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NOTE ON UPPER BOUND SIGNED 2-INDEPENDENCE IN DIGRAPHS

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Abstract. Let D be a finite digraph with the vertex set $V(D)$ and arc set $A(D)$. A two-valued function $f : V(D) \to \{-1, 1\}$ defined on the vertices of a digraph D is called a signed 2-independence function if $f(N^-[v]) \le 1$ for every v in D. The weight of a signed 2-independence function is $f(V(D)) = \sum_{v \in V(D)} f(v)$. The maximum weight of a signed 2-independence function of D is the signed 2-independence number ${\alpha_s}^2(D)$ of D. Recently, Volkmann [3] began to investigate this parameter in digraphs and presented some upper bounds on $\alpha_s^2(D)$ for general digraph D. In this paper, we improve upper bounds on ${\alpha_s}^2(D)$ given by Volkmann [3].

1. Introduction

All digraphs considered in this paper are finite, without loops and multiple arcs. For notation and terminology not defined here, we generally follow [1]. For a digraph D, we denote the vertex set of D and the arc set of D by $V(D)$ and $A(D)$, respectively. We say that u is an *in-neighbor* of v and v is an *out-neighbor* of u if uv is an arc of D. For a vertex $v \in V(D)$, the sets of in-neighbors and out-neighbors of v are called the *open in-neighborhood* and *open out-neighborhood* of v are denoted by $N_D^-(v)$ and $N_D^+(v)$, respectively. The *closed in-neighborhood* of v is $N_D^-[v] = N_D^-(v) \cup \{v\}$. The numbers $d_D^-(v) = |N_D^-(v)|$ and $d_D^+(v) = |N_D^+(v)|$ are the *in-degree* and *out-degree* of v, respectively. We use $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$, and $\Delta^+ = \Delta^+(D)$ to denote the *minimum in-degree, maximum in-degree, minimum out-degree* and *maximum out-degree* of a vertex in D, respectively. For $S \subseteq V(D)$, $D[S]$ denotes the subdigraph induced by S. If $S \subseteq V(D)$ and $v \in V(D)$, then $E(S, v)$ is the set of arcs from S to v. If S and T are two disjoint vertex sets of a digraph D, then $E(S, T)$ is the set of arcs from S to T.

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For a function $f : V(D) \rightarrow \{-1, 1\}$, the *weight* of f is defined $w(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V(D)$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V(D))$. For a vertex $v \in V(D)$, we denote $f(N^-[v])$ by $f[v]$ for notational convenience.

The study of signed 2-independence number of undirected graphs was initiated by Zelink [4] and continued in [2] and elsewhere. Recently, Volkmann [3] began to investigate this parameter in digraphs. Formally, a function $f : V(D) \rightarrow \{-1, 1\}$ is called a *signed 2-independence function* (abbreviated by S2IF) if $f[v] \leq 1$ for every vertex $v \in V(D)$. The *singed 2-independence number*, denoted by $\alpha_s^2(D)$, of D is the maximum weight of a S2IF on D. We call a S2IF of weight $\alpha_s^2(D)$ a $\alpha_s^2(D)$-function on D. Volkmann [3] presented some upper bounds on $\alpha_s^2(D)$ for general digraph D,

Throughout this paper, if f is a S2IF of D, then we let P and M denote the sets of those vertices in D which are assigned under f the value 1 and -1, respectively and let $p = |P|$ and $m = |M|$. Then $|V(D)| = p + m$ and $\alpha_s^2(D) = p - m$.

In this paper, we improve upper bounds on ${\alpha_s}^2(D)$ given by Volkmann [3].

2. Main results

In this section, we study to improve upper bounds on ${\alpha_s}^2(D)$ given by Volkmann [3].

Theorem 2.1. *Let D be a digraph of order n. If n_0 is the number of vertices of odd in-degree of $V(D)$, Then*

$$
{\alpha_s}^2(D) \leq \begin{cases} \dfrac{\{(\Delta^+ + 1) - 2\lceil \frac{\delta^-}{2} \rceil\}n - 2n_0}{\Delta^+ + 1} & if\ \delta^-\ is\ even \\ \dfrac{(\Delta^+ + 1) - 2\lceil \frac{\delta^-}{2} \rceil}{\Delta^+ + 1} n & if\ \delta^-\ is\ odd. \end{cases}
$$

Proof. Let f be a signed 2-independence function on D for which ${\alpha_s}^2(D) = f(V(D))$. Put $P_0 = \{v \in P | d^-(v) \text{ is odd}\}$, $P_e = P - P_0$, $M_0 = \{v \in M | d^-(v) \text{ is odd}\}$, $M_e = M - M_0$, and let $p_0 = |P_0|, p_e = |P_e|, m_0 = |M_0|$ and $m_e = |M_e|$.

By the condition $f[v] \leq 1$ for each $v \in V(D)$, it follows that

$$\text{(1)} \qquad \text{if } v \in P_0, \text{ then } |E(P, v)| \leq |E(M, v)| - 1,$$

$$\text{(2)} \qquad \text{if } v \in P_e, \text{ then } |E(P, v)| \leq |E(M, v)|,$$

$$\text{(3)} \qquad \text{if } v \in M_0, \text{ then } |E(P, v)| \leq |E(M, v)| + 1,$$

$$\text{(4)} \qquad \text{if } v \in M_e, \text{ then } |E(P, v)| \leq |E(M, v)| + 2.$$

Moreover,

$$\text{(5)} \qquad \delta^- \leq d^-(v) = |E(P, v)| + |E(M, v)|.$$

Now, from (1), (2), (3), (4) and (5), we obtain

$$|E(M,v)| \geq \lceil \frac{\delta^- + 1}{2} \rceil \text{ for each } v \in P_0, \tag{6}$$

$$|E(M,v)| \geq \lceil \frac{\delta^-}{2} \rceil \text{ for each } v \in P_e, \tag{7}$$

$$|E(M,v)| \geq \lceil \frac{\delta^- - 1}{2} \rceil \text{ for each } v \in M_0, \tag{8}$$

$$\text{and } |E(M,v)| \geq \lceil \frac{\delta^- - 2}{2} \rceil \text{ for each } v \in M_e. \tag{9}$$

Using (6), (7), (8) and (9), we have

$$\begin{aligned} |E(M,P)| &= \sum_{v \in P} |E(M,v)| = \sum_{v \in P_0} |E(M,v)| + \sum_{v \in P_e} |E(M,v)| \\ &\geq p_0 \lceil \frac{\delta^- + 1}{2} \rceil + (p - p_0) \lceil \frac{\delta^-}{2} \rceil \\ &= (n-m) \lceil \frac{\delta^-}{2} \rceil + p_0 (\lceil \frac{\delta^- + 1}{2} \rceil - \lceil \frac{\delta^-}{2} \rceil) \end{aligned} \tag{10}$$

and

$$\begin{aligned} |E(D[M])| &= \sum_{v \in M} |E(M,v)| = \sum_{v \in M_0} |E(M,v)| + \sum_{v \in M_e} |E(M,v)| \\ &\geq m_0 \lceil \frac{\delta^- - 1}{2} \rceil + (m - m_0) \lceil \frac{\delta^- - 2}{2} \rceil \\ &= m \lceil \frac{\delta^- - 2}{2} \rceil + m_0 (\lceil \frac{\delta^- - 1}{2} \rceil - \lceil \frac{\delta^- - 2}{2} \rceil). \end{aligned} \tag{11}$$

From (11), we get that

$$\begin{aligned} |E(M,P)| &= \sum_{v \in M} d^+(v) - |E(D[M])| \\ &\leq m\Delta^+ - m \lceil \frac{\delta^- - 2}{2} \rceil - m_0 (\lceil \frac{\delta^- - 1}{2} \rceil - \lceil \frac{\delta^- - 2}{2} \rceil). \end{aligned} \tag{12}$$

Now, we consider two cases.

Case1 : $\delta^- =$ even.

It is easy to check that $\lceil \frac{\delta^- + k}{2} \rceil - \lceil \frac{\delta^- + (k-1)}{2} \rceil = 1$ ($k = 1$ or -1) and $\lceil \frac{\delta^-}{2} \rceil - \lceil \frac{\delta^- - 2}{2} \rceil = 1$. It implies $(n-m)\lceil \frac{\delta^-}{2} \rceil + p_0 \leq m\Delta^+ - m \lceil \frac{\delta^- - 2}{2} \rceil - m_0$ from (10) and (12).

Since $n_0 = p_0 + m_0$, we have $m \geq \frac{n \lceil \frac{\delta^-}{2} \rceil + n_0}{\Delta^+ + 1}$. Thus

$$\alpha_s{}^2(D) = n - 2m \leq n - 2\frac{n \lceil \frac{\delta^-}{2} \rceil + n_0}{\Delta^+ + 1} = \frac{\{(\Delta^+ + 1) - 2\lceil \frac{\delta^-}{2} \rceil\} n - 2n_0}{\Delta^+ + 1}.$$

Case2 : $\delta^- =$ odd.

Since $\lceil \frac{\delta^-+k}{2} \rceil - \lceil \frac{\delta^-+(k-1)}{2} \rceil = 0$ ($k = 1$ or -1) and $\lceil \frac{\delta^-}{2} \rceil - \lceil \frac{\delta^--2}{2} \rceil = 1$, we get $(n-m)\lceil \frac{\delta^-}{2} \rceil \leq m\Delta^+ - m\lceil \frac{\delta^--2}{2} \rceil$, from (10)and (12).

Thus $m \geq \frac{n\lceil \frac{\delta^-}{2} \rceil}{\Delta^+ + 1}$ and

$$\alpha_s{}^2(D) = n - 2m \leq n - 2\frac{n\lceil \frac{\delta^-}{2} \rceil}{\Delta^+ + 1} = \frac{(\Delta^+ + 1) - 2\lceil \frac{\delta^-}{2} \rceil}{\Delta^+ + 1} n.$$

□

Corollary 2.2. ([3, Theorem 12]) *Let D be a digraph of order n. Then*

$$\alpha_s{}^2(D) \leq \frac{\Delta^+ + 1 - 2\lceil \frac{\delta^-}{2} \rceil}{\Delta^+ + 1} n.$$

Theorem 2.3. *Let D be a digraph of order n. If n_0 is the number of vertices of odd in-degree of $V(D)$, Then*

$$\alpha_s{}^2(D) \leq \begin{cases} \dfrac{(2\lfloor \frac{\Delta^-}{2} \rfloor + 1 - \delta^+)n - 2n_0}{\delta^+ + 1} & \text{if } \Delta^- \text{ is even} \\ \dfrac{2\lfloor \frac{\Delta^-}{2} \rfloor + 1 - \delta^+}{\delta^+ + 1} n & \text{if } \Delta^- \text{ is odd.} \end{cases}$$

Proof. Let f be a signed 2-independence function on D for which $\alpha_s{}^2(D) = f(V(D))$. Let $P_0, M_0, P_e, M_e, p_0, p_e, m_0$ and m_e be defined as in the proof of Theorem 1. From (1), (2), (3), (4) in the proof of Theorem 1, and

$$\Delta^- \geq d^-(v) = |E(P,v)| + |E(M,v)|. \tag{13}$$

Using (1), (2), (3), (4) in the proof of Theorem 1, and (13), we have

$$|E(P,v)| \leq \lfloor \frac{\Delta^- - 1}{2} \rfloor \text{ for each } v \in P_0, \tag{14}$$

$$|E(P,v)| \leq \lfloor \frac{\Delta^-}{2} \rfloor \text{ for each } v \in P_e, \tag{15}$$

$$|E(P,v)| \leq \lfloor \frac{\Delta^- + 1}{2} \rfloor \text{ for each } v \in M_0, \tag{16}$$

$$\text{and } |E(P,v)| \leq \lfloor \frac{\Delta^- + 2}{2} \rfloor \text{ for each } v \in M_e. \tag{17}$$

From (14), (15), (16) and (17), we get

$$\begin{aligned} |E(P,M)| &= \sum_{v \in M} |E(P,v)| = \sum_{v \in M_0} |E(P,v)| + \sum_{v \in M_e} |E(P,v)| \\ &\leq m_0 \lfloor \frac{\Delta^- + 1}{2} \rfloor + (m - m_0) \lfloor \frac{\Delta^- + 2}{2} \rfloor \\ &= m \lfloor \frac{\Delta^- + 2}{2} \rfloor + m_0 (\lfloor \frac{\Delta^- + 1}{2} \rfloor - \lfloor \frac{\Delta^- + 2}{2} \rfloor) \qquad (18) \end{aligned}$$

and

$$|E(D[P])| = \sum_{v\in P}|E(P,v)| = \sum_{v\in P_0}|E(P,v)| + \sum_{v\in P_e}|E(P,v)|$$

$$\le p_0\lfloor\frac{\Delta^- - 1}{2}\rfloor + p_e\lfloor\frac{\Delta^-}{2}\rfloor = p_0\lfloor\frac{\Delta^- - 1}{2}\rfloor + (p - p_0)\lfloor\frac{\Delta^-}{2}\rfloor$$

$$= p\lfloor\frac{\Delta^-}{2}\rfloor + p_0(\lfloor\frac{\Delta^- - 1}{2}\rfloor - \lfloor\frac{\Delta^-}{2}\rfloor). \tag{19}$$

From (19), we have

$$|E(P,M)| = \sum_{v\in P} d^+(v) - |E(D[P])|$$

$$\ge p\delta^+ - p\lfloor\frac{\Delta^-}{2}\rfloor - p_0(\lfloor\frac{\Delta^- - 1}{2}\rfloor - \lfloor\frac{\Delta^-}{2}\rfloor). \tag{20}$$

Now, we consider two cases. Case1 : Δ^- is even.

Using (18) and (20),

$$p\delta^+ - p\lfloor\frac{\Delta^-}{2}\rfloor + p_0 \le m\lfloor\frac{\Delta^- + 2}{2}\rfloor - m_0. \tag{21}$$

Substitute $p = n - m$ into (21), $(n-m)(\delta^+ - \lfloor\frac{\Delta^-}{2}\rfloor) + p_0 + m_0 \le m\lfloor\frac{\Delta^-+2}{2}\rfloor$.

$m(\delta^+ + 1) \ge n(\delta^+ - \lfloor\frac{\Delta^-}{2}\rfloor) + n_0$, and $m \ge \frac{n(\delta^+ - \lfloor\frac{\Delta^-}{2}\rfloor) + n_0}{\delta^+ + 1}$. It follows that

$$\alpha_s{}^2(D) = n - 2m \le n - \frac{2\{n(\delta^+ - \lfloor\frac{\Delta^-}{2}\rfloor) + n_0\}}{\delta^+ + 1}$$

$$= \frac{(2\lfloor\frac{\Delta^-}{2}\rfloor + 1 - \delta^+)n - 2n_0}{\delta^+ + 1}.$$

Case2 : Δ^- is odd.

Using (18) and (20),

$$p\delta^+ - p\lfloor\frac{\Delta^-}{2}\rfloor \le m\lfloor\frac{\Delta^- + 2}{2}\rfloor. \tag{22}$$

Substitute $p = n - m$ into (22), $(n-m)(\delta^+ - \lfloor\frac{\Delta^-}{2}\rfloor) \le m\lfloor\frac{\Delta^-+2}{2}\rfloor$.
Therefore, $n(\delta^+ - \lfloor\frac{\Delta^-}{2}\rfloor) \le m(\delta^+ + 1)$.
It follows that

$$\alpha_s{}^2(D) = n - 2m \le n - 2\frac{n(\delta^+ - \lfloor\frac{\Delta^-}{2}\rfloor)}{\delta^+ + 1} = \frac{2\lfloor\frac{\Delta^-}{2}\rfloor + 1 - \delta^+}{\delta^+ + 1}n.$$

□

Corollary 2.4. ([3, Theorem 13]) *If D is a digraph of order n, then*

$$\alpha_s{}^2(D) \le \frac{2\lfloor\frac{\Delta^-}{2}\rfloor + 1 - \delta^+}{\delta^+ + 1}n.$$

Theorem 2.5. *Let D be a digraph of order n such that $\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor \geq 0$. Let n_0 be the number of vertices whose in-degree of V(D) is odd and m_0 the number of vertices whose in-degree is odd and assigned value is -1. Then*

$$\alpha_s{}^2(D) \leq \begin{cases} n+1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor \ -2\sqrt{\frac{1}{4}(1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor)^2 + n(\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor) + n_0} \\ \qquad\qquad \text{if } \Delta^- \text{ is even} \\ n+1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor \ -2\sqrt{\frac{1}{4}(1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor)^2 + n(\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor) + m_0} \\ \qquad\qquad \text{if } \Delta^- \text{ is odd.} \end{cases}$$

Proof. Let f be a signed 2-independence function on D for which $\alpha_s{}^2(D) = f(V(D))$, and let $P_0, M_0, P_e, M_e, p_0, p_e, m_0$ and m_e be defined as in the proof of Theorem 1. By the definition of S2IF, each vertex of M_0 has at most m in-neighbors in P and each vertex of M_e has at most $(m+1)$ in-neighbors in P. Thus $|E(P, M_0)| \leq m_0 m$ and $|E(P, M_e)| \leq (m - m_0)(m+1)$. It follows that

$$|E(P, M)| = |E(P, M_0)| + |E(P, M_e)| \leq m(m+1) - m_0.$$

Using (20) in the proof of Theorem 3,

$$(n-m)(\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor) + p_0(\lfloor \frac{\Delta^-}{2} \rfloor - \lfloor \frac{\Delta^- - 1}{2} \rfloor) \leq m^2 + m - m_0.$$

Now, we consider two cases.

Case1 : Δ^- is even.

$$(n-m)(\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor) + p_0 \leq m^2 + m - m_0.$$

Thus

$$m^2 + (1 + \delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor)m - n(\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor) - n_0 \geq 0,$$

and

$$m \geq -\frac{1 + \delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor}{2} + \sqrt{\frac{1}{4}(1 + \delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor)^2 + n(\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor) + n_0}.$$

Now, we get the bound as follows

$$\begin{aligned} \alpha_s{}^2(D) &= n - 2m \\ &\leq n+1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor - 2\sqrt{\frac{1}{4}(1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor)^2 + n(\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor) + n_0}. \end{aligned}$$

Case2 : Δ^- is odd.

Since $(n-m)(\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor) \leq m^2 + m - m_0$, it implies that

$$m \geq -\frac{1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor}{2} + \sqrt{\frac{1}{4}(1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor)^2 + n(\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor) + m_0}.$$

So, we get the bound as follows

$$\begin{aligned} {\alpha_s}^2(D) &= n - 2m \\ &\leq n+1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor - 2\sqrt{\frac{1}{4}(1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor)^2 + n(\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor) + m_0}. \end{aligned}$$

□

Corollary 2.6. ([3, Theorem 14]) *Let D be a digraph of order n such that $\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor \geq 0$. Then*

$${\alpha_s}^2(D) \leq \begin{cases} n+1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor & -2\sqrt{\frac{1}{4}(1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor)^2 + n(\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor)} \\ & \text{if } \Delta^- \text{ is even} \\ n+1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor & -2\sqrt{\frac{1}{4}(1+\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor)^2 + n(\delta^+ - \lfloor \frac{\Delta^-}{2} \rfloor)} \\ & \text{if } \Delta^- \text{ is odd.} \end{cases}$$

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