae. Originally, convolution sums were developed by Liouville and Ramanujan, but recently they have been reworked by Berndt, Hahn, Huard, and Ou et al. For example, in [1, Theorem 1] we can see that

$$(1) \quad \begin{aligned} \sum_{\substack{(k,m) \in \mathbb{N}^2 \\ k+6m=N}} \sigma_1(k) \sigma_1(m) &= \frac{1}{120} \sigma_3(N) + \frac{1}{30} \sigma_3\left(\frac{N}{2}\right) + \frac{3}{40} \sigma_3\left(\frac{N}{3}\right) + \frac{3}{10} \sigma_3\left(\frac{N}{6}\right) \\ &\quad + \left(\frac{1}{24} - \frac{N}{24}\right) \sigma_1(N) + \left(\frac{1}{24} - \frac{N}{4}\right) \sigma_1\left(\frac{N}{6}\right) - \frac{1}{120} c_{1,6}(N) \end{aligned}$$

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and

$$(2) \quad \sum_{\substack{(k,m) \in \mathbb{N}^2 \\ 2k+3m=N}} \sigma_1(k)\sigma_1(m) = \frac{1}{120}\sigma_3(N) + \frac{1}{30}\sigma_3(\frac{N}{2}) + \frac{3}{40}\sigma_3(\frac{N}{3}) + \frac{3}{10}\sigma_3(\frac{N}{6}) \\ + (\frac{1}{24} - \frac{N}{12})\sigma_1(\frac{N}{2}) + (\frac{1}{24} - \frac{N}{8})\sigma_1(\frac{N}{3}) - \frac{1}{120}c_{1,6}(N).$$

Alaca and Williams defined  $c_{1,6}(N)$  as

$$\sum_{N=1}^{\infty} c_{1,6}(N)q^N := q \prod_{N=1}^{\infty} (1-q^N)^2(1-q^{2N})^2(1-q^{3N})^2(1-q^{6N})^2.$$

Williams reported that

$$(3) \quad \sum_{k < N/9} \sigma_1(k)\sigma_1(N-9k) = \frac{1}{216}\sigma_3(N) + \frac{1}{27}\sigma_3(\frac{N}{3}) + \frac{3}{8}\sigma_3(\frac{N}{9}) + \frac{(3-2N)}{72}\sigma_1(N) \\ + \frac{(1-6N)}{24}\sigma_1(\frac{N}{9}) - \frac{1}{54}c_{1,9}(N)$$

in [2] where

$$\sum_{N=1}^{\infty} c_{1,9}(N)q^N := q \prod_{N=1}^{\infty} (1-q^{3N})^8.$$

Further, A. Alaca, S. Alaca, and Williams stated that

$$(4) \quad \sum_{\substack{(k,m) \in \mathbb{N}^2 \\ 2k+9m=N}} \sigma_1(k)\sigma_1(m) = \frac{1}{1080}\sigma_3(N) + \frac{1}{270}\sigma_3(\frac{N}{2}) + \frac{1}{135}\sigma_3(\frac{N}{3}) + \frac{4}{135}\sigma_3(\frac{N}{6}) \\ + \frac{3}{40}\sigma_3(\frac{N}{9}) + \frac{3}{10}\sigma_3(\frac{N}{18}) + (\frac{1}{24} - \frac{N}{36})\sigma_1(\frac{N}{2}) \\ + (\frac{1}{24} - \frac{N}{8})\sigma_1(\frac{N}{9}) - \frac{1}{1080}c_{2,9}(N),$$

where

$$\begin{aligned}
& \sum_{N=1}^{\infty} c_{2,9}(N) q^N \\
&:= 3q \prod_{N=1}^{\infty} (1 + q^{2N})^2 (1 - q^{2N-1})^4 (1 + q^{3N})^4 (1 - q^{6N})^8 (1 - q^{12N-6})^6 \\
&+ 4q \prod_{N=1}^{\infty} (1 - q^N)^2 (1 - q^{2N})^2 (1 - q^{3N})^2 (1 - q^{6N})^2 \\
&- 6q \prod_{N=1}^{\infty} (1 + q^{2N})^2 (1 - q^{2N-1}) (1 + q^{3N})^3 (1 - q^{6N})^8 (1 - q^{12N-6})^6 \\
&- 16q^2 \prod_{N=1}^{\infty} (1 - q^{2N-1})^3 (1 + q^{3N}) (1 - q^{6N})^8 \\
&+ 28q^2 \prod_{N=1}^{\infty} (1 - q^{6N})^8 \\
&+ 12q^3 \prod_{N=1}^{\infty} (1 - q^N)^2 (1 - q^{3N})^2 (1 + q^{4N} + q^{8N})^2 (1 - q^{12N})^4 \\
&- 24q^3 \prod_{N=1}^{\infty} (1 + q^N) (1 - q^{3N})^3 (1 - q^{4N-2})^2 (1 + q^{6N}) (1 - q^{12N})^5.
\end{aligned}$$

In Section 2, we derive the convolution sums  $\sum_{k < N/3} \sigma_1(2^m k) \sigma_1(3^n(N - 3k))$ ,  $\sum_{k < N/3} \sigma_1(3^m k) \sigma_1(2^n(N - 3k))$ , etc., and we consider the following:

Convolution sums	Convolution formula	Reference
$\sum_{k=1}^{N-1} \sigma_1(2^m k) \sigma_1(2^n(N-k))$	$\begin{aligned} & \frac{1}{24} [(3 \cdot 2^{n+m} - 2^m - 2^n - 1) \sigma_3(2N) \\ & - (3 \cdot 2^{n+m} - 2^m - 2^n - 11) \sigma_3(N) \\ & + \{2^n + 2^m - 2 \\ & - 6(2^{n+m+1} - 2^m - 2^n)N\} \sigma_1(2N) \\ & - \{2^n + 2^m - 4 \\ & - 12(2^{n+m} - 2^m - 2^n)N\} \sigma_1(N)] \end{aligned}$	[4, Proposition 2.1]
$\sum_{k < N/3} \sigma_1(2^m k) \sigma_1(3^n(N-3k))$	$\begin{aligned} & \frac{1}{240} [(2^{m+3} - 3)(3^{n+1} - 1) \sigma_3(N) \\ & - 8(2^m - 1)(3^{n+1} - 1) \sigma_3(\frac{N}{2}) \\ & - 3(2^{m+3} - 3)(3^n - 7) \sigma_3(\frac{N}{3}) \\ & + 24(2^m - 1)(3^n - 7) \sigma_3(\frac{N}{6}) \\ & - 5(3^{n+1} - 1)(2^{m+1}N - 1) \sigma_1(N) \\ & - 5\{3^{n+1} - 2^{m+2} - 1 \\ & + 6(2^m \cdot 3^{n+1} - 2 \cdot 3^n + 2^m)N\} \sigma_1(\frac{N}{3}) \\ & + 20(2^m - 1)(2 \cdot 3^{n+1}N - 1) \sigma_1(\frac{N}{6}) \\ & + 2(2^m - 1)(3^{n+1} - 1)c_{1,6}(N)] \end{aligned}$	Theorem 2.2
$\sum_{k < N/3} \sigma_1(3^m k) \sigma_1(2^n(N-3k))$	$\begin{aligned} & \frac{1}{720} [2(4 \cdot 3^m - 1)(2^{n+3} - 3) \sigma_3(N) \\ & - 16(4 \cdot 3^m - 1)(2^n - 1) \sigma_3(\frac{N}{2}) \\ & + (73 \cdot 3^m - 19)(2^{n+3} - 3) \sigma_3(\frac{N}{3}) \\ & - 8(73 \cdot 3^m - 19)(2^n - 1) \sigma_3(\frac{N}{6}) \\ & - 81(3^m - 1)(2^{n+3} - 3) \sigma_3(\frac{N}{9}) \\ & + 648(3^m - 1)(2^n - 1) \sigma_3(\frac{N}{18}) \\ & - 30(2 \cdot 3^m N - 1)(2^{n+1} - 1) \sigma_1(N) \\ & + 60(2 \cdot 3^m N - 1)(2^n - 1) \sigma_1(\frac{N}{2}) \\ & - 15(3^{m+1} - 1)(3 \cdot 2^{n+1}N - 1) \sigma_1(\frac{N}{3}) \\ & + 45(3^m - 1)(3 \cdot 2^{n+1}N - 1) \sigma_1(\frac{N}{9}) \\ & - 2(3^m - 1)(2^n - 1)c_{2,9}(N) \\ & + 20(3^m - 1)(2^{n+1} - 1)c_{1,9}(N) \\ & + 6(3^{m+1} - 1)(2^n - 1)c_{1,6}(N)] \end{aligned}$	Theorem 2.5

TABLE 1. Convolution formulae

for  $N \in \mathbb{N}$  with  $m, n \in \mathbb{N} \cup \{0\}$ . Finally, we consider the sums of even divisor functions or odd divisor functions. In the Appendix section, we propose the values of  $c_{1,6}(N)$ ,  $c_{1,9}(N)$ , and  $c_{2,9}(N)$  ( $1 \leq N \leq 45$ ).

## 2. Generalized Convolution sums

We require the following lemma to prove Theorem 2.2.

**Lemma 2.1.** *Let  $N \in \mathbb{N}$  with  $m \in \mathbb{N} \cup \{0\}$  and  $p$  be a prime. Therefore, we have*

$$\sigma_s(p^m N) = \frac{p^{s(m+1)} - 1}{p^s - 1} \sigma_s(N) + \frac{p^s - p^{s(m+1)}}{p^s - 1} \sigma_s\left(\frac{N}{p}\right).$$

*Proof.* We can state  $\sigma_s(p^m N) = a_m \sigma_s(N) + b_m \sigma_s\left(\frac{N}{p}\right)$  and use induction based on  $m$  to find the coefficients  $a_m$  and  $b_m$ . From the elementary identity we have

$$\sigma_s(pN) = (p^s + 1)\sigma_s(N) - p^s \sigma_s\left(\frac{N}{p}\right),$$

for  $m = 1$  and

$$\begin{aligned} \sigma_s(p^2 N) &= (p^s + 1)\sigma_s(pN) - p^s \sigma_s(N) \\ &= (p^s + 1)\{(p^s + 1)\sigma_s(N) - p^s \sigma_s\left(\frac{N}{p}\right)\} - p^s \sigma_s(N) \\ &= \{(p^s + 1)^2 - p^s\}\sigma_s(N) - (p^s + 1)p^s \sigma_s\left(\frac{N}{p}\right), \end{aligned}$$

for  $m = 2$ . Continuing this process, we derive the rule and generalize the formula.  $\square$

**Theorem 2.2.** Let  $N \in \mathbb{N}$  with  $m, n \in \mathbb{N} \cup \{0\}$ . Then we have

$$\begin{aligned} &\sum_{k < N/3} \sigma_1(2^m k) \sigma_1(3^n(N - 3k)) \\ &= \frac{1}{240} [(2^{m+3} - 3)(3^{n+1} - 1)\sigma_3(N) - 8(2^m - 1)(3^{n+1} - 1)\sigma_3\left(\frac{N}{2}\right) \\ &\quad - 3(2^{m+3} - 3)(3^n - 7)\sigma_3\left(\frac{N}{3}\right) + 24(2^m - 1)(3^n - 7)\sigma_3\left(\frac{N}{6}\right) \\ &\quad - 5(3^{n+1} - 1)(2^{m+1}N - 1)\sigma_1(N) \\ &\quad - 5\{3^{n+1} - 2^{m+2} - 1 + 6(2^m \cdot 3^{n+1} - 2 \cdot 3^n + 2^m)N\}\sigma_1\left(\frac{N}{3}\right) \\ &\quad + 20(2^m - 1)(2 \cdot 3^{n+1}N - 1)\sigma_1\left(\frac{N}{6}\right) + 2(2^m - 1)(3^{n+1} - 1)c_{1,6}(N)]. \end{aligned}$$

*Proof.* Using Lemma 2.1 with  $s = 1$ , we can obtain

$$\begin{aligned}
& \sum_{k < N/3} \sigma_1(2^m k) \sigma_1(3^n(N - 3k)) \\
&= \sum_{k < N/3} \{(2^{m+1} - 1)\sigma_1(k) + (2 - 2^{m+1})\sigma_1(\frac{k}{2})\} \\
&\quad \times \{\frac{3^{n+1} - 1}{2}\sigma_1(N - 3k) + \frac{3 - 3^{n+1}}{2}\sigma_1(\frac{N - 3k}{3})\} \\
(5) \quad &= \frac{1}{2}\{(2^{m+1} - 1)(3^{n+1} - 1) \sum_{k < N/3} \sigma_1(k)\sigma_1(N - 3k) \\
&\quad + (2^{m+1} - 1)(3 - 3^{n+1}) \sum_{k < N/3} \sigma_1(k)\sigma_1(\frac{N}{3} - k) \\
&\quad + (2 - 2^{m+1})(3^{n+1} - 1) \sum_{k < N/3} \sigma_1(\frac{k}{2})\sigma_1(N - 3k) \\
&\quad + (2 - 2^{m+1})(3 - 3^{n+1}) \sum_{k < N/3} \sigma_1(\frac{k}{2})\sigma_1(\frac{N}{3} - k)\}.
\end{aligned}$$

We refer to

$$(6) \quad \sum_{k < N/3} \sigma_1(k)\sigma_1(N - 3k) = \frac{1}{24}\{\sigma_3(N) + (1 - 2N)\sigma_1(N) + 9\sigma_3(\frac{N}{3}) + (1 - 6N)\sigma_1(\frac{N}{3})\}$$

in [3, Theorem 3] and

$$\sum_{k=1}^{N-1} \sigma_1(k)\sigma_1(N - k) = \frac{1}{12}\{5\sigma_3(N) + (1 - 6N)\sigma_1(N)\}$$

in [3, (3.10)]. The last two terms on the right hand side of (5) can be written as

$$\begin{aligned}
\sum_{k < N/3} \sigma_1(\frac{k}{2})\sigma_1(N - 3k) &= \sum_{k < N/6} \sigma_1(k)\sigma_1(N - 6k), \\
\sum_{k < N/3} \sigma_1(\frac{k}{2})\sigma_1(\frac{N}{3} - k) &= \sum_{k < N/6} \sigma_1(k)\sigma_1(\frac{N}{3} - 2k).
\end{aligned}$$

Thus, we use (1) and

$$\sum_{k < N/2} \sigma_1(k) \sigma_1(N - 2k) = \frac{1}{24} \{2\sigma_3(N) + (1 - 3N)\sigma_1(N) + 8\sigma_3(\frac{N}{2}) + (1 - 6N)\sigma_1(\frac{N}{2})\}$$

in [3, (4.4)].  $\square$

### Corollary 2.3.

$$\begin{aligned} & \sum_{k < N/3} \sigma_1(2k) \sigma_1(N - 3k) \\ &= \frac{1}{120} \{13\sigma_3(N) - 8\sigma_3(\frac{N}{2}) + 117\sigma_3(\frac{N}{3}) - 72\sigma_3(\frac{N}{6}) - 5(4N - 1)\sigma_1(N) \\ &\quad - 15(6N - 1)\sigma_1(\frac{N}{3}) + 10(6N - 1)\sigma_1(\frac{N}{6}) + 2c_{1,6}(N)\}. \end{aligned}$$

*Proof.* We take  $m = 1$  and  $n = 0$  in Theorem 2.2.  $\square$

In Theorem 2.4, we deal with the sum of even divisors and odd divisors of  $\sigma_1(2^m k)$  combined with  $\sigma_1(3^n(N - 3k))$ , respectively.

**Theorem 2.4.** *Let  $N \in \mathbb{N}$  with  $m, n \in \mathbb{N} \cup \{0\}$ . Then we obtain*

(a)

$$\begin{aligned} & \sum_{k < N/3} \sigma_{1,0}(2^m k; 2) \sigma_1(3^n(N - 3k)) \\ &= \frac{1}{120} [(2^{m+2} - 3)(3^{n+1} - 1)\sigma_3(N) - 4(2^m - 2)(3^{n+1} - 1)\sigma_3(\frac{N}{2}) \\ &\quad - 3(2^{m+2} - 3)(3^n - 7)\sigma_3(\frac{N}{3}) + 12(2^m - 2)(3^n - 7)\sigma_3(\frac{N}{6}) \\ &\quad - 5(3^{n+1} - 1)(2^m N - 1)\sigma_1(N) \\ &\quad - 5\{3^{n+1} - 2^{m+1} - 1 + 3(2^m \cdot 3^{n+1} - 4 \cdot 3^n + 2^m)N\}\sigma_1(\frac{N}{3}) \\ &\quad + 10(2^m - 2)(2 \cdot 3^{n+1} N - 1)\sigma_1(\frac{N}{6}) + (2^m - 2)(3^{n+1} - 1)c_{1,6}(N)]. \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{k < N/3} \sigma_{1,1}(2^m k; 2) \sigma_1(3^n(N - 3k)) \\ &= \frac{1}{240} [3(3^{n+1} - 1)\sigma_3(N) - 8(3^{n+1} - 1)\sigma_3(\frac{N}{2}) - 9(3^n - 7)\sigma_3(\frac{N}{3}) \\ &\quad + 24(3^n - 7)\sigma_3(\frac{N}{6}) - 5(3^{n+1} - 1)\sigma_1(N) - 5\{4 \cdot 3^{n+1} N - 3^{n+1} + 1\}\sigma_1(\frac{N}{3}) \\ &\quad + 20(2 \cdot 3^{n+1} N - 1)\sigma_1(\frac{N}{6}) + 2(3^{n+1} - 1)c_{1,6}(N)]. \end{aligned}$$

*Proof.* (a) We pay consider

$$\sum_{k < N/3} \sigma_{1,0}(2^m k; 2) \sigma_1(3^n(N - 3k)) = 2 \sum_{k < N/3} \sigma_1(2^{m-1} k) \sigma_1(3^n(N - 3k))$$

by  $\sigma_{1,0}(2k; 2) = 2\sigma_1(k)$ . Then we substitute  $m$  with  $m - 1$  in Theorem 2.2.

(b) We use the fact than

$$\begin{aligned} & \sum_{k < N/3} \sigma_{1,1}(2^m k; 2) \sigma_1(3^n(N - 3k)) \\ &= \sum_{k < N/3} \sigma_1(2^m k) \sigma_1(3^n(N - 3k)) - \sum_{k < N/3} \sigma_{1,0}(2^m k; 2) \sigma_1(3^n(N - 3k)). \end{aligned}$$

Next, we refer to Theorem 2.2 and Theorem 2.4 (a).

□

**Theorem 2.5.** Let  $N \in \mathbb{N}$  with  $m, n \in \mathbb{N} \cup \{0\}$ . Then we have

$$\begin{aligned} & \sum_{k < N/3} \sigma_1(3^m k) \sigma_1(2^n(N - 3k)) \\ &= \frac{1}{720} [2(4 \cdot 3^m - 1)(2^{n+3} - 3)\sigma_3(N) - 16(4 \cdot 3^m - 1)(2^n - 1)\sigma_3(\frac{N}{2}) \\ & \quad + (73 \cdot 3^m - 19)(2^{n+3} - 3)\sigma_3(\frac{N}{3}) - 8(73 \cdot 3^m - 19)(2^n - 1)\sigma_3(\frac{N}{6}) \\ & \quad - 81(3^m - 1)(2^{n+3} - 3)\sigma_3(\frac{N}{9}) + 648(3^m - 1)(2^n - 1)\sigma_3(\frac{N}{18}) \\ & \quad - 30(2 \cdot 3^m N - 1)(2^{n+1} - 1)\sigma_1(N) + 60(2 \cdot 3^m N - 1)(2^n - 1)\sigma_1(\frac{N}{2}) \\ & \quad - 15(3^{m+1} - 1)(3 \cdot 2^{n+1} N - 1)\sigma_1(\frac{N}{3}) + 45(3^m - 1)(3 \cdot 2^{n+1} N - 1)\sigma_1(\frac{N}{9}) \\ & \quad - 2(3^m - 1)(2^n - 1)c_{2,9}(N) + 20(3^m - 1)(2^{n+1} - 1)c_{1,9}(N) \\ & \quad + 6(3^{m+1} - 1)(2^n - 1)c_{1,6}(N)]. \end{aligned}$$

*Proof.* From Lemma 2.1, we can obtain

$$\begin{aligned} & \sum_{k < N/3} \sigma_1(3^m k) \sigma_1(2^n(N - 3k)) \\ &= \frac{1}{2} \left\{ (3^{m+1} - 1)(2^{n+1} - 1) \sum_{k < N/3} \sigma_1(k) \sigma_1(N - 3k) \right. \\ & \quad + (3^{m+1} - 1)(2 - 2^{n+1}) \sum_{k < N/3} \sigma_1(k) \sigma_1\left(\frac{N - 3k}{2}\right) \\ & \quad + (3 - 3^{m+1})(2^{n+1} - 1) \sum_{k < N/3} \sigma_1\left(\frac{k}{3}\right) \sigma_1(N - 3k) \\ & \quad \left. + (3 - 3^{m+1})(2 - 2^{n+1}) \sum_{k < N/3} \sigma_1\left(\frac{k}{3}\right) \sigma_1\left(\frac{N - 3k}{2}\right) \right\}. \end{aligned}$$

Then we can deduce that

$$\begin{aligned} \sum_{k < N/3} \sigma_1(k) \sigma_1\left(\frac{N - 3k}{2}\right) &= \sum_{2k+3m=N} \sigma_1(k) \sigma_1(m), \\ \sum_{k < N/3} \sigma_1\left(\frac{k}{3}\right) \sigma_1(N - 3k) &= \sum_{k < N/9} \sigma_1(k) \sigma_1(N - 9k), \\ \sum_{k < N/3} \sigma_1\left(\frac{k}{3}\right) \sigma_1\left(\frac{N - 3k}{2}\right) &= \sum_{2k+9m=N} \sigma_1(k) \sigma_1(m). \end{aligned}$$

Therefore, we use Eqs. (2), (3), (4), and (6).

□

We can also separate  $\sigma_1(2^n(N - 3k))$  using the sum of the even divisors and the sum of the odd divisors to produce the following theorem.

**Theorem 2.6.** *Let  $N \in \mathbb{N}$  with  $m, n \in \mathbb{N} \cup \{0\}$ . Then we obtain*

(a)

$$\begin{aligned}
& \sum_{k < N/3} \sigma_1(3^m k) \sigma_{1,0}(2^n(N - 3k); 2) \\
&= \frac{1}{360} [2(4 \cdot 3^m - 1)(2^{n+2} - 3)\sigma_3(N) - 8(4 \cdot 3^m - 1)(2^n - 2)\sigma_3(\frac{N}{2}) \\
&\quad + (73 \cdot 3^m - 19)(2^{n+2} - 3)\sigma_3(\frac{N}{3}) - 4(73 \cdot 3^m - 19)(2^n - 2)\sigma_3(\frac{N}{6}) \\
&\quad - 81(3^m - 1)(2^{n+2} - 3)\sigma_3(\frac{N}{9}) + 324(3^m - 1)(2^n - 2)\sigma_3(\frac{N}{18}) \\
&\quad - 30(2 \cdot 3^m N - 1)(2^n - 1)\sigma_1(N) + 30(2 \cdot 3^m N - 1)(2^n - 2)\sigma_1(\frac{N}{2}) \\
&\quad - 15(3^{m+1} - 1)(3 \cdot 2^n N - 1)\sigma_1(\frac{N}{3}) + 45(3^m - 1)(3 \cdot 2^n N - 1)\sigma_1(\frac{N}{9}) \\
&\quad - 2(3^m - 1)(2^n - 2)c_{2,9}(N) + 20(3^m - 1)(2^n - 1)c_{1,9}(N) \\
&\quad + 3(3^{m+1} - 1)(2^n - 2)c_{1,6}(N)].
\end{aligned}$$

(b)

$$\begin{aligned}
& \sum_{k < N/3} \sigma_1(3^m k) \sigma_{1,1}(2^n(N - 3k); 2) \\
&= \frac{1}{720} [6(4 \cdot 3^m - 1)\sigma_3(N) - 16(4 \cdot 3^m - 1)\sigma_3(\frac{N}{2}) + 3(73 \cdot 3^m - 19)\sigma_3(\frac{N}{3}) \\
&\quad - 8(73 \cdot 3^m - 19)\sigma_3(\frac{N}{6}) - 243(3^m - 1)\sigma_3(\frac{N}{9}) + 648(3^m - 1)\sigma_3(\frac{N}{18}) \\
&\quad - 30(2 \cdot 3^m N - 1)\sigma_1(N) + 60(2 \cdot 3^m N - 1)\sigma_1(\frac{N}{2}) - 15(3^{m+1} - 1)\sigma_1(\frac{N}{3}) \\
&\quad + 45(3^m - 1)\sigma_1(\frac{N}{9}) - 2(3^m - 1)c_{2,9}(N) + 20(3^m - 1)c_{1,9}(N) \\
&\quad + 6(3^{m+1} - 1)c_{1,6}(N)].
\end{aligned}$$

*Proof.* (a) Since

$$\sum_{k < N/3} \sigma_1(3^m k) \sigma_{1,0}(2^n(N - 3k); 2) = 2 \sum_{k < N/3} \sigma_1(3^m k) \sigma_1(2^{n-1}(N - 3k)),$$

we substitute  $n$  with  $n - 1$  in Theorem 2.5.

(b) We note that

$$\begin{aligned}
& \sum_{k < N/3} \sigma_1(3^m k) \sigma_{1,1}(2^n(N - 3k); 2) \\
&= \sum_{k < N/3} \sigma_1(3^m k) \sigma_1(2^n(N - 3k)) - \sum_{k < N/3} \sigma_1(3^m k) \sigma_{1,0}(2^n(N - 3k); 2).
\end{aligned}$$

□

**Example 2.7.** We list first nine values for the convolution sums from Theorem 2.6.

<i>Convolution sum</i>	N	2	3	4	5	6	7	8	9	10
$\sum_{k < N/3} \sigma_1(k) \sigma_{1,0}(N - 3k; 2)$	0	0	0	2	0	6	6	8	18	
$\sum_{k < N/3} \sigma_1(k) \sigma_{1,0}(2(N - 3k); 2)$	0	0	2	6	8	20	30	48	66	
$\sum_{k < N/3} \sigma_1(3k) \sigma_{1,0}(N - 3k; 2)$	0	0	0	8	0	24	24	32	72	
$\sum_{k < N/3} \sigma_1(3k) \sigma_{1,0}(2(N - 3k); 2)$	0	0	8	24	32	80	120	192	258	
$\sum_{k < N/3} \sigma_1(k) \sigma_{1,1}(N - 3k; 2)$	0	0	1	1	4	4	9	16	15	
$\sum_{k < N/3} \sigma_1(k) \sigma_{1,1}(2(N - 3k); 2)$	0	0	1	1	4	4	9	16	15	
$\sum_{k < N/3} \sigma_1(3k) \sigma_{1,1}(N - 3k; 2)$	0	0	4	4	16	16	36	64	57	
$\sum_{k < N/3} \sigma_1(3k) \sigma_{1,1}(2(N - 3k); 2)$	0	0	4	4	16	16	36	64	57	

TABLE 2. Some values derived from Theorem 2.6

### 3. Appendix

We can find certain values of  $c_{1,6}(N)$  ( $1 \leq N \leq 45$ ) for  $N \in \mathbb{N}$  in Table 3,  $c_{1,9}(N)$  in Table 4, and  $c_{2,9}(N)$  in Table 5.

N	$c_{1,6}(N)$	N	$c_{1,6}(N)$	N	$c_{1,6}(N)$
1	1	16	16	31	-88
2	-2	17	-126	32	-32
3	-3	18	-18	33	-36
4	4	19	20	34	252
5	6	20	24	35	-96
6	6	21	48	36	36
7	-16	22	-24	37	254
8	-8	23	168	38	-40
9	9	24	24	39	-114
10	-12	25	-89	40	-48
11	12	26	-76	41	42
12	-12	27	-27	42	-96
13	38	28	-64	43	-52
14	32	29	30	44	48
15	-18	30	36	45	54

TABLE 3.  $c_{1,6}(N)$  for  $N$  ( $1 \leq N \leq 45$ )

$N$	$c_{1,9}(N)$	$N$	$c_{1,9}(N)$	$N$	$c_{1,9}(N)$
1	1	16	64	31	308
2	0	17	0	32	0
3	0	18	0	33	0
4	-8	19	56	34	0
5	0	20	0	35	0
6	0	21	0	36	0
7	20	22	0	37	110
8	0	23	0	38	0
9	0	24	0	39	0
10	0	25	-125	40	0
11	0	26	0	41	0
12	0	27	0	42	0
13	-70	28	-160	43	-520
14	0	29	0	44	0
15	0	30	0	45	0

TABLE 4.  $c_{1,9}(N)$  for  $N$  ( $1 \leq N \leq 45$ )

$N$	$c_{2,9}(N)$	$N$	$c_{2,9}(N)$	$N$	$c_{2,9}(N)$
1	1	16	496	31	3872
2	-2	17	-2646	32	1888
3	36	18	216	33	432
4	-116	19	380	34	-2268
5	126	20	504	35	-2016
6	-72	21	-576	36	-432
7	344	22	216	37	-1186
8	-488	23	3528	38	1400
9	-108	24	-288	39	1368
10	108	25	-449	40	432
11	252	26	-4396	41	882
12	144	27	324	42	1152
13	-1042	28	-1024	43	-4732
14	1472	29	630	44	1008
15	216	30	-432	45	-648

TABLE 5.  $c_{2,9}(N)$  for  $N$  ( $1 \leq N \leq 45$ )

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