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ALMOST PRINCIPALLY SMALL INJECTIVE RINGS

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Abstract. Let R be a ring and M a right R-module, $S = \mathrm{End}_R(M)$. The module M is called almost principally small injective (or APS-injective for short) if, for any $a \in J(R)$, there exists an S-submodule X_a of M such that $l_M r_R(a) = Ma \oplus X_a$ as left S-modules. If R_R is an APS-injective module, then we call R a right APS-injective ring. We develop, in this paper, APS-injective rings as a generalization of PS-injective rings and AP-injective rings. Many examples of APS-injective rings are listed. We also extend some results on PS-injective rings and AP-injective rings to APS-injective rings.

1. Introduction

Let R be a ring. A right ideal I of R is called small if, for every proper right ideal K of R, $K+I \neq R$. Recall that a ring R is right principally small injective (or PS-injective) (resp. P-injective, small injective, mininjective) if every R-homomorphism $f : I \to R$, for every principally small (resp. principally, small, minimal) right ideal I, can be extended to R. The detailed discussion of P-injective, small injective and mininjective rings can be found in [2, 3, 4, 8, 9, 10, 12]. The concept of PS-injective rings was first introduced in [14] as a generalization of P-injective rings and small injective rings. It was shown that every right PS-injective ring is also right mininjective. In [11], Page and Zhou introduced AP-injectivity and AGP-injectivity of modules and rings. Given a right R-module M, $S = \mathrm{End}_R(M)$. The module M is called AP-injective if, for any $a \in R$, there exists an S-submodule X_a of M such that $l_M r_R(a) = Ma \oplus X_a$ as left S-modules. The module M is called AGP-injective if, for any $0 \neq a \in R$, there exists a positive integer $n = n(a)$ and an S-submodule X_a of M such that $a^n \neq 0$ and $l_M r_R(a^n) = Ma^n \oplus X_a$ as left S-modules. A ring R is called right AP-injective (resp. AGP-injective) if R_R is an AP-injective (resp. AGP-injective) module. Many of the results on right P-injective rings were obtained for the two classes of right AP-injective rings and right AGP-injective rings. In [17], Zhou continued the study of left AP-injective rings and left AGP-injective rings with various chain conditions.

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In the present paper, we say that a right R-module M is APS-injective if, for any $a \in J(R)$, there exists an S-submodule X_a of M such that $l_M r_R(a) = Ma \oplus X_a$ as left S-modules. A ring R is called right APS-injective if R_R is an APS-injective module. Similarly, we can define a left APS-injective ring. Some examples are listed to show that APS-injective rings are the proper generalization of PS-injective rings and AP-injective rings. It is also shown that there are many similarities between AP-injective rings and APS-injective rings. In light of this fact, some results on PS-injective rings and AP-injective rings are as the corollaries of our results, respectively.

Throughout R is an associative ring with identity and all modules are unitary. $J = J(R)$, $soc(R_R)$ and $Z(R_R)$ denote the Jacobson radical, right socle and right singular ideal of R, respectively. For a right R-module M, let $S = \mathrm{End}_R(M)$, then we have an (S, R)-bimodule M. If X is a subset of R, the right (left) annihilator of X in R is denoted by $r_R(X)$ ($l_R(X)$). We write $a \in L - I$ to indicate that $a \in L$ but $a \notin I$ and $N \leq^e M$ to indicate that N is an essential submodule of M. The notation M^n stands for the direct sum of n copies of the module M, written as column matrices. For the usual notations we refer the reader to [1], [6] and [10].

2. Examples and basic properties

Definition 2.1. Let M be a right R-module, $S = \mathrm{End}_R(M)$. The module M is called *almost principally small injective* (or *APS-injective* for short) if, for any $a \in J(R)$, there exists an S-submodule X_a of M such that $l_M r_R(a) = Ma \oplus X_a$ as left S-modules. If R_R is an APS-injective module, then we call R a right APS-injective ring. Similarly, we can define the concept of left APS-injective rings.

For an R-module N and a submodule P of N, we will identify $\mathrm{Hom}_R(N, M)$ with the set of homomorphisms in $\mathrm{Hom}_R(P, M)$ that can be extended to N, and hence $\mathrm{Hom}_R(N, M)$ can be seen as a left S-submodule of $\mathrm{Hom}_R(P, M)$.

Lemma 2.2. *Let M_R be a module, $S = \mathrm{End}_R(M)$ and $a \in J(R)$.*

(1) *If $l_M r_R(a) = Ma \oplus X$ for some $X \subseteq M$ as left S-modules, then $\mathrm{Hom}_R(aR, M) = \mathrm{Hom}_R(R, M) \oplus \Gamma$ as left S-modules, where $\Gamma = \{f \in \mathrm{Hom}_R(aR, M) : f(a) \in X\}$.*
(2) *If $\mathrm{Hom}_R(aR, M) = \mathrm{Hom}_R(R, M) \oplus Y$ as left S-modules, then $l_M r_R(a) = Ma \oplus X$ as left S-modules, where $X = \{f(a) : f \in Y\}$.*
(3) *Ma is a direct summand of $l_M r_R(a)$ as left S-modules if and only if $\mathrm{Hom}_R(R, M)$ is a direct summand of $\mathrm{Hom}_R(aR, M)$ as left S-modules.*

Proof. The proof is similar to that of [11, Lemma 1.2]. □

From Lemma 2.2, we have the following corollary.

Corollary 2.3. *Let M_R be a module and $a \in J(R)$. Then $l_M r_R(a) = Ma$ if and only if every R-homomorphism of aR into M extends to R.*

Remark 2.4. (1) Obviously, right PS-injective modules are right APS-injective. But the converse is false in general. For example, let $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ with F a field and $M_R = \left(\begin{smallmatrix} 0 & F \\ 0 & F \end{smallmatrix}\right)$. Then M is right APS-injective but not right PS-injective. In fact, choose $0 \neq x \in F$. Then $a = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) \in J(R)$ and $l_M r_R(a) = M \neq Ma = 0$. By the preceding corollary, M is not right PS-injective. Note that $J(R) = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$. Thus, $l_M r_R(a) = Ma \oplus M$ for any $a \in J(R)$. Therefore, M is right APS-injective.

(2) Right AP-injective modules are right APS-injective.

(3) Right APS-injective rings are right almost mininjective [13] (A ring R is called right almost mininjective if, for any minimal right ideal kR of R, there exists an S-submodule X_k of R such that $l_R r_R(k) = Rk \oplus X_k$ as left S-modules). In fact, in view of [6, Lemma 10.22], every minimal right ideal of R is either nilpotent or a direct summand of R.

Example 2.5. The three examples of [11, Examples 1.5] are commutative APS-injective but not PS-injective.

Example 2.6. Let $R = \mathbb{Z}$ be the ring of integers. Then R is APS-injective but not AGP-injective.

Example 2.7. Let K be a field and L be a proper subfield of K such that $\rho : K \to L$ is an isomorphism, e.g., let $K = F(y_1, y_2, \ldots)$ with F a field, $\rho(y_i) = y_{i+1}$ and $\rho(c) = c$ for all $c \in F$. Let $K[x_1, x_2; \rho]$ be the ring of twisted right polynomials over K where $kx_i = x_i\rho(k)$ for all $k \in K$ and for $i = 1, 2$. Set $R = K[x_1, x_2; \rho]/(x_1^2, x_2^2)$. In view of [3, Example 1 and Proposition 1], R is a left AGP-injective but not APS-injective.

Theorem 2.8. *Let R be a right APS-injective ring. Then.*

(1) $J(R) \subseteq Z(R_R)$.

(2) $soc(R_R) \subseteq r_R(J)$.

Proof. (1) Take any $a \in J(R)$. If $a \notin Z(R_R)$, then there exists a nonzero right ideal I of R such that $r_R(a) \cap I = 0$. So there exists $b \in I$ such that $ab \neq 0$. Note that $ab \in J(R)$, by hypothesis, there exists $0 \neq u \in abR$ such that $l_R r_R(u) = Ru \oplus X_u$, where $X_u \subseteq {}_R R$. Write $u = abc$ for some $c \in R$. If $t \in r_R(abc)$, then $abct = 0$, implying $ct \in r_R(ab) = r_R(b)$ since $r_R(a) \cap I = 0$. Hence, $(bc)t = b(ct) = 0$, and so $t \in r_R(bc)$. This shows that $r_R(bc) = r_R(abc)$. Note that $bc \in l_R r_R(bc) = l_R r_R(abc) = Ru \oplus X_u$. Write $bc = dabc + x$, where $dabc \in Ru - X_u$ and $x \in X_u - Ru$. Then $x = (1 - da)bc$, and so $bc = (1 - da)^{-1}x \in X_u$ since $1 - da$ is invertible, contradicting with $dabc \in Ru - X_u$.

(2) Let kR be a simple right ideal of R. Suppose $jk \neq 0$ for some $j \in J(R)$, then $r_R(jk) = r_R(k)$. Note that $jk \in J(R)$ and R is right APS-injective. Then there exists a left ideal X_{jk} of R such that $l_R r_R(jk) = Rjk \oplus X_{jk}$. Since $k \in l_R r_R(jk)$, write $k = rjk + x$, where $rjk \in Rjk - X_{jk}$ and $x \in X_{jk} - Rjk$. Then $x = (1 - rj)k$, and hence $k = (1 - jk)^{-1}x \in X_{jk}$ since $1 - jk$ is invertible, contradicting with $rjk \in Rjk - X_{jk}$. □

The following example shows that a right mininjective ring need not be right APS-injective.

Example 2.9. Let R be the ring of all $\mathbb{N}$-square upper triangular matrices over a field F that are constant on the diagonal and have only finitely many nonzero entries off the diagonal. By [16, Example 1.7], $soc(R_R) = Z(R_R) = 0$ and $J(R) \neq 0$. So R is right mininjective. However, R is not right APS-injective by Theorem 2.8(1).

A ring R is called semiregular if $R/J(R)$ is von Neumann regular and idempotents lift modulo $J(R)$, equivalently if, for any $a \in R$, there exists $e^2 = e \in Ra$ such that $a(1-e) \in J(R)$ (cf. [10, Lemma B.40]).

Proposition 2.10. *If R is semiregular, then R is right AP-injective if and only if R is right APS-injective.*

Proof. It is enough to prove sufficient condition. Since R is semiregular, for any $a \in R$, $Ra = Re \oplus Rb$, where $e^2 = e \in R$ and $b \in J(R)$. By hypothesis, $l_R r_R(b) = Rb \oplus X_b$ for some left ideal X_b of R. Then $Ra \oplus X_b = Re \oplus Rb \oplus X_b = l_R(1-e) \oplus l_R r_R(b) = l_R((1-e)R \oplus r_R(b)) = l_R(r_R(Re) \oplus r_R(Rb)) = l_R r_R(Re \oplus Rb) = l_R r_R(Ra) = l_R r_R(a)$. Therefore, R is right AP-injective. □

Remark 2.11. There exists a ring that is semiregular but not right APS-injective. Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$, where $\mathbb{Z}_2$ is the ring of integers modulo 2. Then $J(R) = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$, and $Z(R_R) = 0$. By Theorem 2.8, R is not right APS-injective. But $R/J(R) \cong \begin{pmatrix} \mathbb{Z}_2 & 0 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ is von Neumann regular and any idempotent of $R/J(R)$ can be lifted to R, so R is semiregular.

By Proposition 2.10 and [11, Theorem 2.16], we have the following result.

Corollary 2.12. *If R is a semiperfect and right APS-injective ring, then $R = R_1 \times R_2$, where R_1 is semisimple and every simple right ideal of R_2 is nilpotent.*

Clearly, a semiprimitive ring ($J(R) = 0$) is left and right APS-injective. But the converse is not true as Example 2.5. Next, we shall consider when a right APS-injective ring is semiprimitive. Following [7], A ring R is called a right $J-PP$ ring if aR is projective for any $a \in J(R)$.

Proposition 2.13. *Let R be a ring. Then the following are equivalent*:

(1) *R is semiprimitive.*
(2) *R is right $J-PP$ and right APS-injective.*
(3) *R is a right APS-injective ring whose every simple singular right R-module is PS-injective.*

Proof. (1)⇒(2) and (1)⇒(3) are trivial.

(2)⇒(1). Suppose $0 \neq a \in J(R)$. Since R is right $J-PP$, aR is projective. So the exact sequence $0 \to r_R(a) \to R \to aR \to 0$ splits. Then $r_R(a) = eR$ for some $e^2 = e \in R$. It follows that $l_R r_R(a) = l_R(eR) = R(1-e)$. Note that

R is also right APS-injective, so there exists a left ideal X_a of R such that $l_R r_R(a) = Ra \oplus X_a$. Then Ra is a direct summand of $R(1-e)$, and hence a direct summand of ${}_RR$, which implies $a = 0$, a contradiction.

(3)⇒(1). We first show that $J \bigcap Z(R_R) = 0$. Take any $b \in J \bigcap Z(R_R)$. If $b \neq 0$, then $r_R(b) + RbR$ is an essential right ideal of R. If $r_R(b) + RbR \neq R$, there exists a maximal essential right ideal T of R such that $r_R(b) + RbR \subseteq T$. By hypothesis, R/T is PS-injective. Note that $r_R(b) \subseteq T$, then the R-homomorphism $f : bR \to R/T$ by $br \mapsto r + T$ is well defined. So $f = (c + T)\cdot$ for some $c \in R$. Then $f(b) = 1 + T = cb + T$. Note that $cb \in RbR \subseteq T$, so $1 \in T$, a contradiction. This proves that $r_R(b) + RbR = R$, and hence $r_R(b) = R$ because RbR is a small ideal of R. This implies $b = 0$, which is required contradiction. Therefore, $J(R) = J \cap Z(R_R) = 0$ by Theorem 2.8(1). □

Now we construct a right APS-injective ring that is not left APS-injective.

Example 2.14. Let $R = \left(\begin{smallmatrix} K & K \\ 0 & A \end{smallmatrix}\right)$, where $K = \mathbb{Z}_2$ and

$$A = \{(a_1, a_2, \ldots, a_n, a, a, \ldots) \mid a, a_1, a_2, \ldots \in K, n \in \mathbb{N}\}.$$

If $k \in K$ and $(a_1, a_2, \ldots, a_n, a, a, \ldots) \in A$, let $k \cdot (a_1, a_2, \ldots, a_n, a, a, \ldots) = ka$.

Following [2, Example 1], R is right P-injective, and hence right APS-injective. But $J(R) = \left(\begin{smallmatrix} 0 & K \\ 0 & 0 \end{smallmatrix}\right) \neq 0$, so R is not semiprimitive. We claim that R is not left APS-injective. By Proposition 2.13, it is enough to show that every simple singular left R-module is PS-injective. In fact, $M = \begin{pmatrix} K & K \\ 0 & \mathbb{Z}_2^{(\mathbb{N})} \end{pmatrix}$ is the unique maximal essential right ideal of R, where

$$\mathbb{Z}_2^{(\mathbb{N})} = \{(a_1, a_2, \ldots, a_n, 0, 0, \ldots) \mid a_1, a_2, \ldots \in K, n \in \mathbb{N}\}.$$

In view of [15, p. 5], $\overline{R} = R/M$ is left P-injective, and hence left PS-injective.

Proposition 2.15. *If R is a right APS-injective ring and $R/soc(R_R)$ satisfies the* ACC *on right annihilators, then $J(R)$ is nilpotent.*

Proof. Write $S = soc(R_R)$ and $\overline{R} = R/S$. For any sequence $a_1, a_2, a_3, \ldots \in J(R)$, there is an ascending chain

$$r_{\overline{R}}(\overline{a_1}) \subseteq r_{\overline{R}}(\overline{a_2}\ \overline{a_1}) \subseteq r_{\overline{R}}(\overline{a_3}\ \overline{a_2}\ \overline{a_1}) \subseteq \cdots,$$

by hypothesis, there exists a positive integer m such that

$$r_{\overline{R}}(\overline{a_m} \cdots \overline{a_2}\ \overline{a_1}) = r_{\overline{R}}(\overline{a_{m+k}} \cdots \overline{a_m} \cdots \overline{a_2}\ \overline{a_1}),\ k = 1, 2, \ldots.$$

Since $a_{n+1}a_n \cdots a_1 \in J(R) \subseteq Z(R_R)$ by Theorem 2.8(1), $r_R(a_{n+1}a_n \cdots a_1)$ is the essential right ideal of R. Then $S \subseteq r_R(a_{n+1}a_n \cdots a_1)$.

Now we prove that

$$(1) \qquad r_{\overline{R}}(\overline{a_n} \cdots \overline{a_2}\ \overline{a_1}) \subseteq r_R(a_{n+1}a_n \cdots a_1)/S \subseteq r_{\overline{R}}(\overline{a_{n+1}}\ \overline{a_n} \cdots \overline{a_1}).$$

In fact, for any $b + S \in r_{\overline{R}}(\overline{a_n} \cdots \overline{a_2}\ \overline{a_1})$, $a_n \cdots a_1 b \in S$. Then $a_{n+1}a_n \cdots a_1 b = 0$ because $S \subseteq r_R(a_{n+1})$. So $b \in r_R(a_{n+1}a_n \cdots a_1)$, and hence $b + S \in r_R(a_{n+1}a_n \cdots a_1)/S$. But the second inclusion is clear.

Since $r_{\overline{R}}(\overline{a_m}\cdots\overline{a_2}\ \overline{a_1}) = r_{\overline{R}}(\overline{a_{m+2}}\ \overline{a_{m+1}}\cdots\overline{a_2}\ \overline{a_1})$, by (1),

$$r_R(a_{m+1}a_m\cdots a_1)/S = r_R(a_{m+2}a_{m+1}\cdots a_1)/S.$$

Then $r_R(a_{m+1}a_m\cdots a_1) = r_R(a_{m+2}a_{m+1}\cdots a_1)$, and so $(a_{m+1}a_m\cdots a_1)R\cap r_R(a_{m+2}) = 0$. Note that $r_R(a_{m+2})$ is also an essential right ideal of R, then $a_{m+1}a_m\cdots a_1 = 0$. So $J(R)$ is a right T-nilpotent ideal and the ideal $J(R)+S/S$ of $\overline{R}$ is also a right T-nilpotent. By [1, Proposition 29.1], $J(R)+S/S$ is nilpotent. Then there exists a positive integer t such that $(J(R))^t\subseteq S$, so $(J(R))^{t+1}\subseteq J(R)S = 0$, as desired. □

Proposition 2.16. *If R is a right APS-injective (resp. PS-injective, AP-injective) ring, so is eRe for all $e^2 = e\in R$ such that $ReR = R$.*

Proof. Let $S = eRe$ and let $a\in J(S) = eJe$. Then $a = ae\in J(R)$, so there exists a left ideal X_a of R such that $l_Rr_R(a) = Ra\oplus X_a$. Since $1-e\in r_R(a)$, we see that $t(1-e) = 0$ for any $t\in X_a$, which implies $X_a = X_ae$. Thus $eRae\cap eX_ae = 0$. Clearly, $eRae\subseteq l_Sr_S(a)$ and $eX_ae\subseteq l_Sr_S(a)$ since $Rae = Ra$ and $X_ae = X_a$. Now we prove the other inclusion. Take $x\in l_Sr_S(a)$, and write $1 = \sum_{i=1}^n a_ieb_i$ for some a_i, b_i in R. Then for any $y\in r_R(a)$, we get $aeya_ie = aya_ie = 0$ for each i. This implies that $xeya_ie = 0$ for each i, which gives $xy = xey = xey\sum_{i=1}^n a_ieb_i = 0$ since $x\in S$. So $x\in l_Rr_R(a)$, and hence $l_Sr_S(a)\subseteq l_Rr_R(a)$. Take $x = s+t$, where $s\in Ra$ and $t\in X_a$. Hence, $x = exe = ese+ete\in eRae+eX_ae$. This shows that $l_Sr_S(a) = eRae\oplus eX_ae = Sa\oplus eX_a$, where eX_a is a left ideal of S. Therefore, S is right APS-injective. □

Remark 2.17. The condition that $ReR = R$ in Proposition 2.16 is needed. For example, let R be the algebra of matrices, over a field F, of the form

$$R = \begin{pmatrix} a & x & 0 & 0 & 0 & 0\\ 0 & b & 0 & 0 & 0 & 0\\ 0 & 0 & c & y & 0 & 0\\ 0 & 0 & 0 & a & 0 & 0\\ 0 & 0 & 0 & 0 & b & z\\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}.$$

By [5, Example 9], R is a QF-ring, and hence it is right APS-injective. Let $e = e_{11}+e_{22}+e_{44}+e_{55}$ be a sum of canonical matrix units. Then e is an idempotent of R such that $ReR\neq R$ and $eRe\cong S = \left(\begin{smallmatrix} F & F\\ 0 & F\end{smallmatrix}\right)$. We claim that S is not right APS-injective. In fact, $J(S) = \left(\begin{smallmatrix} 0 & F\\ 0 & 0\end{smallmatrix}\right)$. Then for any $\overline{d} = \left(\begin{smallmatrix} 0 & d\\ 0 & 0\end{smallmatrix}\right)\in J(S)$, $l_Sr_S(\overline{d}) = \left(\begin{smallmatrix} 0 & F\\ 0 & F\end{smallmatrix}\right)$ and $S\overline{d} = \left(\begin{smallmatrix} 0 & F\\ 0 & 0\end{smallmatrix}\right)$. So it does not exist a left ideal $X_{\overline{d}}$ of S such that $l_Sr_S(\overline{d}) = S\overline{d}\oplus X_{\overline{d}}$.

Corollary 2.18. *If the matrix ring $M_n(R)$ over a ring R is right APS-injective ($n\geq 1$), then so is R.*

Proof. If $S = M_n(R)$ is right APS-injective, so is $R\cong e_{11}Se_{11}$ by Proposition 2.16 because $Se_{11}S = S$ (here e_{ij} is the matrix unit). □

We do not know if the converse of Corollary 2.18 holds. However, we have the following result motivated by [11, Theorem 3.8].

Theorem 2.19. *Let R be a ring and $n \geq 1$. Then the following are equivalent*:

(1) $M_n(R)$ *is right APS-injective.*

(2) $\mathrm{Hom}_R(R^n, R)$ *is a direct summand of* $\mathrm{Hom}_R(I, R)$ *as left R-modules for any n-generated R-submodule I of J^n.*

Proof. (1)$\Rightarrow$(2). Let $S = M_n(R)$ and let $I = \overline{a_1}R + \cdots + \overline{a_n}R \in J^n$. Write $(\overline{a_1}, \ldots, \overline{a_n}) = A$, then $A \in J(S)$. By hypothesis, we have $l_S r_S(A) = SA \oplus X_A$ for some left ideal X_A of S. Let

$$\Gamma = \left\{ f \in \mathrm{Hom}_R(I, R) : \begin{pmatrix} f(\overline{a_1}) & \cdots & f(\overline{a_n}) \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in X_A \right\}.$$

It is easy to verify that Γ is a left R-submodule of $\mathrm{Hom}_R(I, R)$. We claim that $\mathrm{Hom}_R(I, R) = \mathrm{Hom}_R(R^n, R) \oplus \Gamma$ as left R-modules. In fact, for any $g \in \mathrm{Hom}_R(I, R)$, write

$$B = \begin{pmatrix} g(\overline{a_1}) & \cdots & g(\overline{a_n}) \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Then $B \in l_S r_S(A)$, and hence $B = (c_{ij})A + (d_{ij})$, where $(c_{ij}) \in S$ and $(d_{ij}) \in X_A$. Let $h : R^n \to R$, $\sum_{i=1}^n \overline{e_i} r_i \mapsto \sum_{i=1}^n c_{1i} r_i$, where $\overline{e_i}$ is the standard basis of R^n over R, and let $k : I \to R$, $\sum_{i=1}^n \overline{a_i} r_i \mapsto \sum_{i=1}^n d_{1i} r_i$. Then $g = h + k$. Note that

$$\begin{pmatrix} d_{11} & \cdots & d_{1n} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} (d_{ij}) \in X_A.$$

So $k \in \Gamma$. Therefore, we have $\mathrm{Hom}_R(I, R) = \mathrm{Hom}_R(R^n, R) + \Gamma$. Suppose $l \in \mathrm{Hom}_R(R^n, R) \cap \Gamma$. Then there exists $(c_1, \ldots, c_n) \in R^n$ such that $(l(\overline{a_1}), \ldots, l(\overline{a_n})) = (c_1, \ldots, c_n)A$. Thus,

$$\begin{pmatrix} l(\overline{a_1}) & \cdots & l(\overline{a_n}) \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} c_1 & \cdots & c_n \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} A \in SA \cap X_A = 0.$$

Therefore, $\mathrm{Hom}_R(I, R) = \mathrm{Hom}_R(R^n, R) \oplus \Gamma$.

(2)⇒(1). Suppose $A = (a_{ij}) \in J(S)$. Let $I = \overline{a_1}R + \cdots + \overline{a_n}R$, where $\overline{a_i}$ is i-th column of A. Then $I \in J^n$. By hypothesis, we have $\mathrm{Hom}_R(I, R) = \mathrm{Hom}_R(R^n, R) \oplus \Gamma$ for some left R-submodule Γ of $\mathrm{Hom}_R(I, R)$. Let

$$X_A = \left\{ \begin{pmatrix} f_1(\overline{a_1}) & \cdots & f_1(\overline{a_n}) \\ f_2(\overline{a_1}) & \cdots & f_2(\overline{a_n}) \\ \vdots & & \vdots \\ f_n(\overline{a_1}) & \cdots & f_n(\overline{a_n}) \end{pmatrix} : f_i \in \Gamma, i = 1, 2, \ldots, n \right\}.$$

Then X_A is a left ideal of S. Now we show that $l_S r_S(A) = SA \oplus X_A$ as left S-modules. It is easy to check that $X_A \subseteq l_S r_S(A)$. If $B = (b_{ij}) \in l_S r_S(A)$, then $r_S(A) \subseteq r_S(B)$. So $f : AS \to BS$, $A(s_{ij}) \mapsto B(s_{ij})$, $(s_{ij}) \in S$ is a well-defined S-homomorphism, which induces an R-homomorphism $f_i : \sum_{j=1}^n \overline{a_j} r_j \mapsto \sum_{j=1}^n b_{ij} r_j$ from I to R for each $1 \leq i \leq n$. Write $f_i = g_i + h_i$, where $g_i \in \mathrm{Hom}_R(R^n, R)$ and $h_i \in \Gamma$. Then, for each i, there exists $(c_{i1}, \ldots, c_{in}) \in R^n$ such that $(g_i(\overline{a_1}), \ldots, g_i(\overline{a_n})) = (c_{i1}, \ldots, c_{in})A$. So,

$$B = (b_{ij}) = (c_{ij})A + \begin{pmatrix} h_1(\overline{a_1}) & \cdots & h_1(\overline{a_n}) \\ h_2(\overline{a_1}) & \cdots & h_2(\overline{a_n}) \\ \vdots & & \vdots \\ h_n(\overline{a_1}) & \cdots & h_n(\overline{a_n}) \end{pmatrix} \in SA + X_A,$$

showing $l_S r_S(A) = SA + X_A$. Let $C \in SA \cap X_A$. Then for some $(d_{ij}) \in S$ and some $k_i \in \Gamma (i = 1, 2, \ldots, n)$,

$$C = \begin{pmatrix} k_1(\overline{a_1}) & \cdots & k_1(\overline{a_n}) \\ k_2(\overline{a_1}) & \cdots & k_2(\overline{a_n}) \\ \vdots & & \vdots \\ k_n(\overline{a_1}) & \cdots & k_n(\overline{a_n}) \end{pmatrix} \in (d_{ij})A.$$

Then, for each i, $(k_i(\overline{a_1}), \ldots, k_i(\overline{a_n})) = (d_{i1}, \ldots, d_{in})A$, which shows that $k_i \in \mathrm{Hom}_R(R^n, R) \cap \Gamma = 0$. Thus, each $k_i = 0$, and hence $C = 0$. Therefore, $l_S r_S(A) = SA \oplus X_A$. □

The following theorem is a generalization of [17, Theorem 2.1].

Theorem 2.20. *Let R be a right Noetherian, left APS-injective ring. Then*
(1) $l_R(J) \leq^e {}_R R$.
(2) *J is nilpotent.*
(3) $l_R(J) \leq^e R_R$.

Proof. (1) For any $0 \neq x \in R$, it is enough to show that $l_R(J) \cap Rx \neq 0$. Since R has ACC on right annihilators, choose $y \in R$ such that $yx \neq 0$ and $r_R(yx)$ is maximal in $\{r_R(ax) | a \in R,\ ax \neq 0\}$. Now we prove that $yxJ = 0$. Otherwise, there exists a $t \in J$ such that $yxt \neq 0$. Note that $yxt \in J$ and R is left APS-injective, then $r_R l_R(yxt) = yxtR \oplus X$ for some right ideal X of R. We proceed with the following two cases.

Case 1. $r_Rl_R(yx) = r_Rl_R(yxt)$. Then $yx \in r_Rl_R(yxt) = yxtR \oplus X$. Write $yx = yxtr + z$, where $yxtr \in yxtR - X$ and $z \in X - yxtR$. So $z = yx(1-tr)$, and hence $yx = z(1-tr)^{-1}$ since $1-tr$ is invertible, contradicting with $yxtr \notin X$.

Case 2. $r_Rl_R(yx) \neq r_Rl_R(yxt)$. Then $l_R(yx) \neq l_R(yxt)$. It follows that there exists $u \in l_R(yxt)$ but $u \notin l_R(yx)$. Thus $uyxt = 0$ and $uyx \neq 0$. This gives that $t \in r_R(uyx)$ and $t \notin r_R(yx)$. So $r_R(yx) \subset r_R(uyx)$, contradicting the maximality of $r_R(yx)$.

Then $yxJ = 0$, and so $0 \neq yx \in l_R(J) \cap Rx$. Therefore, $l_R(J) \leq^e {}_RR$.

(2) There exists $k \geq 1$ such that $l_R(J^k) = l_R(J^{k+1}) = \cdots$. If J is not nilpotent, choose $r_R(x)$ to be maximal in $\{r_R(y) | yJ^k \neq 0\}$. Then $xJ^{2k} \neq 0$ because $l_R(J^{2k}) = l_R(J^k)$, so there exists $b \in J^k$ with $xbJ^k \neq 0$. Since $l_R(J) \leq l_R(J^k)$, we have $l_R(J^k) \leq^e {}_RR$ by (1). Thus, $Rxb \cap l_R(J^k) \neq 0$, say $0 \neq cxb \in l_R(J^k)$. Hence, $r_R(x) \subset r_R(cx)$ because $xbJ^k \neq 0$, contradicting the maximality of $r_R(x)$.

(3) If $0 \neq d \in R$, we must show that $dR \cap l_R(J) \neq 0$. It is clear if $dJ = 0$. Otherwise, since J is nilpotent by (2), there exists $m \geq 1$ such that $dJ^m \neq 0$ but $dJ^{m+1} = 0$. Then $0 \neq dJ^m \subseteq dR \cap l_R(J)$, as desired. □

In [17], a module M is said to satisfy the generalized C2-condition (GC2) if, for any $N \subseteq M$ and $N \cong M$, N is a summand of M.

Corollary 2.21. *If R is a right Noetherian, left APS-injective ring such that R_R satisfies* (GC2)*, then it is right Artinian.*

Proof. Note that R is right finitely dimensional. By [17, Lemma 1.1], R is semilocal. By Theorem 2.18, $J(R)$ is nilpotent. Thus, R is semiprimary, and hence it is right Artinian by the Hopkins-Levitzki theorem. □

The condition that R_R satisfies (GC2) can not be omitted. For example, the ring $R = \mathbb{Z}$ is a Noetherian and APS-injective ring but not Artinian. From [10, Proposition 1.46], a left Kasch ring is right $C2$. Thus, we have the following corollary.

Corollary 2.22. *If R is a right Noetherian, left Kasch and left APS-injective ring, then it is right Artinian.*

3. Trivial extensions

Let R be a ring and M a bimodule over R. The trivial extension of R and M is

$$R \propto M = \{(a, x) \mid a \in R, x \in M\}$$

with addition defined componentwise and multiplication defined by

$$(a, x)(b, y) = (ab, ay + xb).$$

For convenience, we write $I \propto X = \{(a, x) \mid a \in I, x \in X\}$, where I is a subset of R and X is a subset of M. It is easy to check that $J(R \propto M) = J(R) \propto M$.

Proposition 3.1. *Let R be a ring and X_a a left ideal of R for any $a \in R$, $S = R \propto R$. Then the following are equivalent*:

(1) $l_R r_R(a) = Ra \oplus X_a$.
(2) $l_S r_S(0,a) = S(0,a) \oplus X_{(0,a)}$, *where* $X_{(0,a)} = 0 \propto X_a$ *is a left ideal of* S.
(3) $l_S r_S(a,0) = S(a,0) \oplus X_{(a,0)}$, *where* $X_{(a,0)} = X_a \propto 0$ *is a left ideal of* S.
(4) $l_S r_S(a,a) = S(a,a) \oplus X_{(a,a)}$, *where* $X_{(a,a)} = X_a \propto X_a$ *is a left ideal of* S.

Proof. (1)⇒(2). For any $(b,c) \in l_S r_S(0,a)$, $r_S(0,a) \subseteq r_S(b,c)$. Since $(0,1) \in r_S(0,a)$, $0 = (b,c)(0,1) = (0,b)$, showing $b = 0$. If $x \in r_R(a)$, then $(x,0) \in r_S(0,a) \subseteq r_S(b,c)$, showing that $0 = (0,c)(x,0) = (0,cx)$. So $x \in r_R(c)$, and hence $r_R(a) \subseteq r_R(c)$. Thus, $c \in l_R r_R(c) \subseteq l_R r_R(a) = Ra \oplus X_a$. Write $c = ra + y$, where $ra \in Ra - X_a$ and $y \in X_a - Ra$. Then $(b,c) = (0, ra+y) = (r,0)(0,a) + (0,y) \in S(0,a) + X_{(0,a)}$, where $X_{(0,a)} = 0 \propto X_a$ is a left ideal of S. It is easy to prove that $S(0,a) \cap X_{(0,a)} = 0$, so $l_S r_S(0,a) \subseteq S(0,a) \oplus X_{(0,a)}$. Conversely, for any $(m,n) \in S(0,a) \oplus X_{(0,a)}$, where $X_{(0,a)} = 0 \propto X_a$ is a left ideal of S. Then $(m,n) = (r_1,r_2)(0,a) + (0,y) = (0, r_1 a + y)$, where $(r_1,r_2)(0,a) \in S(0,a) - X_{(0,a)}$ and $(0,y) \in X_{(0,a)} - S(0,a)$. Note that $r_1 a \in Ra - X_a$ and $y \in X_a - Ra$, so $m = 0$, $n = r_1 a + y \in Ra \oplus X_a = l_R r_R(a)$. Then $r_R(a) \subseteq r_R(n)$. For any $(k,l) \in r_S(0,a)$, $0 = (0,a)(k,l) = (0,ak)$, showing $k \in r_R(a)$, and hence $nk = 0$. Then $(m,n)(k,l) = (0,n)(k,l) = (0,nk) = 0$, proving $(m,n) \in l_S r_S(0,a)$.

(2)⇒(1). For any $b \in l_R r_R(a)$, $r_R(a) \subseteq r_R(b)$. If $(x,y) \in r_S(0,a)$, then $ax = 0$. So $x \in r_R(a) \subseteq r_R(b)$, showing $(0,b)(x,y) = 0$. Thus, $(x,y) \in r_S(0,b)$. So $r_S(0,a) \subseteq r_S(0,b)$. This gives that $(0,b) \in l_S r_S(0,b) \subseteq l_S r_S(0,b) = S(0,a) \oplus X_{(0,a)}$. Write $(0,b) = (r_1,r_2)(0,a) + (0,y) = (0, r_1 a + y)$, where $(r_1,r_2)(0,a) \in S(0,a) - X_{(0,a)}$ and $(0,y) \in X_{(0,a)} - S(0,a)$. Note that $r_1 a \in Ra - X_a$ and $y \in X_a - Ra$, so $b = r_1 a + y \in Ra \oplus X_a$, proving $l_R r_R(a) \subseteq Ra \oplus X_a$. Now we show the other inclusion. For any $c \in Ra \oplus X_a$, write $c = ra + z$, where $ra \in Ra - X_a$ and $z \in X_a - Ra$. Then $(0,c) = (0,ra) + (0,z) = (r,0)(0,a) + (0,z) \in S(0,a) \oplus X_{(0,a)} = l_S r_S(0,a)$. So $r_S(0,a) \subseteq r_S(0,c)$. If $x \in r_R(a)$, then $(x,0) \in r_S(0,a)$, showing $0 = (0,c)(x,0) = (0,cx)$, and hence $x \in r_R(c)$. Thus, $r_R(a) \subseteq r_R(c)$. This implies that $c \in l_R r_R(c) \subseteq l_R r_R(a)$.

The proofs of (1)⇔(3) and (1)⇔(4) are similar to that of (1)⇔(2). $\square$

Corollary 3.2. *Let R be a ring and $a \in R$, $S = R \propto R$. Then the following are equivalent*:

(1) $l_R r_R(a) = Ra$.
(2) $l_S r_S(0,a) = S(0,a)$.
(3) $l_S r_S(a,0) = S(a,0)$.
(4) $l_S r_S(a,a) = S(a,a)$.

Corollary 3.3. *Let R be a ring. If $R \propto R$ is right APS-injective, then R is right AP-injective.*

Proof. Let $S = R \propto R$. For any $0 \neq a \in R$, $(0,a) \in J(S)$. So there exists a left ideal $X_{(0,a)}$ of S such that $l_S r_S(0,a) = S(0,a) \oplus X_{(0,a)}$. By the proof of (1)⇒(2) in Proposition 3.1, if $(b,c) \in l_S r_S(0,a)$ and $(m,n) \in S(0,a)$, then $b = 0$ and $m = 0$. So $X_{(0,a)} = 0 \propto X_a$, where X_a is a left ideal of R. By Proposition 3.1 again, we have $l_R r_R(a) = Ra \oplus X_a$, proving that R is right AP-injective. □

Remark 3.4. We claim that R being right APS-injective can not imply $R \propto R$ being right APS-injective. For example, let $R = \mathbb{Z}$ be the ring of integers. Suppose that $S = \mathbb{Z} \propto \mathbb{Z}$ is APS-injective, then $\mathbb{Z}$ is AP-injective by Corollary 3.3, a contradiction.

For $f, g \in \mathrm{Hom}_R(R,R)$, define $\alpha = (f,g)$ such that $\alpha(a,b) = (f,g)(a,b) = (f(a), f(b) + g(a))$, where $(a,b) \in S = R \propto R$. It is easy to check that $\alpha \in \mathrm{Hom}_S(S,S)$. Conversely, for any $\alpha \in \mathrm{Hom}_S(S,S)$, let $\alpha(1,0) = (p,q)$. Define $f(1) = p$, $g(1) = q$, then $f, g \in \mathrm{Hom}_R(R,R)$ and $\alpha = (f,g)$. In the following theorem, we shall discuss when $R \propto R$ is right APS-injective.

Theorem 3.5. *Let R be a ring. If, for any $a \in J(R)$, $b \in R$, $\mathrm{Hom}_R(aR + br_R(a), R) = \mathrm{Hom}_R(R,R) \oplus X$ as left R-modules for some submodules X of $\mathrm{Hom}_R(aR + br_R(a), R)$, then $R \propto R$ is right APS-injective.*

Proof. Let $S = R \propto R$ and any $A \in J(S)$. By Lemma 2.2, it is enough to show that $\mathrm{Hom}_S(AS,S) = \mathrm{Hom}_S(S,S) \oplus Y$ for some left S-submodules Y of $\mathrm{Hom}_S(AS,S)$. Write $A = (a,b)$, then $a \in J(R)$, $b \in R$. For any $f \in \mathrm{Hom}_S(AS,S)$, say $f(A) = (p,q)$, $p,q \in R$. Define $g : aR + br_R(a) \to R$, $ar_1 + br_2 \mapsto pr_1 + qr_2$. If $ar_1 + br_2 = 0$, then $(a,b)(r_2,r_1) = (ar_2, ar_1 + br_2) = 0$ since $ar_2 = 0$, and hence $0 = f((a,b)(r_2,r_1)) = (p,q)(r_2,r_1) = (pr_2, pr_1 + qr_2)$, which implies $pr_1 + qr_2 = 0$. So $g \in \mathrm{Hom}_R(aR + br_R(a), R)$. By hypothesis, $g = h \oplus k$, where $h \in \mathrm{Hom}_R(R,R)$ and $k \in X$. In particular, $p = g(a) = h(a) + k(a) = h(1)a + k(a)$.

If $a = 0$, then $r_R(a) = R$. So R is right AP-injective. Define $l : aR \to R$, $ar \mapsto qr - h(1)br - k(br)$, $r \in R$. If $ar = 0$, then $r \in r_R(a)$, so $h(1)br = h(br) = g(br) - k(br) = qr - k(br)$, and hence $qr - h(1)br - k(br) = 0$. Thus, $l \in \mathrm{Hom}_R(aR,R)$. Then $l = h' \oplus k'$, where $h' \in \mathrm{Hom}_R(R,R)$ and $k' \in K \subseteq \mathrm{Hom}_R(aR,R)$. We have $q - h(1)b - k(b) = l(a) = h'(a) + k'(a)$, so $q = h(1)b + k(b) + h'(1)a + k'(a)$. Then $f(A) = (p,q) = (h(1)a + k(a), h(1)b + k(b) + h'(1)a + k'(a)) = (h(1), h'(1))(a,b) + (k(a), k(b) + k'(a)) = (h,h')(a,b) + (k,k')(a,b) = (h,h')A + (k,k')A$. Note that $(h,h') \in \mathrm{Hom}_S(S,S)$ and $(k,k') \in Y = \{(i,j) | i \in X, j \in K\} \subseteq \mathrm{Hom}_S(AS,S)$, then $\mathrm{Hom}_S(AS,S) \subseteq \mathrm{Hom}_S(S,S) + Y$. Now we show that $\mathrm{Hom}_S(S,S) \cap Y = 0$. For any $y \in \mathrm{Hom}_S(S,S) \cap Y$, write $y = (m,n)$, then $m \in \mathrm{Hom}_R(R,R) \cap X = 0$ and $n \in \mathrm{Hom}_R(R,R) \cap K = 0$, proving $y = 0$. Therefore, $\mathrm{Hom}_S(AS,S) \subseteq \mathrm{Hom}_S(S,S) \oplus Y$. But the other inclusion is clear, as desired. □

Following the preceding theorem, we immediately deduce the following corollaries.

Corollary 3.6. *Let R be a ring. If, for any $a,\ b \in R$, $\mathrm{Hom}_R(aR+br_R(a), R) = \mathrm{Hom}_R(R, R) \oplus X$ as left R-modules for some submodules X of $\mathrm{Hom}_R(aR + br_R(a), R)$, then $R \propto R$ is right AP-injective.*

Corollary 3.7. *Let R be a ring. If, for any $a \in J(R)$, $b \in R$, any R-homomorphism $aR + b \cdot r(a) \to R$ can be extended to R, then $R \propto R$ is right PS-injective.*

Corollary 3.8. *If R is semiprimitive and right AP-injective, then $R \propto R$ is right APS-injective.*

Remark 3.9. As Remark 3.4, the condition that R is right AP-injective in Corollary 3.8 can not be omitted.

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