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VISCOSITY APPROXIMATION METHODS FOR NONEXPANSIVE SEMINGROUPS AND MONOTONE MAPPPINGS

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Abstract. Let C be a nonempty closed convex subset of real Hilbert space H and $\mathfrak{F} = \{S(t) : t \geq 0\}$ a nonexpansive self-mapping semigroup of C, and $f : C \to C$ is a fixed contractive mapping. Consider the process $\{x_n\}$:

$$\begin{cases} x_{n+1} = \beta_n x_n + (1-\beta_n) z_n \\ z_n = \alpha_n f(x_n) + (1-\alpha_n) S(t_n) P_C(x_n - r_n A x_n). \end{cases}$$

It is shown that $\{x_n\}$ converges strongly to a common element of the set of fixed points of nonexpansive semigroups and the set of solutions of the variational inequality for an inverse strongly-monotone mapping which solves some variational inequality.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, C be a nonempty closed convex subset of H. $S : C \to C$ is nonexpansive if $\|S(x) - S(y)\| \leq \|x - y\|$, for all $x, y \in C$. The set of fixed points of S is $F(S) = \{x \in C : x = Sx\}$. We assume that $F(S) \neq \phi$, it is well known that $F(S)$ is closed convex.

A nonexpansive semigroup is a family $\mathfrak{F} = \{S(t) : t \geq 0\}$ of self-mapping of C if the following conditions are satisfied:
(a) $S(0)x = x$ for all $x \in C$;
(b) $S(s + t) = S(s)S(t)$ for all $s, t \geq 0$;
(c) For each $t > 0$, $\|S(t)x - S(t)y\| \leq \|x - y\|, x, y \in C$.
(d) For each $x \in C$, the mapping $S(\cdot)x$ is continuous.
In this paper, we use F to denote the set of common fixed points of $\mathfrak{F}$; that is,

$$F = \{x \in C : S(t)x = x, t \geq 0\} = \bigcap_{t \geq 0} F(S(t)).$$

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The family $\{S(t) : t \geq 0\}$ is said to be uniformly asymptotically regular if for any $t \geq 0$ and for any bounded subset D of C,

$$\lim_{s\to\infty} \sup_{x\in D} \|S(t+s)x - S(s)x\| = 0.$$

A mapping $A : C \to H$ be a α-inverse strongly mapping, if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C$$

then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. Recall that the classical variational inequality problem, denoted by $VI(C, A)$, is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \ \forall y \in C.$$

In 2001, Moudafi [2] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on C, starting with an arbitrary initial $x_0 \in C$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sx_n, n \geq 0.$$

In 2007, Song and Xu [4] introduced the iterative process given as follow:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)S(t_n)x_n, n \geq 0.$$

They given an example of uniform asymptotically regular operator semigroup.

In this paper, motivated and inspired by the above results, we introduce an iterative scheme given as follows: $x_1 \in C$ and

$$\begin{cases} x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n \\ z_n = \alpha_n f(x_n) + (1 - \alpha_n)S(t_n)P_C(x_n - r_n Ax_n). \end{cases} \tag{1.1}$$

for all $n \in N$, where $\{\alpha_n\} \in [0, 1]$ and $\{r_n\} \subset (0,\infty)$ satisfy some appropriate conditions. We will prove that $\{x_n\}$ converges strongly to $q \in F \cap VI(C, A)$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot, \cdot\rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the metric projection of H onto C. It is well known that P_C satisfies :

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \ \forall x, y \in H, \tag{2.1}$$

and P_C is characterized by the following properties:

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \tag{2.2}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \forall x \in H, y \in C. \tag{2.3}$$

In the context of the variational inequality problem, this implies

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda Au), \forall \lambda > 0, \tag{2.4}$$

We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} &\|(I - \lambda A)x - (I - \lambda A)y\|^2 \\ &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 \end{aligned}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C to H and let $N_C v$ be normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\},$$

and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ [3].

Lemma 2.1. ([5]) *Let $\{x_n\}$, $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$, Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n, \forall\, n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.2. ([6]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers such that:*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \beta_n, n \geq 0,$$

where $\{\lambda_n\}, \{\beta_n\}$ satisfy the conditions:

(i) $\{\lambda_n\} \subset (0, 1)$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$,

(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\lambda_n} \leq 0$ or $\sum_{n=1}^{\infty} |\beta_n| < \infty$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. ([1]) *Let C a nonempty bounded closed convex subset of H and $\{S(t) : t \geq 0\}$ be a nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $\limsup_t \limsup_n \|x_n - S(t)x_n\| \to 0$, then $z \in F(S)$.*

3. Main result

Theorem 3.1. *Let H be a real Hilbert space, and C be a nonempty closed convex subset of H. Let $\{S(t) : t \geq 0\}$ be a uniform asymptotically regular nonexpansive semigroup from C to C, and A an α-inverse-strongly monotone mapping of C into H such that $F \cap VI(C, A) \neq \emptyset$. $f : C \to C$ is a contraction with coefficient $k \in (0, 1)$, Suppose $\{x_n\}$ be sequences generated by (1.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ is sequence in $(0, 1)$ and $\lim_{n\to\infty} t_n = \infty$. If $\{r_n\}$ are chosen so that $r_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} |r_n - r_{n+1}| = 0, \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Then $\{x_n\}$ converges strongly to $q \in F \cap VI(C, A)$, which is the unique solution in the $F \cap VI(C, A)$ to the following variational inequality

$$\langle f(q) - q, p - q\rangle \leq 0, \qquad \forall\ p \in F \cap VI(C, A).$$

Proof. It is easy to see that the uniqueness of a solution of the above variational inequality. Since $I - f$ is strongly monotone, so the variational inequality has only one solution.

Put $y_n = P_C(x_n - r_n Ax_n)$ for every $n = 0, 1, 2, ...$ Let $p \in F \cap VI(C, A)$. Notice that

$$\begin{aligned} \|y_n - p\| &= \|P_C(x_n - r_n Ax_n) - P_C(p - r_n Ap)\| \\ &\leq \|x_n - r_n Ax_n - (p - r_n Ap)\| \\ &\leq \|x_n - p\|. \end{aligned}$$

Then we compute that

$$\begin{aligned} \|z_n - p\| &\leq \alpha_n \|f(x_n) - p\| + (1 - \alpha_n)\|S(t_n)y_n - p\| \\ &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n)\|y_n - p\| \\ &\leq \alpha_n k\|x_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\ &= [1 - \alpha_n(1 - k)]\|x_n - p\| + \alpha_n \|f(p) - p\|. \end{aligned}$$

so we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \beta_n \|x_n - p\| + (1 - \beta_n)\|z_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n)[1 - \alpha_n(1 - k)]\|x_n - p\| \\ &\quad + (1 - \beta_n)\alpha_n \|f(p) - p\| \\ &\leq [1 - (1 - \beta_n)\alpha_n(1 - k)]\|x_n - p\| + (1 - \beta_n)\alpha_n \|f(p) - p\|. \end{aligned}$$

By induction

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1 - k}\|f(p) - p\|\}, n \geq 0.$$

Therefore $\{x_n\}$ is bounded, we have $\{z_n\}$, $\{y_n\}$, $\{S(t_n)y_n\}$, $\{Ax_n\}$, $\{f(x_n)\}$

are also bounded. Since $I - rA$ is nonexpansive, we have

$$\begin{aligned}\|y_{n+1} - y_n\| &\le \|x_{n+1} - r_{n+1}Ax_{n+1} - (x_n - r_nAx_n)\| \\ &\le \|x_{n+1} - r_{n+1}Ax_{n+1} - (x_n - r_{n+1}Ax_n)\| + |r_n - r_{n+1}|\|Ax_n\| \\ &\le \|x_{n+1} - x_n\| + |r_n - r_{n+1}|\|Ax_n\|.\end{aligned}$$

So we obtain (for some appropriate constant $M > 0$)

$$\begin{aligned}\|z_{n+1} - z_n\| &\le \alpha_{n+1}\|f(x_{n+1}) - f(x_n)\| + |\alpha_{n+1} - \alpha_n|\|f(x_n) - S(t_{n+1})y_n\| \\ &\quad + (1 - \alpha_{n+1})\|S(t_{n+1})(y_{n+1} - y_n)\| \\ &\quad + (1 - \alpha_n)\|S(t_{n+1})y_n - S(t_n)y_n\| \\ &\le \alpha_{n+1}k\|x_{n+1} - x_n\| + |\alpha_n - \alpha_{n+1}|M + (1 - \alpha_{n+1})\|y_{n+1} - y_n\| \\ &\quad + (1 - \alpha_n)\|S(t_{n+1})y_n - S(t_n)y_n\| \\ &\le (1 - \alpha_{n+1} + \alpha_{n+1}k)\|x_{n+1} - x_n\| + |\alpha_n - \alpha_{n+1}|M \\ &\quad + |r_n - r_{n+1}|M + (1 - \alpha_n)\|S(t_{n+1})y_n - S(t_n)y_n\|\end{aligned}$$

So we have

$$\begin{aligned}&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\le (\alpha_{n+1}k - \alpha_{n+1})\|x_{n+1} - x_n\| + |\alpha_n - \alpha_{n+1}|M \\ &\quad + |r_n - r_{n+1}|M + (1 - \alpha_n)\|S((t_{n+1} - t_n) + t_n)y_n - S(t_n)y_n\| \\ &\le (\alpha_{n+1}k - \alpha_{n+1})\|x_{n+1} - x_n\| + |\alpha_n - \alpha_{n+1}|M \\ &\quad + |r_n - r_{n+1}|M + (1 - \alpha_n)\sup_{y \in \{y_n\}, t \ge 0}\|S(t + t_n)y - S(t_n)y\|\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$ and the uniform asymptotic regularity of the nonexpansive semigroup, we have

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) = 0.$$

In view of Lemma 2.1, we have $\lim_{n\to\infty} \|z_{n+1} - x_n\| = 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$. From the definition of x_n, we have

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + (1 - \alpha_n)\|S(t_n)y_n - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + (1 - \alpha_n)\|y_n - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + (1 - \alpha_n)[\|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2] \\ &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 + (1 - \alpha_n)a(b - 2\alpha)\|Ax_n - Ap\|^2\end{aligned}$$

For $p \in F \cap VI(C, A)$, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 \\ &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)[\alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 \\ &\quad + (1 - \alpha_n)a(b - 2\alpha)\|Ax_n - Ap\|^2] \\ &\le \|x_n - p\|^2 + (1 - \alpha_n)(1 - \beta_n)a(b - 2\alpha)\|Ax_n - Ap\|^2 \\ &\quad + \alpha_n\|f(x_n) - p\|^2.\end{aligned}$$

So, we obtain

$$
\begin{aligned}
&(1-\alpha_n)(1-\beta_n)a(2\alpha-b)\|Ax_n-Ap\|^2\\
&\le \alpha_n\|f(x_n)-p\|^2+(\|x_{n+1}-p\|+\|x_n-p\|)\|x_n-x_{n+1}\|.
\end{aligned}
$$

Since $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, $\alpha_n\to 0$ and $\|x_n-x_{n+1}\|\to 0$, then $\|Ax_n-Ap\|\to 0, n\to\infty$.

Further, from (2.1) we have

$$
\begin{aligned}
\|y_n-p\|^2 &= \|P_C(x_n-r_nAx_n)-P_C(p-r_nAp)\|^2\\
&\le \langle x_n-r_nAx_n-(p-r_nAp), y_n-p\rangle\\
&= \frac{1}{2}\{\|(x_n-r_nAx_n-(p-r_nAp)\|^2+\|y_n-p\|^2\\
&\quad -\|x_n-r_nAx_n-(p-r_nAp)-(y_n-p)\|^2\}\\
&\le \frac{1}{2}\{\|x_n-p\|^2+\|y_n-p\|^2-\|x_n-y_n-r_n(Ax_n-Ap)\|^2\}\\
&= \frac{1}{2}\{\|x_n-p\|^2+\|y_n-p\|^2-\|x_n-y_n\|^2\\
&\quad +2r_n\langle x_n-y_n, Ax_n-Ap\rangle-r_n^2\|Ax_n-Ap\|^2\}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|y_n-p\|^2 &\le \|x_n-p\|^2-\|x_n-y_n\|^2+2r_n\langle x_n-y_n, Ax_n-Ap\rangle\\
&\quad -r_n^2\|Ax_n-Ap\|^2\\
&\le \|x_n-p\|^2-\|x_n-y_n\|^2+2r_n\|x_n-y_n\|\|Ax_n-Ap\|.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|z_n-p\|^2\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)[\alpha_n\|f(x_n)-p\|^2\\
&\quad +(1-\alpha_n)\|S(t_n)y_n-p\|^2]\\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2+\alpha_n\|f(x_n)-p\|^2\\
&\le \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2-(1-\beta_n)\|x_n-y_n\|^2\\
&\quad +2r_n\|x_n-y_n\|\|Ax_n-Ap\|.
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(1-\beta_n)\|x_n-y_n\|^2 \le\ & 2r_n\|x_n-y_n\|\|Ax_n-Ap\|+\alpha_n\|f(x_n)-p\|^2\\
&+\|x_n-p\|^2-\|x_{n+1}-p\|^2
\end{aligned}
$$

Since $\alpha_n\to 0$, $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, $\|x_n-x_{n+1}\|\to 0$ and $\|Ax_n-Ap\|\to 0$, we have $\|x_n-y_n\|\to 0$. Since

$$
\begin{aligned}
\|x_n-S(t_n)x_n\| &\le \|x_n-z_n\|+\|z_n-S(t_n)y_n\|+\|S(t_n)y_n-S(t_n)x_n\|\\
&\le \|x_n-z_n\|+\alpha_n\|f(x_n)-S(t_n)y_n\|+\|y_n-x_n\|.
\end{aligned}
$$

From $\alpha_n \to 0$, we have $\lim_{n\to\infty} \|x_n - S(t_n)x_n\| = 0$. Further

$$\begin{aligned}\|x_n - S(t)x_n\| &\leq \|x_n - S(t_n)x_n\| + \|S(t_n)x_n - S(t)S(t_n)x_n\| \\ &\quad + \|S(t)S(t_n)x_n - S(t)x_n\| \\ &\leq 2\|x_n - S(t_n)x_n\| + \sup_{x\in\{x_n\},t\geq 0} \|S(t+t_n)x - S(t_n)x\|.\end{aligned}$$

Using this and the uniform asymptotic regularity of the nonexpansive semigroup, we get

$$\|x_n - S(t)x_n\| \to 0, \forall t \geq 0. \tag{3.1}$$

Choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle f(q) - q, S(t_n)y_n - q\rangle = \lim_{i\to\infty}\langle f(q) - q, S(t_{n_i})y_{n_i} - q\rangle.$$

As $\{y_{n_i}\}$ is bounded, Without loss of generality that $y_{n_i} \rightharpoonup z \in C$. Since $\|S(t_n)y_n - y_n\| \to 0$, we obtain $S(t_{n_i})y_{n_i} \rightharpoonup z \in C$. In fact, Let

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v, w) \in G(T)$. Since $w - Av \in N_C v$ and $y_n \in C$ we have

$$\langle v - y_n, w - Av\rangle \geq 0.$$

On the other hand, from $y_n = P_C(x_n - r_n Ax_n)$, we have $\langle v - y_n, y_n - (x_n - r_n Ax_n)\rangle \geq 0$ and hence

$$\langle v - y_n, \frac{y_n - x_n}{r_n} + Ax_n\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}\langle v - y_{n_i}, w\rangle &\geq \langle v - y_{n_i}, Av\rangle \\ &\geq \langle v - y_{n_i}, Av\rangle - \langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_n}} + Ax_{n_i}\rangle \\ &= \langle v - y_{n_i}, Av - Ay_{n_i}\rangle + \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i}\rangle \\ &\quad - \langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle \\ &\geq \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i}\rangle - \langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle\end{aligned}$$

Hence we have $\langle v - z, w\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $z \in T^{-1}0$ and $z \in VI(C, A)$. From (3.1) and Lemma 2.3, we have $z \in F$.

Further we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(q) - q, S(t_n)y_n - q\rangle &= \lim_{i\to\infty}\langle f(q) - q, S(t_{n_i})y_{n_i} - q\rangle \\ &= \langle f(q) - q, z - q\rangle \leq 0\end{aligned}$$

We compute that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \beta_n\|x_n-q\|^2+(1-\beta_n[2\alpha_n(1-\alpha_n)\langle f(x_n)-q,S(t_n)y_n-q\rangle \\
&\quad +\alpha_n^2\|f(x_n)-q\|^2+(1-\alpha_n)^2\|S(t_n)y_n-q\|^2] \\
&\le \beta_n\|x_n-q\|^2+2(1-\beta_n)\alpha_n(1-\alpha_n)\langle f(x_n)-f(q),S(t_n)y_n-q\rangle \\
&\quad +2(1-\beta_n)\alpha_n(1-\alpha_n)\langle f(q)-q,S(t_n)y_n-q\rangle \\
&\quad +(1-\beta_n)\alpha_n^2\|f(x_n)-q\|^2+(1-\beta_n)(1-\alpha_n)^2\|y_n-q\|^2 \\
&\le \beta_n\|x_n-q\|^2+2k(1-\beta_n)\alpha_n(1-\alpha_n)\|x_n-q\|^2 \\
&\quad +2(1-\beta_n)\alpha_n(1-\alpha_n)\langle f(q)-q,S(t_n)y_n-q\rangle \\
&\quad +(1-\beta_n)\alpha_n^2\|f(x_n)-q\|^2+(1-\beta_n)(1-\alpha_n)^2\|x_n-q\|^2 \\
&\le \{1-(1-\beta_n)\alpha_n[2-\alpha_n-2k(1-\alpha_n)]\}\|x_n-q\|^2 \\
&\quad +(1-\beta_n)\alpha_n^2\|f(x_n)-q\|^2 \\
&\quad +2(1-\beta_n)\alpha_n(1-\alpha_n)\langle f(q)-q,S(t_n)y_n-q\rangle \\
&=(1-\bar{\alpha_n})\|x_n-q\|^2+\bar{\alpha_n}\bar{\beta_n}.
\end{aligned}$$

where

$$\bar{\alpha_n}=(1-\beta_n)\alpha_n[2-\alpha_n-2k(1-\alpha_n)],$$

$$\bar{\beta_n}=\frac{\alpha_n\|f(x_n)-q\|^2+(1-\alpha_n)\langle f(q)-q,S(t_n)y_n-q\rangle}{2-\alpha_n-2k(1-\alpha_n)}.$$

It is easily seen that $\bar{\alpha_n}\to 0, \sum_{n=1}^{\infty}\bar{\alpha_n}=\infty$, and $\limsup_{n\to\infty}\bar{\beta_n}\le 0$, by Lemma 2.2, we obtain $x_n\to q$. □

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