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UNIVERSAL QUATERNARY LATTICES OVER $\mathbb{F}_q[x]$

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Abstract. In this paper, we show that any definite lattice over $\mathbb{F}_q[x]$ is universal if and only if it is quaternary and its discriminant is of degree 2, where $ch(\mathbb{F}_q) \neq 2$. The Four Conjecture follows as an immediate consequence.

1. Introduction

After Conway and Schneeberger proved the Fifteen Theorem, which provides a criterion for determining universality of positive definite integral quadratic forms over $\mathbb{Z}$, the theorem was generalized in various ways. One variation is the 'Finiteness Theorem' proved by B.M. Kim, M.-H. Kim and B.-K. Oh [5]. *Any infinite set $S \subset \mathbb{N}$ contains a finite subset S_0 satisfying the following: any positive definite integral quadratic form representing S_0 represents S.* Another is the 'Four Conjecture' by L. Gerstein in [2], which was recently proved by M.-H. Kim, Y. Wang and F. Xu in [6] and by W.K. Chan and J. Daniels in [9], independently.

Four Conjecture A definite quadratic form over $\mathbb{F}_q[x]$ represents every polynomial in $\mathbb{F}_q[x]$ if it represents $1, \delta, x$ and δx, where $ch(\mathbb{F}_q) \neq 2$ and δ is a non-square element in $\mathbb{F}_q$.

The aim of this paper is to give another proof of the Four Conjecture. In fact, we can find another criterion for universality which is so simple that we only have to check the degree of discriminant. It will give us the Four Conjecture as an immediate corollary.

Theorem. *A definite $\mathbb{F}_q[x]$-lattice L is universal if and only if it is quaternary and its discriminant is of degree 2, where $ch(\mathbb{F}_q) \neq 2$.*

Within this paper, we adopt geometric terminology and notations. From now on, by a lattice, we mean a lattice over $\mathbb{F}_q[x]$, where $\mathbb{F}_q$ is a finite field with q elements where $q = p^r$ for some odd prime p, unless stated otherwise.

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We denote the quadratic map and the bilinear map on L by ϕ_L and B_L, respectively. We pick a non-square element $\delta \in \mathbb{F}_q$ and fix it. We refer [8] to the readers for unexplained terminology and basic facts.

2. General forms

In [2, Proposition, p.132], Gerstein showed that if L is a definite quaternary lattice representing $1, \delta, x$ and δx, then

$$L \simeq \langle 1, -\delta \rangle \perp \begin{pmatrix} \alpha x + \beta & \gamma \\ \gamma & -\delta(\alpha x + \beta) + \xi \end{pmatrix} \tag{A}$$

for some $\alpha, \beta, \gamma, \xi \in \mathbb{F}_q$, $\alpha \neq 0$. In this section, we prove that the definite quaternary lattice over $\mathbb{F}_q[x]$ with its discriminant of degree 2 also satisfies (A).

Lemma 2.1. *If L is a definite quaternary lattice with discriminant of degree 2, then (A) holds.*

Proof. Let L be a definite quaternary lattice, let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be a basis of L such that $\{\phi_L(\mathbf{e}_1), \cdots, \phi_L(\mathbf{e}_4)\}$ is the set of successive minima which gives the Gram matrix of L $M = (b_{ij})$. Then, by assuming that M is reduced, the degree of the discriminant of L is the sum of degrees of diagonal entries of M.

Since $\deg(\mathrm{d}L) = 2$, there are only two possibilities:

$$\deg(b_{11}) = \deg(b_{22}) = 0, \quad \deg(b_{33}) = \deg(b_{44}) = 1$$

or

$$\deg(b_{11}) = \deg(b_{22}) = \deg(b_{33}) = 0, \quad \deg(b_{44}) = 2.$$

However, in the latter case, L contains a unimodular ternary sublattice L', which is isotropic:

$$L' = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle.$$

This contradicts to the definiteness of L and hence both $\deg b_{33}$ and $\deg b_{44}$ must be 1. This conclusion coincides with the condition in the proof of the first Proposition [2, §2]. Therefore, we get the expected result. □

Remark 2.2. Observe that we may assume that α is either 1 or δ: either $\alpha = \omega^2$ or $\alpha\delta^{-1} = \omega^2$ for some $\omega \in \mathbb{F}_q$. Take $\mathcal{B}' = \{\mathbf{e}_1, \mathbf{e}_2, \omega^{-1}\mathbf{e}_3, \omega^{-1}\mathbf{e}_4\}$ to get

$$\phi_L(\omega^{-1}\mathbf{e}_3) = x + \alpha^{-1}\beta, \quad \phi_L(\omega^{-1}\mathbf{e}_4) = \delta(x + \alpha^{-1}\beta) + \xi.$$

Furthermore, letting $y = \alpha x + \beta$ leads $\mathbb{F}_q[x] = \mathbb{F}_q[y]$ and hence, without loss of generality, we can replace (A) with the following:

$$L \simeq L_0 \perp L_1 = \langle 1, -\delta \rangle \perp \begin{pmatrix} y & \gamma \\ \gamma & -\delta y + \xi \end{pmatrix}, \tag{B}$$

where $\delta, \xi \in \mathbb{F}_q$.

In fact, the discriminant of L is an element of $\mathbb{F}_q/\mathbb{F}_q^2$. For convenience, we choose a representative of $\mathrm{d}\,L$:

Definition 2.3. Let L be a definite quaternary lattice of the form (B). We choose a representative of the discriminant of L to be the determinant of the Gram matrix of the form (B) and denote

$$\mathrm{d}\, L := \delta\left(\delta y^2 - \xi y + \gamma^2\right).$$

For notational convenience, we denote

$$k_L(y) = \mathrm{d}\, L_1 = -\delta y^2 + \xi y - \gamma^2. \tag{C}$$

3. Reducible discriminant case

Let L be a definite quaternary lattice such that $\mathrm{d}\, L$ is of degree 2. Then we have two possibilities - $\mathrm{d}\, L$ is an irreducible polynomial or not. In this section, we treat the reducible discriminant case. We prove that if L is a definite quaternary lattice with reducible discriminant of degree 2, then L is diagonalizable and the class number of L is one.

Lemma 3.1. *Let L be a definite quaternary lattice with reducible discriminant of degree 2. Then L is diagonalizable.*

Proof. Suppose that L is of the form (B) with respect to a basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$:

$$L \simeq \langle 1, -\delta \rangle \perp \begin{pmatrix} y & \gamma \\ \gamma & -\delta y + \xi \end{pmatrix},$$

where $y = \alpha x + \beta$ for some $\alpha, \beta \in \mathbb{F}_q$.

By assumption, $\mathrm{d}\, L$ is reducible:

$$\mathrm{d}\, L = \delta\left(\delta y^2 - \xi y + \gamma^2\right) = \delta^2 (y + \sigma)(y + \tau)$$

for some $\sigma, \tau \in \mathbb{F}_q$. Thus, we get

$$\sigma + \tau = -\delta^{-1}\xi \quad \text{and} \quad \sigma\tau = \delta^{-1}\gamma^2.$$

If $\tau = \sigma$, then $\delta\sigma^2 = \gamma^2$ and hence δ is a square, which is a contradiction. So we may assume that $\tau \neq \sigma$.

Let

$$\mathbf{b}_3 = \gamma \mathbf{e}_3 + \sigma \mathbf{e}_4 \quad \text{and} \quad \mathbf{b}_4 = \delta\sigma \mathbf{e}_3 + \gamma \mathbf{e}_4$$

and let $\mathcal{D} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{b}_3, \mathbf{b}_4\}$. Then, $\mathcal{D}$ is also a basis of L because $\gamma^2 - \delta\sigma^2 \neq 0$. Furthermore, we get

$$\begin{aligned} B_L(\mathbf{b}_3, \mathbf{b}_4) &= \delta\sigma\gamma y + \delta\sigma^2\gamma + \gamma^3 + \gamma\sigma(-\delta y + \xi) \\ &= \gamma\sigma\delta(\sigma + \tau + \delta^{-1}\xi) = 0 \end{aligned}$$

and hence L is diagonalizable. □

Corollary 3.2. *Let L be a definite quaternary lattice such that* $\mathrm{d}\, L$ *is of degree 2. Then it is diagonalizable if and only if dL is reducible.*

Proof. Lemma 3.1 shows one direction. So, we only need to show 'only if' direction: suppose L is diagonalizable:

$$L \simeq \langle c_1, c_2, c_3, c_4\rangle$$

for some $c_1, \cdots, c_4 \in \mathbb{F}_q[x]$. Then, $\mathrm{d}\, L = \theta^2 c_1c_2c_3c_4$ for some $\theta \in \mathbb{F}_q^\times$. Since $\deg \mathrm{d}\, L = 2$, at least two of them are of degree 0. If three of them are of degree 0, then there is a ternary sublattice $L' \simeq \langle c_1, c_2, c_3\rangle$, which is isotropic. It contradicts to the definiteness of L. Therefore, only two of them are of degree 0 and the others are of degree 1. □

Lemma 3.3. *Let K, K' be definite binary lattices such that*

$$K \simeq \langle y, cy + d\rangle\,, \quad K' \simeq \langle \alpha y, \alpha(cy + d)\rangle\,,$$

where $c, d, \alpha \in \mathbb{F}_q$ with $c, \alpha \neq 0$. If $d = 0$ or α is a square, then $K \simeq K'$. Otherwise, K and K' are not in the same genus.

Proof. If α is a square, then $K \simeq K'$ and hence they are in the same genus trivially.

Suppose $d = 0$. Let K is of the form $\langle y, cy + d\rangle$. with respect to a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. Since K, K' are definite, $-c$ can't be a square in $\mathbb{F}_q$: if $-c = \omega^2$, then $\phi_K(\omega\mathbf{e}_1 + \mathbf{e}_2) = 0$. Thus, $-c\delta^{-1}$ is a square in $\mathbb{F}_q$: $c = -\delta\omega^2$ for some $\omega \in \mathbb{F}_q^\times$. Hence

$$K \simeq \langle y, -\delta y\rangle \quad \text{and} \quad K' \simeq \langle \alpha y, -\alpha\delta y\rangle\,.$$

Since we know that

$$\langle 1, -\delta\rangle \simeq \langle \alpha, -\alpha\delta\rangle\,,$$

we get $K \simeq \langle y, -\delta y\rangle \simeq \langle \alpha y, -\alpha\delta y\rangle \simeq K'$ immediately.

Suppose now that $d \neq 0$ and α is not a square. If K, K' are in the same genus, then K, K' are isometric for all prime spots $v \in M_{\mathbb{F}_q[y]}$. Take a prime spot $v \in M_{\mathbb{F}_q[y]}$ such that $\mathrm{ord}_v(cy + d) = 1$. Since $\mathrm{ord}_v\, y = 0$, $\langle y, cy + d\rangle$ and $\langle \alpha y, \alpha(cy + d)\rangle$ are the Jordan decompositions of K_v and K'_v, respectively. Moreover, y is a unit in $\mathbb{F}_q[x]_v$, we have

$$\langle y\rangle \simeq \langle \alpha y\rangle \quad \text{and} \quad \langle cy + d\rangle \simeq \langle \alpha(cy + d)\rangle\,.$$

However, this implies that α is a square, which contradicts to our assumption. □

Theorem 3.4. *Let L be a definite quaternary lattice such that* $\mathrm{d}\, L$ *is of degree 2. Then the class number of L is one.*

Proof. Let L, L' be definite quaternary lattices in the same genus. Take a basis which makes L of the form (B) and get

$$L \simeq \langle 1, -\delta, y, -\delta y + \xi\rangle \quad \text{and} \quad L' \simeq \langle 1, -\delta, z, -\delta z + \xi'\rangle\,,$$

for some $y = \alpha x + \beta, z = \alpha' x + \beta'$. Let $z = \mu y + \lambda$ to get

$$L' \simeq \langle 1, -\delta, \mu y + \lambda, -\delta\mu y + \zeta\rangle\,,$$

for some $\mu, \lambda, \xi, \zeta \in \mathbb{F}_q$ with $\mu \neq 0$. Since they are in the same genus, $\mathrm{d}\,L/\,\mathrm{d}\,L'$ is a square of a unit:

$$-\delta y^2 + \xi y = \omega^2(-\delta\mu^2 y^2 + (\mu\zeta - \delta\mu\lambda)y + \lambda\zeta)$$

for some $\omega \in \mathbb{F}_q^\times$. Hence, we get

$$\omega^2\mu^2 = 1, \quad \frac{\xi}{\mu\omega^2} = \zeta - \delta\lambda \quad \text{and} \quad \lambda\zeta = 0.$$

In particular, we get

$$\mu\xi = \zeta - \delta\lambda. \tag{D}$$

Suppose $\lambda = 0$ first. Then, we get $\mu\xi = \zeta$ from (D) and hence

$$L' \simeq \langle 1, -\delta, \mu y, \mu(-\delta y + \xi)\rangle .$$

Since L, L' are in the same genus and $L \simeq \langle 1, -\delta, y, -\delta y + \xi\rangle$, two binary lattices

$$\langle \mu y, \mu(-\delta y + \xi)\rangle , \langle \mu y, \mu(-\delta y + \xi)\rangle$$

are in the same genus, too. Then, by Lemma 3.3, they are globally isometric. Therefore, $L \simeq L'$.

Suppose $\zeta = 0$. Then, we get $\mu\xi = -\delta\lambda$ from (D) and hence

$$\begin{aligned} L' &\simeq \langle 1, -\delta, \mu y + \lambda, -\delta\mu y\rangle \simeq \left\langle 1, -\delta, \delta^2(\mu y + \lambda), -\delta\mu y\right\rangle \\ &\simeq \langle 1, -\delta, -\delta\mu(-\delta y + \xi), -\delta\mu y\rangle . \end{aligned}$$

If $-\delta\mu$ is not a square, then L and L'. cannot be in the same genus by Lemma 3.3. Therefore, $-\delta\mu$ must be a square and hence $L \simeq L'$. $\square$

4. Irreducible discriminant case

In this section, we treat the irreducible discriminant case. If $\mathrm{d}\,L$ is irreducible of degree 2, then L can't be globally diagonalized by Corollary 3.2. However, since all spots on $\mathbb{F}_q(x)$ are non-dyadic, L has an orthogonal basis at every prime spot $v \in \mathbb{F}_q[x]$.

Lemma 4.1. *Let L be a definite quaternary lattice such that $\mathrm{d}\,L$ is an irreducible polynomial of degree 2. Suppose that L is of the form (B):*

$$L \simeq \langle 1, -\delta\rangle \perp \begin{pmatrix} y & \gamma \\ \gamma & -\delta y + \xi \end{pmatrix}, \quad \gamma \neq 0.$$

Let $k_L(y)$ be the polynomial defined in Definition 2.3. Then the following are the Jordan decompositions of L at a prime spot $v \in \mathbb{F}_q[y]$:

If $\xi \neq 0$,

$$L_v \simeq \begin{cases} \left\langle 1, -\delta, \dfrac{\gamma^2}{y}, \dfrac{k_L(y)}{y}\right\rangle & \text{if} \quad v \neq y, \infty \\ \left\langle 1, -\delta, \dfrac{\gamma^2}{-\delta y + \xi}, \dfrac{k_L(y)}{-\delta y + \xi}\right\rangle & \text{if} \quad v = y \\ \langle 1, -\delta, y, -\delta y\rangle & \text{if} \quad v = \infty; \end{cases}$$

If $\xi = 0$,

$$L_v \simeq \begin{cases} \left\langle 1, -\delta, \dfrac{\gamma^2}{y}, \dfrac{k_L(y)}{y} \right\rangle & \text{if} \quad v \neq y, \infty \\ \langle 1, -\delta, y, -\delta y \rangle & \text{if} \quad v = \infty. \end{cases}$$

Proof. When $v = \infty$, it is done in [2, Lemma, p.132]. So, we assume that v is a finite prime spot.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the basis of L yielding the Gram matrix of L of the form (B).

Assume first that $v \neq y$. Let

$$\mathbf{b}_3 = \frac{\gamma}{y}\mathbf{e}_3 \quad \text{and} \quad \mathbf{b}_4 = -\frac{\gamma}{y}\mathbf{e}_3 + \mathbf{e}_4.$$

Then, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{b}_3, \mathbf{b}_4\}$ is a basis of L_v because $\gamma \neq 0$ and y is a unit in $\mathbb{F}_q[y]_v$. Then, we get the desired result:

$$\phi_L(\mathbf{b}_3) = \frac{\gamma^2}{y}, \quad \phi_L(\mathbf{b}_4) = \frac{k_L(y)}{y} \quad \text{and} \quad B_L(\mathbf{b}_3, \mathbf{b}_4) = 0.$$

If $v = y$ and $\xi \neq 0$, then take a set $\left\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{b}_3 = \dfrac{\gamma}{-\delta y + \xi}\mathbf{e}_4, \mathbf{b}_4 = \mathbf{e}_3 - \dfrac{\gamma}{-\delta y + \xi}\mathbf{e}_4\right\}$ of L_v: it is a basis of L_v because $\gamma \neq 0$ and $-\delta y + \xi$ is a unit in $\mathbb{F}_q[y]_v$. Then, we get

$$\phi_L(\mathbf{b}_3) = \frac{\gamma^2}{-\delta y + \xi}, \quad \phi_L(\mathbf{b}_4) = \frac{k_L(y)}{-\delta y + \xi} \quad \text{and} \quad B_L(\mathbf{b}_3, \mathbf{b}_4) = 0.$$

□

Theorem 4.2. *Let L be a definite quaternary lattice such that* $\mathrm{d}\,L$ *is an irreducible polynomial of degree 2. Then the class number of L is one.*

Proof. Let L be a lattice described above and L' be a lattice in the same genus with L. Then, by Lemma 2.1, L, L' are of the form (B) with respect to some $y = \alpha x + \beta$ and $z = \alpha' x + \beta'$. Let $z = \mu y + \lambda$ and get

$$L \simeq \langle 1, -\delta \rangle \perp \begin{pmatrix} y & \gamma \\ \gamma & -\delta y + \xi \end{pmatrix} \tag{E}$$

and

$$L' \simeq \langle 1, -\delta \rangle \perp \begin{pmatrix} z & \gamma' \\ \gamma' & -\delta z + \xi' \end{pmatrix} \simeq \langle 1, -\delta \rangle \perp \begin{pmatrix} \mu y + \lambda & \gamma' \\ \gamma' & -\delta \mu y + \zeta \end{pmatrix} \tag{F}$$

for some $\mu, \lambda, \gamma, \gamma', \xi, \zeta \in \mathbb{F}_q$. Recall

$$k_L(y) = -\delta y^2 + \xi y - \gamma^2$$

and

$$k_{L'}(z) = -\delta z^2 + \xi' z - (\gamma')^2.$$

For convenience, we rewrite $k_{L'}$ with respect to y:

$k_{L'}(y) = -\delta(\mu y+\lambda)^2 + \xi'(\mu y+\lambda) - (\gamma')^2 = -\delta\mu^2 y^2 + (\zeta - \delta\lambda)\mu y + (\lambda\zeta - \gamma'^2)$ where $\zeta = \xi' - \delta\lambda$.

Since $\mathrm{d}\, L$ is irreducible, we can take a prime spot v where $\mathrm{ord}_v\, \mathrm{d}\, L = 1$. By Remark 2.2, we observe that we can assume that μ is 1 or δ.

If $\mu = 1$, we have

$$L_v \simeq \left\langle 1, -\delta, \frac{\gamma^2}{y}, \frac{k_L(y)}{y} \right\rangle \simeq \left\langle 1, -\delta, \frac{\gamma'^2}{y+\lambda}, \frac{k_{L'}(y)}{y+\lambda} \right\rangle \simeq L'_v.$$

Since L and L' are in the same genus, $\mathrm{d}L/\mathrm{d}L'$ is a unit square. Thus, we have

$$\mathrm{d}\, L/\, \mathrm{d}\, L' = u \in (\mathbb{F}_q^\times)^2$$

where u is the ratio of the leading coefficients of $\mathrm{d}\, L$ and $\mathrm{d}\, L'$. Furthermore, the assumption $\mu = 1$ guarantees that the leading coefficients of $\mathrm{d}\, L$ and $\mathrm{d}\, L'$ are same and hence $\mathrm{d}\, L = \mathrm{d}\, L'$. From (E) and (F), we have

$$k_L(y) = -\delta^{-1}\, \mathrm{d}\, L = k_{L'}(y)$$

and hence

$$\xi = \zeta - \delta\lambda \quad \text{and} \quad \gamma^2 = -\lambda\zeta + \gamma'^2.$$

The order 1 components of the Jordan decompositions of L_v and L'_v are isometric. Moreover, since $y, y+\lambda$ are units in $\mathbb{F}_q[x]_v$, we get $\mathrm{ord}_v \dfrac{\gamma^2}{y} = \mathrm{ord}_v \dfrac{\gamma'^2}{y+\lambda} = 0$ and $\mathrm{ord}_v \dfrac{k_L(y)}{y} = \mathrm{ord}_v \dfrac{k_{L'}(y)}{y+\lambda} = 1$, we get

$$\left\langle \frac{k_L(y)}{y} \right\rangle \simeq \left\langle \frac{k_{L'}(y)}{y+\lambda} \right\rangle = \left\langle \frac{k_L(y)}{y+\lambda} \right\rangle.$$

Thus, $y(y+\lambda)$ must be a square in $F_q[x]_v$ and hence so is

$$\frac{\zeta y - \gamma^2}{\delta} = y(y+\lambda) + \frac{1}{\delta} k_L(y) \in \left(\mathbb{F}_q[x]_v^\times\right)^2$$

by the Local Square Theorem [8, 63:1].

Let

$$\frac{\zeta y - \gamma^2}{\delta} = \left(\sum f_i v^i\right)^2,$$

where f_i's are polynomials of degree 1. In particular, let $f_0 = ay + b$. Then, $\dfrac{\gamma^2 - \zeta y}{\delta} - f_0^2$ must be divided by v. Since $\mathrm{ord}_v\, \mathrm{d}\, L = 1$ and $k_L(y) = -\delta^{-1}\, \mathrm{d}\, L$, we have

$$k_L(y) \,\Bigg|\, \left((ay+b)^2 - \frac{\zeta y - \gamma^2}{\delta} \right).$$

By comparing coefficients, we get

$$2ab\delta - \zeta = -\xi a^2 \quad \text{and} \quad \delta b^2 + \gamma^2 = \gamma^2 a^2$$

and hence

$$\zeta = \xi a^2 + 2ab\delta \quad \text{and} \quad \lambda = \frac{\zeta - \xi}{\delta} = 2ab + \xi \frac{a^2 - 1}{\delta} = 2ab + \frac{\xi b^2}{\gamma^2}.$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be a global basis of L yielding the Gram matrix (E) and let

$$\mathbf{b}_3 = a\mathbf{e}_3 + \frac{b}{\gamma}\mathbf{e}_4 \quad \text{and} \quad \mathbf{b}_4 = \frac{\delta b}{\gamma}\mathbf{e}_3 + a\mathbf{e}_4.$$

Then, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{b}_3, \mathbf{b}_4\}$ is also a global basis of L and

$$\phi_L(\mathbf{b}_3) = a^2 y + 2ab + \frac{b^2}{\gamma^2}(-\delta y + \xi) = \frac{\gamma^2 a^2 - \delta b^2}{\gamma^2} y + \left(2ab + \xi \frac{b^2}{\gamma^2}\right) = y + \lambda,$$

$$\phi_L(\mathbf{b}_4) = \left(\frac{\delta b}{\gamma}\right)^2 y + 2ab\delta + a^2(-\delta y + \xi) = -\delta \frac{\gamma^2 a^2 - \delta b^2}{\gamma^2} y + (\xi a^2 + 2ab\delta) = -\delta y + \zeta$$

and

$$B_L(\mathbf{b}_3, \mathbf{b}_4) = \frac{ab\delta}{\gamma} y + \gamma a^2 + \frac{\delta b^2}{\gamma} + \frac{ab}{\gamma}(-\delta y + \xi) = \frac{\gamma^2 a^2 + \delta b^2 + ab\xi}{\gamma}.$$

Therefore, we have

$$L \simeq \langle 1, -\delta \rangle \perp \begin{pmatrix} y + \lambda & \dfrac{\gamma^2 a^2 + \delta b^2 + ab\xi}{\gamma} \\ \dfrac{\gamma^2 a^2 + \delta b^2 + ab\xi}{\gamma} & -\delta y + \zeta \end{pmatrix}$$

and hence

$$\begin{aligned} \mathrm{d}\, L &= -\delta \left((y + \lambda)(-\delta y + \zeta) - \left(\frac{\gamma^2 a^2 + \delta b^2 + ab\xi}{\gamma} \right)^2 \right) \\ &= -\delta \left(y^2 + (\zeta - \lambda\delta) y - \left(\frac{\gamma^2 a^2 + \delta b^2 + ab\xi}{\gamma} \right)^2 \right). \end{aligned}$$

Still, we have $\mathrm{d}\, L = \mathrm{d}\, L'$ and hence we get

$$\frac{\gamma^2 a^2 + \delta b^2 + ab\xi}{\gamma} = \pm \gamma',$$

which confirms $L \simeq L'$ globally.

If $\mu = \delta$, by similar calculation, we also have the desired result. □

5. Universal lattices

Let L be a definite quaternary lattice. If L is universal, then the successive minima of L are $1, \delta, x$ and δx. Then L must be of the form (A) [2, Proposition, p.132] and hence $\deg \mathrm{d}\, L = 2$. Furthermore, as stated in [2], L must be quaternary. In this section, we will prove that the converse is also true.

Theorem 5.1. *Let L be a definite quaternary lattice such that* $\mathrm{d}\,L$ *is of degree 2. Then L is universal.*

Proof. We only have to show that the definite lattice L of the form (B) is universal. Moreover, because the class number of L is one, it's enough to show that L is universal at every prime spot v.
Case 1 : $\mathrm{d}\,L$ is irreducible.

If $v \neq y, \infty$, then we have the Jordan decomposition of L by Lemma 4.1:

$$L_v \simeq \left\langle 1, -\delta, \frac{\gamma^2}{y}, \frac{k_L(y)}{y} \right\rangle,$$

where $k_L(y) = -\delta y^2 + \xi y - \gamma^2$. Since $1, -\delta$ and $\dfrac{\gamma^2}{y}$ are units, $\left\langle 1, -\delta, \dfrac{\gamma^2}{y} \right\rangle$ is universal and hence so is L_v.

If $v = y$, suppose that the Jordan decomposition of L is of the form

$$\langle 1, -\delta, f(y), g(y) \rangle ,$$

where $f(y), g(y) \in \mathbb{F}_q[y]_y$. Because $-\delta f(y)g(y) = \omega^2 k_L(y)$ and $k_L(y)$ is not divided by y, $f(y)$ and $g(y)$ are units in $\mathbb{F}_q[y]_y$. Therefore $\langle 1, -\delta, f(y) \rangle$ is universal and hence so is L_y.

If $v = \infty$, take an arbitrary polynomial $f(y) \in \mathbb{F}_q[y]$ and M to be a lattice defined as follows:

$$M := \langle 1, -\delta, y, -\delta y + \xi, -f(y) \rangle .$$

Since $\mathrm{rank}(M) = 5$, M is indefinite and hence M_∞ is isotropic. So, there is a vector

$$\mathbf{b} = (b_1, b_2, b_3, b_4, b_5) \in (\mathbb{F}_q[y])^5$$

such that $\phi_M(\mathbf{b}) = 0$. However, since L is definite, L_∞ is anisotropic and hence b_5 must be non-zero. This result guarantees that L_∞ represents $f(y)$ over $\mathbb{F}_q[x]_\infty$. Therefore L_∞ is universal.

Case 2 : $\mathrm{d}\,L$ is reducible.

If $\mathrm{d}\,L$ is reducible, then L is diagonal by Lemma 3.1: if $\mathrm{d}\,L = -\delta^2(y+\sigma)(y+\tau)$, we get

$$L \simeq \langle 1, -\delta, \alpha'(y+\sigma), -\delta\alpha'(y+\sigma-\beta') \rangle ,$$

where $\alpha' = \gamma^2 - \delta\sigma^2$ and $\beta' = \delta^2\sigma(\sigma-\tau)^2$. Replacing $\alpha'(y+\sigma)$ by y again, we obtain

$$L \simeq \langle 1, -\delta, y, -\delta y + \zeta \rangle .$$

(1) If $v \neq y$, $-\delta y + \zeta$ or ∞, then both y and $-\delta y+\zeta$ are units. Therefore L_v contains universal sublattice $\langle 1, -\delta, y \rangle$ and $\langle 1, -\delta, -\delta y + \zeta \rangle$ and hence L_v is also universal.
(2) If $v = \infty$, the same argument as in the irreducible case proves the universality of L_∞.
(3) If $v = y$ and $\zeta \neq 0$. Then $-\delta y + \zeta$ is a unit. Therefore $\langle 1, -\delta, -\delta y + \zeta \rangle$ is universal and hence so is L_v.

(4) Suppose that $v = y$ and $\zeta = 0$. Let K be a sublattice of L defined as follows:

$$K := \langle 1, -\delta, -\delta y \rangle .$$

If K_v represents all $f(y) \in \mathbb{F}_q[y]$, we are done. Suppose there is a polynomial

$$f(y) = a_0 + a_0 y + \cdots + a_m y^m \in \mathbb{F}_q[y]$$

which is not represented by K_v. Then, by Lemma 2.1 (2) in [6, §2], $v_v(f(y)) = m$ is odd and $-a_m \in (\mathbb{F}_q)^2$ for $v \neq \infty$. (Note that Lemma 2.1 (1) in [6, §2] is the case not represented by K_∞.)

Let $\eta = (-a_m)^{-1}$ and define $K' := \left\langle 1, -\delta, -\delta(-\delta^{-1}\eta y) \right\rangle$. Then since $L \simeq \langle 1, -\delta, y, -\delta y \rangle$ and $\langle y, -\delta y \rangle$ represents εy for all $\varepsilon \in \mathbb{F}_q^\times$, K' is also a sublattice of L. If we rewrite $f(y)$ in the form

$$f(y) = b_0(-\delta^{-1}\eta y)^0 + \cdots + b_m(-\delta^{-1}\eta y)^m,$$

then, since $b_m(-\delta^{-1}\eta y)^m = a_m y^m$ and m is odd,

$$-b_m = -(-a_m\delta)^m a_m = (a_m^{(m+1)/2}\delta^{(m-1)/2})^2\delta.$$

Therefore, $-b_m$ is not a square. So $f(y)$ is represented by K'_v and hence by L_v.

(5) If $v = -\delta y + \zeta$, then replacing $-\delta y + \zeta$ by z and get the same result by (3) and (4).

□

Example 5.2. Let $q = 5$ and

$$L = \begin{pmatrix} 1 + 3x^2 & 2x - 2x^2 \\ 2x - 2x^2 & -2x^2 - 2x^4 \end{pmatrix} \perp \begin{pmatrix} -x & x \\ x & x \end{pmatrix}.$$

Then,

$$L_\infty \simeq \langle 1, -2, x, 2x \rangle$$

and hence L is definite. Moreover, since $\mathrm{d}\, L = x^2$, So, L is universal and diagonalizable.

Corollary 5.3. *The Four Conjecture is true.*

Proof. Let L be a definite quaternary lattice which represents $1, -\delta, x$, and $-\delta x$. Then, by [2],

$$L \simeq \langle 1, -\delta \rangle \perp \begin{pmatrix} \alpha x + \beta & \gamma \\ \gamma & -\delta\alpha x + \xi \end{pmatrix}.$$

Since $\deg(\mathrm{d}\, L) = 2$, L is universal. □

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