

TRANSFORMATION OF LOCAL BIFURCATIONS UNDER COLLOCATION METHODS

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ABSTRACT. Numerical schemes are routinely used to predict the behavior of continuous dynamical systems. All such schemes transform flows into maps, which can possess dynamical behavior deviating from their continuous counterparts.

Here the common bifurcations of scalar dynamical systems are transformed under a class of algorithms known as linearized one-point collocation methods. Through the use of normal forms, we prove that each such bifurcation in an originating flow gives rise to an exactly corresponding one in its discretization. The conditions for spurious period doubling behavior under this class of algorithm are derived. We discuss the global behavioral consequences of a singular set induced by the discretizing methods, including loss of monotonicity of solutions, intermittency, and distortion of attractor basins.

1. Introduction

Numerical integration schemes for computing approximate solutions of differential equations typically involve discretizing steps. It is well known that these discretizing algorithms can cause spurious behavior in the resulting approximate solution map [5], [8], [9], [11], [13], [22], [23], [27]. The study of such spurious behavior and general behavioral differences caused by discretization has come to be known as the dynamics of numerics.

Such behavioral differences typically take the form of spurious fixed points and bifurcations to spurious asymptotes [5], [11], [14]. Runge-Kutta and predictor-corrector schemes generally possess spurious fixed points for nonlinear problems, but this is not the case with linear multistep methods [14]. A number of authors have studied the properties of spurious fixed points. One important aspect of these studies is to ascertain whether spurious fixed points can exist and be stable below the linearized stability limit of the methods for the genuine fixed points. It has been demonstrated through extensive numerical studies [24], [25], [26], [27] that this is the case, not only for fixed points but for fixed

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points of higher periods. On the other hand, it has been shown that, under certain smoothness conditions, spurious fixed points for Runge-Kutta schemes and period 2 solutions for Runge-Kutta and multistep methods do not exist for stepsize h arbitrarily small, and will either become unbounded or approach a true fixed point as $h \rightarrow 0$ [12].

Here we investigate the behavior of maps resulting from the application of collocation schemes to parameter-dependent autonomous flows $d\mathbf{x}/dt = f(\mathbf{x}, \alpha)$, using a dynamical systems approach. We are particularly interested in whether there is an exact correspondence in behavior and fixed point stability between a flow and its approximate solution map near the common scalar bifurcations: saddle-node, transcritical and pitchfork. We also wish to discover the defining conditions for any spurious bifurcation behavior. This work differs somewhat from that reported in Iserles et al. [14] and Griffiths et al. [5], in which the ODEs under discussion were not parameter-dependent and all spurious bifurcations resulted from variation of h .

Many applied researchers using differential equations are not fully aware of the possible spurious results introduced by the numerical integration of their systems. Automated continuation software for tracking bifurcating branches of fixed points is becoming increasingly prevalent in applied research involving parametrized nonlinear ODE systems. In computational fluid dynamics and other areas, iterative schemes constructed from ODE methods are commonly used to obtain steady state numerical solutions of nonlinear PDEs. Theoretical study of spurious behavior invoked by numerical schemes can yield results to be immediately incorporated into such algorithms [10], [18], increasing overall confidence in their predictions.

2. The bifurcations

In this section, the common scalar bifurcations for continuous dynamical systems (flows) and for discrete dynamical systems (maps) will be defined in the context of their normal forms. The normal forms will then be employed to prove generic behavior for the discretized systems. The results will be obtained by considering one dimensional parameter-dependent flows and maps. However, the results apply quite generally to higher dimensional systems as the bifurcations considered here exist on a one dimensional local center manifold in such systems; see Guckenheimer and Holmes [6] for details. It is expected that generalization to solutions of functional differential equations in Banach space via Fréchet differentiation will be quite straightforward.

The numerical schemes investigated here are *linearized one-point collocation methods*. Introduced by Loscalzo and Talbot [20], collocation methods are based on an approximation of the solution $\mathbf{x}(t)$ of the flow in $[t_n, t_{n+1}]$, by a piecewise continuous polynomial. Collocation of this polynomial at the points $t_i = t_n + c_i h$ is then required, where $\{c_i\}$ are chosen points such that $0 \leq c_i \leq 1$. The resulting methods generally belong to a class of implicit Runge-Kutta

methods; see Hairer et al. [9] for an extensive discussion of these methods. Local superconvergence of the methods is attained at the mesh points $\{t_n\}$ if the collocation points $\{c_i\}$ are Gauss points [7].

In one-point collocation methods, there is a single collocation point and the approximating function is a piecewise linear function. The resulting methods are given by

$$(2.1) \quad \mathbf{x}_{n+1} = \mathbf{x}_n + hf((1 - c_1)\mathbf{x}_n + c_1\mathbf{x}_{n+1}, \alpha),$$

where $c_1 \in [0, 1]$. All the methods are globally first-order; however, order 2 local superconvergence is attained when $c_1 = 1/2$. We choose to base our discussion on these methods because of their extensive use in recent decades. They include, as special cases, the well known explicit Euler method ($c_1 = 0$), implicit midpoint method ($c_1 = 1/2$) and implicit Euler method ($c_1 = 1$).

For $c_1 > 0$, the methods are implicit and in order to solve for \mathbf{x}_{n+1} at each step, we linearize. This is the preferable alternative to solving by iteration [25]. Furthermore, since linearized methods fall between explicit and implicit methods, it is of great interest to discuss the dynamics of linearized methods and ascertain whether they are qualitatively superior to their explicit counterparts.

Performing linearization of (2.1) about the point (t_n, \mathbf{x}_n) results in the family of maps

$$(2.2) \quad \mathbf{x} \mapsto \mathbf{x} + h(I - hc_1J(\mathbf{x}, \alpha))^{-1}f(\mathbf{x}, \alpha),$$

where I is the $m \times m$ identity matrix and J is the Jacobian matrix of f .

A fixed point is defined to be $f(\bar{\mathbf{x}}, \alpha) = \mathbf{0}$ in a flow, and is nonhyperbolic if in addition it has at least one eigenvalue with zero real part. In a map, a fixed point is defined to be $f(\bar{\mathbf{x}}, \mu) = \bar{\mathbf{x}}$, and is nonhyperbolic if it has at least one eigenvalue with modulus equal to unity. The following are simple consequences of the form of (2.2), where $\dot{\mathbf{x}} = d\mathbf{x}/dt$:

Lemma 2.1. *If $\mathbf{x} = \bar{\mathbf{x}}$ is a fixed point of $\dot{\mathbf{x}} = f(\mathbf{x}, \alpha)$, and $hc_1J(\bar{\mathbf{x}}, \alpha) \neq I$, then $\bar{\mathbf{x}}$ is a fixed point of (2.2).*

Lemma 2.2. *If $\mathbf{x} = \bar{\mathbf{x}}$ is a fixed point of (2.2), then it is a fixed point of its generating vector field $\dot{\mathbf{x}} = f(\mathbf{x}, \alpha)$.*

Normal forms are obtained by reducing general dynamical systems, subject to specified nondegeneracy conditions, to a ‘‘simplest’’ (polynomial in \mathbf{x}_i) form by means of a sequence of analytic transformations of coordinates of the type $\mathbf{x} = \eta + p_r(\eta)$ where $p_r \in P^r$, $r \geq 2$, and P^r is the space of vector fields whose components are homogeneous polynomials of degree r . The usefulness of normal forms relies on the notions of topological equivalence and topological conjugacy. Two flows f and g are said to be topologically equivalent if there exists a homeomorphism h which takes the orbits of f onto those of g , while preserving the direction of time. This preserves the orbit structure of flows with finite numbers of fixed points. Similarly, two maps F and G are said to be topologically conjugate if there exists a homeomorphism H such that

$H \circ F \circ H^{-1} = G$. That is, the homeomorphism takes the orbits of F onto the orbits of G .

Extending these notions, a parameter-dependent flow $\dot{\mathbf{x}} = f(\mathbf{x}, \alpha)$ is said to be *locally* topologically equivalent in $U_\alpha \subset \mathbb{R}^n$ to a parameter-dependent flow $\dot{\eta} = f(\eta, \beta)$ in $V_\beta \subset \mathbb{R}^n$ if U_α and V_β are small but nonvanishing neighborhoods of the origin in their respective systems, there is a homeomorphism q of the parameter space such that $q(\mathbf{0}) = \mathbf{0}$, and there is a parameter-dependent homeomorphism $h_\alpha(U_\alpha) = V_{q(\alpha)}$ of the phase space such that $h_0(\mathbf{0}) = \mathbf{0}$, which takes the orbits of f in U_α onto orbits of g with $\beta = q(\alpha)$ in $V_{q(\alpha)}$ for all α , preserving the direction in time. Local topological conjugacy of maps is defined analogously, preserving the parameter n .

In the following, we will use the notation $\tilde{s} = \pm 1$ as a binary tag, to represent the sign of a term. Also, we will require only the scalar version of (2.2):

$$(2.3) \quad x \mapsto x + \frac{hf}{1 - hc_1 f_x}.$$

2.1. Saddle-node bifurcation

The following lemmas supply well known topological normal forms [16] for saddle-node (fold, tangent) bifurcation in scalar dynamical systems. The normal forms are generic, but not unique.

Lemma 2.3. *Let $\dot{x} = f(x, \alpha)$ be a one-parameter family of real scalar flows such that f is C^2 in x and C^1 in α on some sufficiently large open set containing the origin. Let the origin be a nonhyperbolic fixed point: $f(0, 0) = 0$, $f_x(0, 0) = 0$. Let the nondegeneracy conditions $f_{xx}(0, 0) \neq 0$, $f_\alpha(0, 0) \neq 0$ apply. Then a normal form for saddle-node bifurcation is*

$$(2.4) \quad \dot{x} = \alpha + \tilde{s}x^2.$$

The normal form (2.4) has fixed points $\bar{x} = \pm\sqrt{-\tilde{s}\alpha}$. For $\tilde{s}\alpha < 0$, $\bar{x} = \tilde{s}\sqrt{-\tilde{s}\alpha}$ is unstable and $\bar{x} = -\tilde{s}\sqrt{-\tilde{s}\alpha}$ is stable. At the bifurcation value $\alpha = 0$, there is a unique unstable fixed point $\bar{x} = 0$. There are no fixed points for $\tilde{s}\alpha > 0$.

Lemma 2.4. *Let $x \mapsto F(x, \mu)$ be a one-parameter family of real scalar maps such that F is locally C^2 in x and C^1 in μ . Let $F(0, 0) = 0$ and $F_x(0, 0) = 1$, $F_{xx}(0, 0) \neq 0$ and $F_\mu(0, 0) \neq 0$. Then a normal form for saddle-node bifurcation is*

$$(2.5) \quad x \mapsto \mu + x + \tilde{s}x^2.$$

The fixed points of normal forms (2.4) and (2.5) correspond exactly in location and stability type.

Theorem 2.5. *If a saddle-node bifurcation exists in $\dot{x} = f(x, \alpha)$, then a corresponding saddle-node bifurcation exists in its transformation by linearized one-point collocation.*

Proof. The normal form (2.4) produces the family of maps

$$(2.6) \quad F = x + \frac{h(\alpha + \tilde{s}x^2)}{1 - 2\tilde{s}hc_1x}$$

under linearized one-point collocation. Expanding as $F(x, \alpha) = x + \sum F_i(\alpha)x^i$, we obtain

$$F = h\alpha + (1 + 2\tilde{s}h^2c_1\alpha)x + (\tilde{s}h + 4h^3c_1^2\alpha)x^2 + O(x^3).$$

Let $\xi = x + \delta$, where $\delta = \delta(\alpha)$ is to be defined. Then the image of ξ under the action of the map is

$$\begin{aligned} \xi = x + \delta \mapsto F(x, \alpha) + \delta &= F(\xi - \delta, \alpha) + \delta \\ &= [F_0 - F_1\delta + F_2\delta^2 + O(\delta^3)] \\ &\quad + [1 + F_1 - 2F_2\delta + O(\delta^2)]\xi \\ &\quad + [F_2 + O(\delta)]\xi^2 + O(\xi^3). \end{aligned}$$

Because $F_1(0) = 0$ and $F_2(0) \neq 0$, the Implicit Function Theorem can be invoked to annihilate the α -dependent part of the linear term (in ξ) for all sufficiently small $|\alpha|$. Hence we define $\delta(\alpha) = \frac{F_1'(0)}{2F_2(0)}\alpha + O(\alpha^2) = hc_1\alpha + O(\alpha^2)$, obtaining

$$(2.7) \quad \xi \mapsto [h\alpha - \tilde{s}h^3c_1^2\alpha^2 + \alpha^3\Phi(\alpha)] + \xi + [\tilde{s}h + O(\alpha)]\xi^2 + O(\xi^3),$$

where $\Phi(\alpha)$ is a smooth function. Let $\gamma(\alpha)$ be the ξ -independent term of (2.7). Since $\gamma(0) = 0$ and $\gamma'(0) = h > 0$, the Inverse Function Theorem assures local existence and uniqueness of a smooth inverse $\alpha(\gamma)$ increasing through the origin. Then $\xi \mapsto \gamma + \xi + a_2(\gamma)\xi^2 + O(\xi^3)$, where $a_2(\gamma)$ is smooth with $a_2(0) = \tilde{s}h \neq 0$. A final change of variable $\eta = |a_2(\gamma)|\xi$ gives

$$\eta \mapsto \mu + \eta + \tilde{s}\eta^2 + O(\eta^3),$$

where $\mu = |a_2(\gamma)|\gamma$. The higher degree terms can be eliminated due to the local topological conjugacy of these continuously parameter-dependent maps [1].

Thus we obtain the normal form (2.5) for saddle-node bifurcation at $(\eta, \mu) = (0, 0)$. Direct consideration of the above transformation sequence, restricted to a neighborhood of $(x, \alpha) = (0, 0)$, confirms that $(\eta, \mu) = (0, 0)$ if and only if $(x, \alpha) = (0, 0)$. Furthermore, the derived map and the original flow have exactly corresponding normal forms and μ is x -independent. Therefore, the phase location, orientation and stability properties of the saddle-node bifurcation are preserved under the map. \square

Obviously, location and stability can be calculated explicitly for the fixed points of the family of maps (2.6). Singularities exist for $2\tilde{s}hc_1x = 1$; the consequences of this singular set are discussed in §3. Fixed points $\bar{x} = \pm\sqrt{-\tilde{s}\alpha}$ exist for $\tilde{s}\alpha \leq 0$, $\alpha \neq -\tilde{s}/(4h^2c_1^2)$. Their stability types near the saddle-node point are those of the originating flow. A spurious period doubling bifurcation

occurs; however, by Theorem 2.5 it cannot exist in the neighborhood of the saddle-node point. Values of $c_1 = 0$ (explicit Euler method) and $c_1 = 1/2$ (linearized implicit midpoint) give valid limiting cases, in which the singular set and the spurious period doubling point, respectively, are removed to infinity. The cases $c_1 \in (0, 1/2)$ and $c_1 \in (1/2, 1]$ are illustrated as bifurcation diagrams in Figs. 1 and 2. All bifurcation diagrams in this study use the following legend: (—) stable branch, (---) unstable branch, and (- · - · -) singular set.

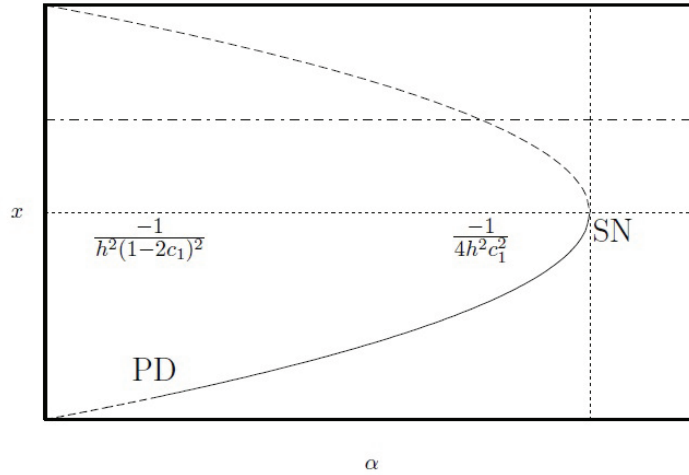


FIGURE 1. Saddle-node bifurcation: $f = \alpha + x^2$, $c_1 < 1/2$.

Normal form construction is an advantageous approach to understanding the dynamics of numerics. It can illuminate the general action of numerical methods near bifurcation for parameter-dependent flows; for example, the variable x of Theorem 2.5 is shown to have a “natural” local discretized analog hx . Properties such as the asymptotics of orbits and global behavior near bifurcation value, and spurious or nonequivalent bifurcation can also be examined through normal forms.

Theorem 2.6. *If a saddle-node bifurcation exists in a map $x \mapsto F(x, \mu)$ constructed by linearized one-point collocation, it results from an exactly corresponding saddle-node bifurcation in the originating flow $\dot{x} = f(x, \alpha)$.*

Proof. Assume (2.3) possesses a saddle-node bifurcation at $(x, \mu) = (0, 0)$, where $f = f(x, \mu)$ is an undetermined flow function given to be C^2 in x and C^1 in μ . Equating this family of maps to the normal form (2.5), and expanding $f(x, \mu) = \sum a_i(\mu)x^i$, we match coefficients to obtain

$$f(x, \mu) = \frac{\mu}{h} + 2c_1^2\mu^2a_2(\mu) - 2c_1\mu a_2(\mu)x + a_2(\mu)x^2 + O(x^3).$$

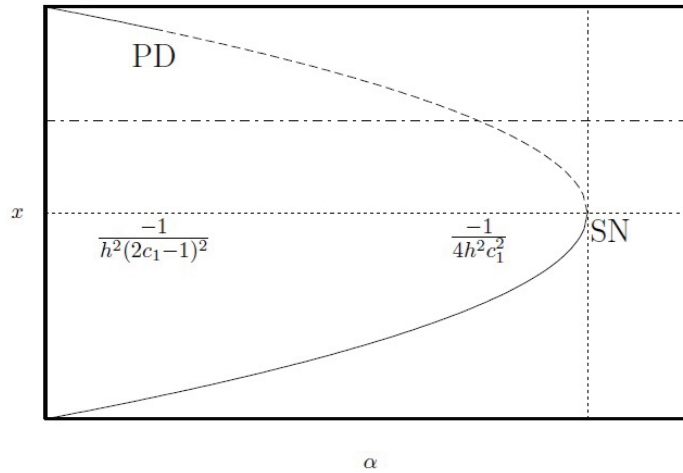


FIGURE 2. Saddle-node bifurcation: $f = \alpha + x^2$, $c_1 > 1/2$.

A change of variable $\xi = x + \delta(\mu)$ yields the originating flow as

$$\begin{aligned} \dot{\xi} &= \dot{x} = f(x, \mu) = f(\xi - \delta, \mu) \\ &= [a_0 - a_1\delta + a_2\delta^2 + O(\delta^3)] + [a_1 - 2a_2\delta + \delta^2\Psi(\mu, \delta)]\xi \\ &\quad + [a_2 + O(\delta)]\xi^2 + O(\xi^3), \end{aligned}$$

where $\Psi(\mu, \delta)$ is a smooth function. If $a_1(0) = 0$ and $a_2(0) \neq 0$, then the linear term in ξ can be annihilated for all sufficiently small $|\mu|$ by the Implicit Function Theorem. These conditions are satisfied whenever $\beta_0 \neq 0$ in $a_2(\mu) = \beta_0 + O(\mu)$, equivalent to the nondegeneracy condition $f_{xx}(0, 0) \neq 0$ for saddle-node bifurcation. We can then define $\delta(\mu) = \frac{a_1'(0)}{2a_2(0)}\mu + O(\mu^2) = -c_1\mu + O(\mu^2)$, to obtain

$$(2.8) \quad f(x, \mu) = \left[\frac{\mu}{h} + \beta_0 c_1^2 \mu^2 + \mu^3 \Phi(\mu) \right] + [\beta_0 + O(\mu)] \xi^2 + O(\xi^3),$$

where $\Phi(\mu)$ is smooth.

Now define $\gamma = \gamma(\mu)$ as the ξ -independent term of (2.8). Since $\gamma(0) = 0$ and $\gamma'(0) \neq 0$, the Inverse Function Theorem guarantees local existence and uniqueness of a smooth inverse $\mu = \mu(\gamma)$ with $\mu(0) = 0$. Therefore, $\dot{\xi} = \gamma + a_2(\gamma)\xi^2 + O(\xi^3)$, with $a_2(\gamma)$ a smooth function such that $a_2(0) = \beta_0$.

Finally, we scale the variable as $\eta = |\beta_0| \xi$, which gives

$$\dot{\eta} = \alpha + s\eta^2 + O(\eta^3),$$

where $\alpha = |\beta_0| \left(\frac{\mu}{h} + \beta_0 c_1^2 \mu^2 + \mu^3 \Phi(\mu) \right)$ and $s = \text{sign}(\beta_0)$.

The higher degree terms for η can be eliminated due to local topological equivalence of the flows, to attain the normal form (2.4) in η . The origin is locally preserved under the transformation $(x, \mu) \rightarrow (\eta, \alpha)$, so the location of the bifurcation is preserved. Lemma 2.2 implies the direction of the saddle-node bifurcation is preserved under the transformation (*n.b.*, direction is not preserved by the normal forms themselves). The derived normal form is identical to (2.4), and α increases with increasing μ near the origin, so the stability type is preserved as well. Thus, no spurious saddle-node bifurcation can occur from linearized one-point collocation. \square

2.2. Transcritical bifurcation

In practical applications, dynamical systems often possess symmetry such that a trivial fixed point persists through bifurcation. Thus, a restriction of $f_\alpha(0, 0) = 0$ will ensure that $x = 0$ remains a fixed point for all parameter values in the topological normal form [6].

Lemma 2.7. *Let $\dot{x} = f(x, \alpha)$ be a one-parameter family of scalar flows such that f is locally C^2 in x and C^1 in α . Assume the generating conditions $f(0, 0) = 0$, $f_x(0, 0) = 0$, $f_\alpha(0, 0) = 0$, and the nondegeneracy conditions $f_{xx}(0, 0) \neq 0$, $f_{x\alpha}(0, 0) \neq 0$. Then a normal form for transcritical bifurcation is*

$$(2.9) \quad \dot{x} = \alpha x + \tilde{s}x^2.$$

Lemma 2.8. *Let $x \mapsto F(x, \mu)$ be a one-parameter family of scalar maps such that F is locally C^2 in x and C^1 in μ . Assume $F(0, 0) = 0$, $F_x(0, 0) = 1$, $F_\mu(0, 0) = 0$, $F_{xx}(0, 0) \neq 0$ and $F_{x\mu}(0, 0) \neq 0$. Then a normal form for transcritical bifurcation is*

$$(2.10) \quad x \mapsto (1 + \mu)x + \tilde{s}x^2.$$

Theorem 2.9. *If a transcritical bifurcation exists in $\dot{x} = f(x, \alpha)$, then a corresponding transcritical bifurcation exists in its transformation by linearized one-point collocation.*

Proof. Linearized one-point collocation methods convert normal form (2.9) into the following family of maps:

$$(2.11) \quad F = x + \frac{h(\alpha x + \tilde{s}x^2)}{1 - hc_1(\alpha + 2\tilde{s}x)}.$$

Expanding in powers of x gives

$$F = x + \frac{h\alpha}{1 - hc_1\alpha}x + \frac{\tilde{s}h(1 + hc_1\alpha)}{(1 - hc_1\alpha)^2}x^2 + O(x^3).$$

Consider a new parameter, defined by

$$(2.12) \quad \mu = \mu(\alpha) = \frac{h\alpha}{1 - hc_1\alpha}.$$

Since $\mu(0) = 0$ and $\mu'(0) = h > 0$, the Inverse Function Theorem justifies existence and uniqueness of a smooth local inverse function $\alpha = \alpha(\mu)$ with $\alpha(0) = 0$, increasing through the origin. Therefore, $F = x + \mu x + \tilde{s}a_2(\mu)x^2 + O(x^3)$, where $a_2(\mu)$ is a smooth function such that $a_2(0) = h \neq 0$.

The change of variable $\eta = |a_2(\mu)|x$ yields

$$\eta \mapsto (1 + \mu)\eta + s\eta^2 + O(\eta^3),$$

where $s = \text{sign}(\tilde{s}a_2(0))$. But $a_2(0) > 0$, and thus the final scaling transformation preserves the direction of bifurcation. Due to local topological conjugacy, we can eliminate the higher degree terms to obtain the normal form (2.10) for transcritical bifurcation at $(\eta, \mu) = (0, 0)$. The origin is locally preserved under the transformation $(x, \alpha) \rightarrow (\eta, \mu)$, so the location and orientation of bifurcation are preserved as well. \square

The behavior of the family of maps (2.11) can be found explicitly. Figs. 3 and 4 illustrate this behavior for $c_1 \in (0, 1/2)$ and $c_1 \in (1/2, 1]$. The fixed points $\bar{x} = 0$ and $\bar{x} = -\tilde{s}\alpha$ have the same stability properties as the originating flow near the origin. They undergo spurious period doubling bifurcation, but at a distance removed from the origin. The singular set $hc_1(\alpha + 2\tilde{s}x) = 1$ does not affect the dynamics in the vicinity of the transcritical point for any $\{h, c_1\}$, by Theorem 2.9. Values of $c_1 = 0$ and $c_1 = 1/2$ give limiting cases wherein the singularities and the period doubling locations, respectively, are removed to infinity.

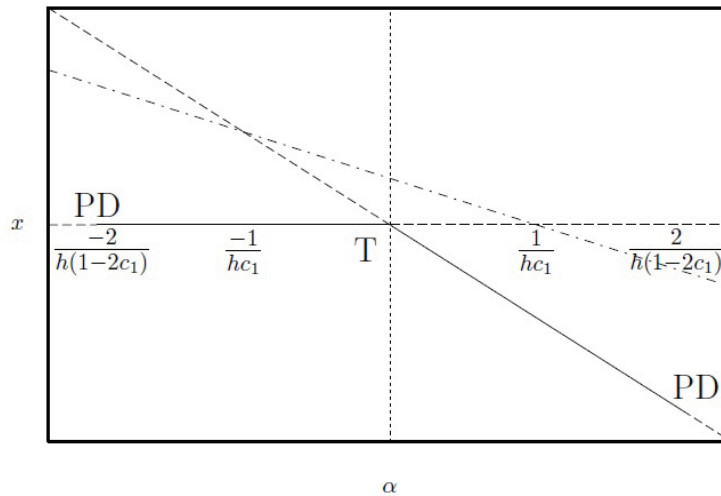


FIGURE 3. Transcritical bifurcation: $f = \alpha x + x^2$, $c_1 < 1/2$.

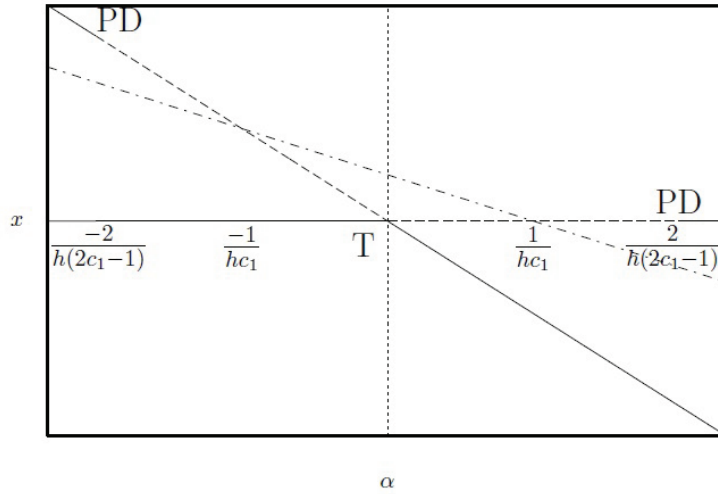


FIGURE 4. Transcritical bifurcation: $f = \alpha x + x^2$, $c_1 > 1/2$.

Theorem 2.10. *If a transcritical bifurcation exists in a scalar map $x \mapsto F(x, \mu)$ constructed by linearized one-point collocation, it results from a corresponding transcritical bifurcation in the originating flow $\dot{x} = f(x, \alpha)$.*

Proof. Assume the scalar family of maps (2.3) has normal form (2.10) for transcritical bifurcation. Expanding the undetermined flow function as $f(x, \mu) = \sum a_i(\mu)x^i$ and matching coefficients of powers of x in the identity

$$(1 + \mu)x + \tilde{s}x^2 = x + \frac{hf}{1 - hc_1f_x}$$

yields

$$f(x, \mu) = \frac{\mu}{h(1 + c_1\mu)}x + \frac{\tilde{s}}{h(1 + c_1\mu)(1 + 2c_1\mu)}x^2 + O(x^3).$$

Define

$$(2.13) \quad \alpha = \alpha(\mu) = \frac{\mu}{h(1 + c_1\mu)}.$$

Then $\alpha(0) = 0$ and $\alpha'(0) = 1/h \neq 0$, so $\alpha(\mu)$ possesses a unique smooth local inverse according to the Inverse Function Theorem. Furthermore, $\alpha'(0) > 0$ so the function and its inverse must increase in a neighborhood of the origin. We can write $f = \alpha x + \tilde{s}a_2(\alpha)x^2 + O(x^3)$, where $a_2(\alpha)$ is a smooth function with $a_2(0) = 1/h \neq 0$.

Finally, a change of variable $\eta = |a_2(\alpha)|x$ yields $\dot{\eta} = \alpha\eta + s\eta^2 + O(\eta^3)$, where $s = \text{sign}(\tilde{s}a_2(0)) = \tilde{s}$. Eliminating the higher degree terms by a local topological equivalence construction, we obtain the normal form (2.9) for transcritical bifurcation at $(\eta, \alpha) = (0, 0)$. It is easily verified that $(\eta, \alpha) = (0, 0)$ if and only if $(x, \alpha) = (0, 0)$. Thus the location, direction and orientation of transcritical bifurcation are preserved by the transformation sequence. \square

The resonance singularities encountered during transformation to normal form are $\mu = -1/c_1$ and $\mu = -1/(2c_1)$. Due to the restriction $0 \leq c_1 \leq 1$, these singularities cannot exist in the vicinity of the bifurcation point. Therefore, their existence does not influence the theorem.

2.3. Pitchfork bifurcation

Physical systems often contain certain symmetries. A scalar flow $\dot{x} = f(x, \alpha)$ is said to be equivariant with respect to the representation $\{-1, 1\}$ of the group \mathbb{Z}_2 if $f(-x, \alpha) = -f(x, \alpha)$. Thus, a \mathbb{Z}_2 -equivariant vector field is one for which $f(x, \alpha)$ is an odd function of x . The generic unfolding of such a reflection symmetric system results in a pitchfork bifurcation.

Lemma 2.11. *Let $\dot{x} = f(x, \alpha)$ be a one-parameter family of scalar flows such that f is locally C^3 in x and C^1 in α . Assume $f(0, 0) = 0$, $f_x(0, 0) = 0$, $f_\alpha(0, 0) = 0$, $f_{xx}(0, 0) = 0$, and the nondegeneracy conditions $f_{x\alpha}(0, 0) \neq 0$, $f_{xxx}(0, 0) \neq 0$. Then a normal form [16] for pitchfork bifurcation is*

$$(2.14) \quad \dot{x} = \alpha x - \tilde{s}x^3.$$

A supercritical pitchfork ($\tilde{s} = +1$) involves a stable fixed point bifurcating into an unstable fixed point flanked by a stable pair. A subcritical pitchfork ($\tilde{s} = -1$) involves the same situation with all stabilities reversed.

Lemma 2.12. *Let $x \mapsto F(x, \mu)$ be a one-parameter family of scalar maps such that F is locally C^3 in x and C^1 in μ . Assume $F(0, 0) = 0$, $F_x(0, 0) = 1$, $F_\mu(0, 0) = 0$, $F_{xx}(0, 0) = 0$, $F_{x\mu}(0, 0) \neq 0$ and $F_{xxx}(0, 0) \neq 0$. Then a normal form [16] for pitchfork bifurcation is*

$$(2.15) \quad x \mapsto (1 + \mu)x - \tilde{s}x^3.$$

Theorem 2.13. *If a pitchfork bifurcation exists in $\dot{x} = f(x, \alpha)$, then a corresponding pitchfork bifurcation exists in its transformation by linearized one-point collocation.*

Proof. The family of maps generated from the normal form (2.14) by linearized one-point collocation is

$$(2.16) \quad F = x + \frac{h(\alpha x - \tilde{s}x^3)}{1 - hc_1(\alpha - 3\tilde{s}x^2)}.$$

The proof method is identical to that of Theorem 2.9, yielding the normal form

$$\eta \mapsto (1 + \mu)\eta - \tilde{s}\eta^3,$$

where μ is given by (2.12), $\eta = \sqrt{|a_3(\mu)|}x$ and $a_3 = h + \mu\Phi(\mu)$ is a smooth function. Location of the pitchfork bifurcation is locally preserved at the origin. The orientation and stability type of bifurcation are preserved as well by the transformation to normal form. \square

The case $c_1 \in (0, 1/2)$ for (2.16) is illustrated in Fig. 5. The family of singularities $hc_1(\alpha - 3\tilde{s}x^2) = 1$ arising from the imposition of linearized one-point collocation cannot cause spurious behavior near the origin, by Theorem 2.13. The limiting cases $c_1 = 0$ and $c_1 = 1/2$ merely remove the singularities and spurious period doubling bifurcations, respectively, to infinity.

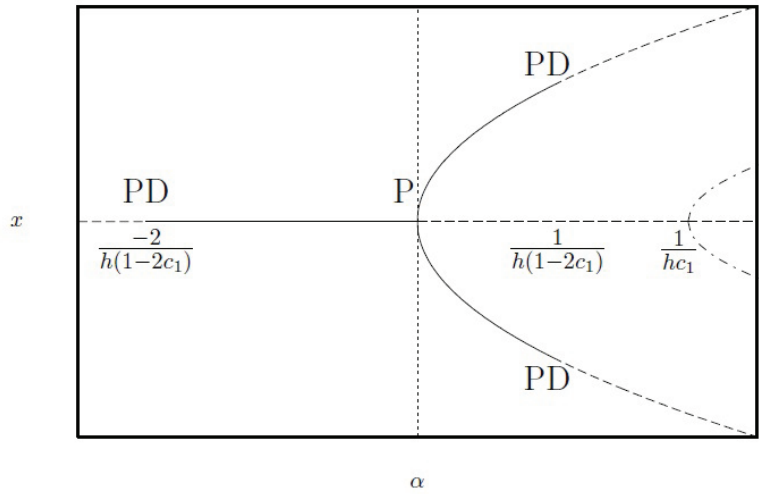


FIGURE 5. Supercritical pitchfork: $f = \alpha x - x^3$, $c_1 < 1/2$.

Theorem 2.14. *If a pitchfork bifurcation exists in a scalar map $x \mapsto F(x, \mu)$ constructed by linearized one-point collocation, it results from a corresponding pitchfork bifurcation in the originating flow $\dot{x} = f(x, \alpha)$.*

Proof. Assume the scalar map has normal form (2.15) for pitchfork bifurcation. Proceeding as in Theorem 2.10, we can transform the originating flow to normal form $\dot{\eta} = \alpha\eta - \tilde{s}\eta^3$ for the pitchfork bifurcation, where α is given by (2.13), $\eta = \sqrt{|a_3(\alpha)|}x$ and $a_3(\alpha) = (1/h) + \alpha\Phi(\alpha)$ is a smooth function. The location, direction and stability type of bifurcation are preserved locally by the transformation sequence. \square

Resonance singularities of the form $\mu = -1/(nc_1)$, $n = 1, 2, 3$, arise during the transformation to normal form. Due to the restriction $0 \leq c_1 \leq 1$, the

singularities cannot be located local to $\mu = 0$, and thus do not affect the normal form results.

2.4. Period doubling bifurcation

The appearance of a simple eigenvalue $\lambda = -1$ in a map typically gives rise to a period doubling (flip) bifurcation. This bifurcation is fundamentally different because it does not have an analog in one-dimensional vector field dynamics, and thus is necessarily spurious in our context.

Lemma 2.15. *Let $x \mapsto F(x, \mu)$ be a one-parameter family of scalar maps such that F is locally C^3 in x and C^1 in μ . Assume $F(0, 0) = 0$, $F_x(0, 0) = -1$, and the nondegeneracy conditions $F_{x\mu}(0, 0) \neq 0$, $F_{xxx}^2(0, 0) \neq 0$. Then a normal form [1] for period doubling bifurcation is*

$$(2.17) \quad x \mapsto -(1 + \mu)x + \tilde{s}x^3.$$

The map at the origin has a nonhyperbolic fixed point with $\lambda = -1$, at which the second iterate undergoes a pitchfork bifurcation. The bifurcation is said to be supercritical if the bifurcating solution of period two is stable and subcritical if the bifurcating solution is unstable.

The final nondegeneracy condition of Lemma 2.15 is awkward to apply because it is expressed in terms of the second iterate (F^2). An equivalent condition [16] using F is $G(x, \mu) \neq 0$, where

$$(2.18) \quad G(x, \mu) = \frac{1}{2}F_{xx}(0, 0)^2 + \frac{1}{3}F_{xxx}(0, 0).$$

If the Schwarzian derivative [3] of F is negative, any period doubling bifurcation must be supercritical. The form of (2.18) is intimately related to the Schwarzian derivative, and we can equivalently state the condition for supercritical period doubling as $G(x, \mu) > 0$.

Theorem 2.16. *Spurious period doubling behavior can result from the application of linearized one-point collocation for all $c_1 \neq 1/2$. A normal form for scalar flow causing spurious period doubling bifurcation ($c_1 \neq 1/6, c_1 \neq 1/2$) is $\dot{x} = \left(\frac{2}{h(2c_1-1)} - \alpha\right)x + \tilde{s}x^3$.*

Proof. Assume the induced scalar map will have normal form (2.17) for period doubling bifurcation at $(x, \mu) = (0, 0)$. Expanding the undetermined flow function $f(x, \mu) = \sum a_i(\mu)x^i$ and solving for coefficients of powers of x in $-(1 + \mu)x + \tilde{s}x^3 = x + hf/(1 - hc_1f_x)$ such that $a_2 = 0$ yields

$$f(x, \mu) = \frac{2 + \mu}{h(2c_1 + c_1\mu - 1)}x + \frac{\tilde{s}}{h(2c_1 + c_1\mu - 1)(6c_1 + 3c_1\mu - 1)}x^3 + O(x^4).$$

Define

$$\alpha = \alpha(\mu) = \frac{\mu}{h(2c_1 + c_1\mu - 1)(2c_1 - 1)}.$$

Then $\alpha(0) = 0$ and $\alpha'(0) > 0$ for $c_1 \neq 1/2$, so a unique smooth local inverse function $\mu(\alpha)$ exists by the Inverse Function Theorem, and the function and its inverse increase monotonically in a neighborhood of the origin. We have

$$f = (\alpha_0 - \alpha)x + \tilde{s}a_3(\alpha)x^3 + O(x^4), \quad \alpha_0 = \frac{2}{h(2c_1 - 1)},$$

where $a_3(\alpha)$ is smooth with $a_3(0) = [h(2c_1 - 1)(6c_1 - 1)]^{-1} \neq 0$, but $a_3(0)$ is singular for $c_1 = 1/6$ and $c_1 = 1/2$.

Finally, let $\eta = \sqrt{|a_3(\alpha)|}x$, yielding

$$\dot{\eta} = (\alpha_0 - \alpha)\eta + s\eta^3 + O(\eta^4),$$

where $s = \text{sign}(\tilde{s}a_3(0)) = \text{sign}(\tilde{s}(2c_1 - 1)(6c_1 - 1))$. Higher degree terms can be eliminated due to local topological equivalences [1]. The origin $(x, \mu) = (0, 0)$ is preserved under the transformations $(x, \mu) \rightarrow (\eta, \alpha)$. Therefore, we interpret this as a normal form for scalar flow causing a spurious period doubling bifurcation at the origin under linearized one-point collocation. \square

The value $c_1 = 1/6$ gives the transitional boundary between supercritical and subcritical bifurcation in the normal form. Period doubling bifurcation is possible at $c_1 = 1/6$, but the normal form does not apply.

Singularities $\mu = -1/c_1$ and $\mu = -1/(2c_1)$ are encountered during the construction of normal form. Due to the restriction $0 \leq c_1 \leq 1$, the singularities are always removed from $\mu = 0$, and therefore they have no effect on the normal form results.

By direct calculation using (2.18), if a spurious period doubling bifurcation exists under linearized one-point collocation, it will be *supercritical* whenever $g(x, \alpha) > 0$, where

$$(2.19) \quad g(x, \alpha) = 3hf_{xx}f_{xx}(1 - 4c_1)(1 - 2c_1)^3 + 2f_{xxx}(1 - 6c_1)(1 - 2c_1).$$

The normal form for scalar vector fields giving rise to spurious period doubling behavior here is $f(x, \alpha) = (\alpha_0 - \alpha)x + \tilde{s}x^3$, where $\alpha_0 = \frac{2}{h(2c_1 - 1)}$. From (2.19), we determine supercritical bifurcation to occur in the induced map for $1/6 < c_1 < 1/2$ in the $\tilde{s} = -1$ case, and subcritical bifurcation to occur for $1/6 < c_1 < 1/2$ in the $\tilde{s} = +1$ case. We can explicitly calculate the family of maps arising from the transformation by considering

$$(2.20) \quad F = x + \frac{h(\alpha_0 x - \alpha x + \tilde{s}x^3)}{1 - hc_1(\alpha_0 - \alpha + 3\tilde{s}x^2)}.$$

See Figs. 6–9. The fixed points are $\bar{x} = 0$ and $\bar{x} = \pm\sqrt{\tilde{s}(\alpha - \alpha_0)}$. The limiting case $c_1 = 0$ removes the singular set to infinity.

A quantity of interest in the dynamics of numerics is the critical stepsize h_c , defined as the maximum stepsize of an ODE solver such that no spurious period doubling behavior can occur. To find this quantity for our numerical schemes, we regard h as the bifurcation parameter for fixed μ in the map $x \mapsto F(x, \mu; h)$.

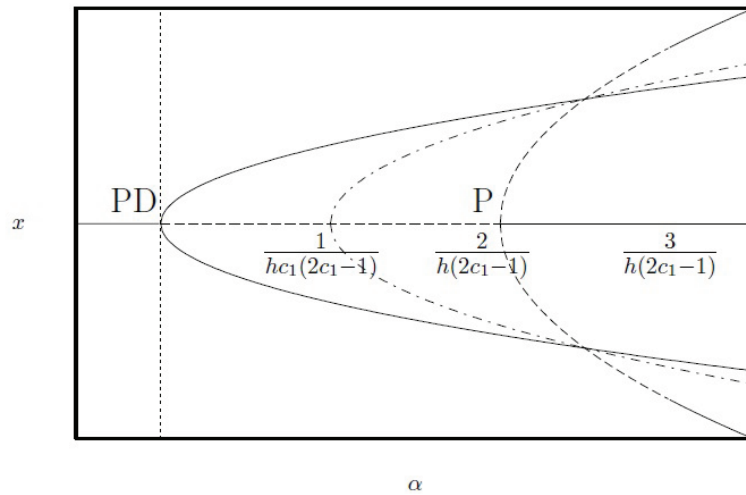


FIGURE 6. Period doubling bifurcation: $f = (\alpha_0 - \alpha)x + x^3$, $\alpha_0 = \frac{2}{h(2c_1-1)}$, $c_1 > 1/2$.

Then it is a simple matter to impose the conditions of Lemma 2.15 on our induced family of maps (2.3).

The first generating condition $F(\bar{x}, \mu; h_c) = \bar{x}$ yields the fixed points $f(\bar{x}, \alpha) = 0$. The defining condition $F_x(\bar{x}, \mu; h_c) = -1$ then immediately yields the critical stepsize:

$$h_c = \frac{2}{(2c_1 - 1)f_x(\bar{x}, \alpha)}.$$

Note that this expression is equivalent to $f_x(\bar{x}, \alpha) = \alpha_0(h_c)$, and implies that no spurious period doubling can occur when $c_1 = 1/2$. The first nondegeneracy condition $F_{xh}(\bar{x}, \mu; h_c) \neq 0$ gives an identity for all $c_1 \neq 1/2$, and the final nondegeneracy condition yields the subcritical/supercritical boundaries $g(\bar{x}, \alpha) \neq 0$, where g is given by (2.19).

For a general linear multistep method [14],

$$h_c = \frac{\rho(-1)}{f_x \sigma(-1)}.$$

Here $\rho(x)$ and $\sigma(x)$ are the first and second characteristic polynomials of the method, respectively [19]. It is trivial to observe that the expressions above for h_c coincide for the explicit Euler and implicit Euler methods, which are both linear multistep.

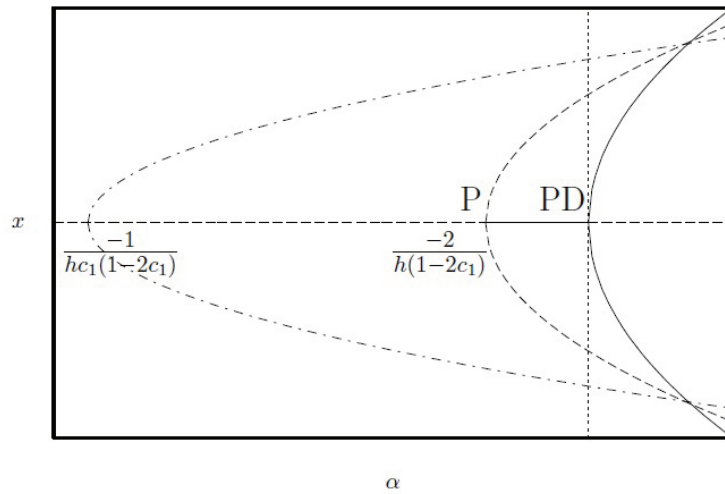


FIGURE 7. Period doubling bifurcation: $f = (\alpha_0 - \alpha)x + x^3$, $\alpha_0 = \frac{2}{h(2c_1-1)}$, $0 < c_1 < \frac{1}{6}$.

3. The singular set

It is noted that the mathematical form of the linearized one-step collocation methods contains a family of singularities. This family has no effect on the dynamics of the methods local to the bifurcations studied; however, it has major effects on asymptotics and global dynamics.

Monotonicity is a fundamental property of scalar autonomous flows. Discretizing such differential equations by linearized collocation methods can result in loss of monotonicity in the orbits of the induced map. This is due to the fact that the singular set acts as a local repeller, causing spurious pole type behavior [15]. A consequence of this phenomenon is the distortion of basins of attraction for local fixed points. Solutions which tend monotonically to infinity in the flow may not do so in the map induced by the methods. More precisely, the basin of attraction for infinity in the neighborhood of the bifurcation contains only the singular set and its preiterates, a set of measure zero. Other orbits can exist as spurious unstable periodic cycles, tend to the stable fixed points, or exhibit more complicated behavior described below.

Another consequence is spurious intermittency. The notion of intermittency transition to a chaotic attractor [21] pertains to the asymptotics of orbits with

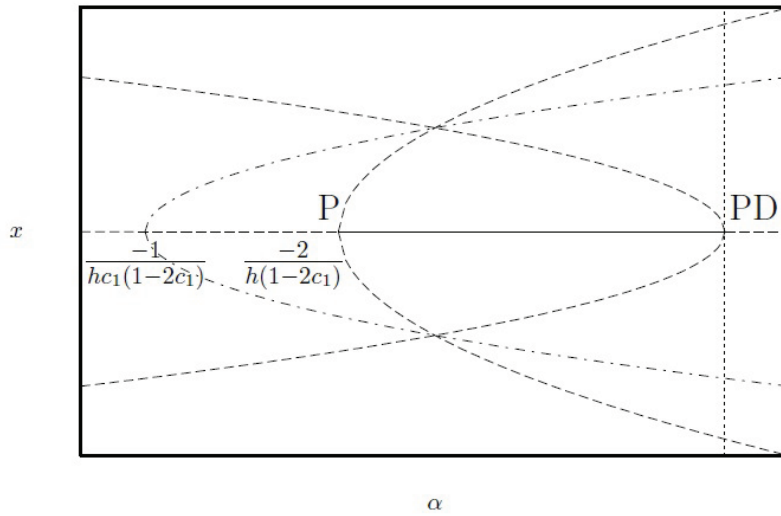


FIGURE 8. Period doubling bifurcation: $f = (\alpha_0 - \alpha)x + x^3$, $\alpha_0 = \frac{2}{h(2c_1-1)}$, $\frac{1}{6} < c_1 < \frac{1}{2}$.

change in bifurcation parameter. As the parameter increases through bifurcation value, the attractor changes from a stable fixed point or stable periodic orbit to a complicated orbit consisting of laminar phases during which it is monotonic and sporadic bursting phases during which it appears chaotic. Pomeau and Manneville [21] identified three bifurcations as generic for producing intermittency. These are the saddle-node bifurcation (Type I intermittency), the subcritical Hopf bifurcation of a periodic orbit (Type II), and the inverse period doubling bifurcation (Type III). Each type has distinguishing scaling behavior on the average time between the seemingly random bursting phases.

We now consider the effects of the singular set on each of the bifurcation normal forms, in the neighborhood of the bifurcation.

3.1. The saddle-node normal form

Figure 10 depicts the bifurcation diagram ($c_1 = 1, h = 0.1$) of the map (2.6) resulting from a discretization of the saddle-node normal form by linearized one-point collocation. The L_2 norm for closest approach of the singular set to the saddle-node point $(x, \alpha) = (0, 0)$ for the induced map is $d = 1/(2h)$, at $(x, \alpha; c_1) = (\tilde{s}/(2hc_1), 0; 1)$.

The singular set in the derived family of maps is $1 - 2\tilde{s}hc_1x = 0$. At the saddle-node point, the singularity destroys the monotonicity of orbits and resembles a snapback repeller, where the singular point $x = \tilde{s}/(2hc_1)$ and

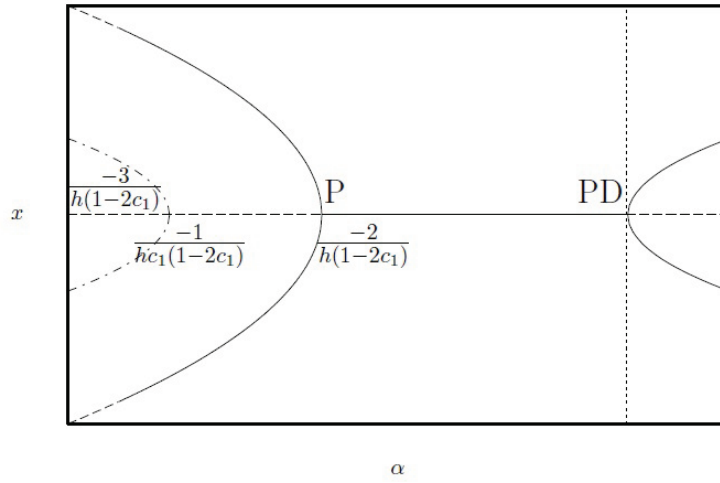


FIGURE 9. Period doubling bifurcation: $f = (\alpha_0 - \alpha)x - x^3$, $\alpha_0 = \frac{2}{h(2c_1-1)}$, $\frac{1}{6} < c_1 < \frac{1}{2}$.

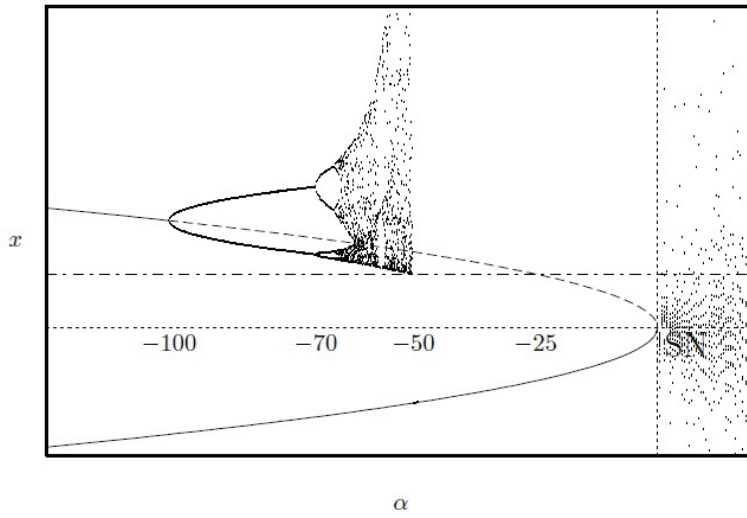


FIGURE 10. Saddle-node: $f = \alpha + x^2$, $c_1 = 1$, $h = 0.1$.

its preimages are mapped to infinity and all other points are mapped to the

origin. For $\alpha > 0$, again the locus of points mapped to infinity consists solely of $x = \tilde{s}/(2hc_1)$ and its preimages; for small $\alpha > 0$, almost all initial values yield the “chaotic” orbits with laminar phase which distinguish Type I intermittency.

3.2. The transcritical normal form

The singular set is given by $1 - hc_1(\alpha + 2\tilde{s}x) = 0$, and is removed from the origin for all c_1 . The complete bifurcation diagram ($c_1 = 1, h = 0.1$) of the map (2.11) is shown in Figure 11. The L_2 minimum distance from the singular set to the transcritical bifurcation point in the induced map is $d = 1/(\sqrt{5}h)$, at $(x, \alpha; c_1) = (2\tilde{s}/(5hc_1), 1/(5hc_1); 1)$.

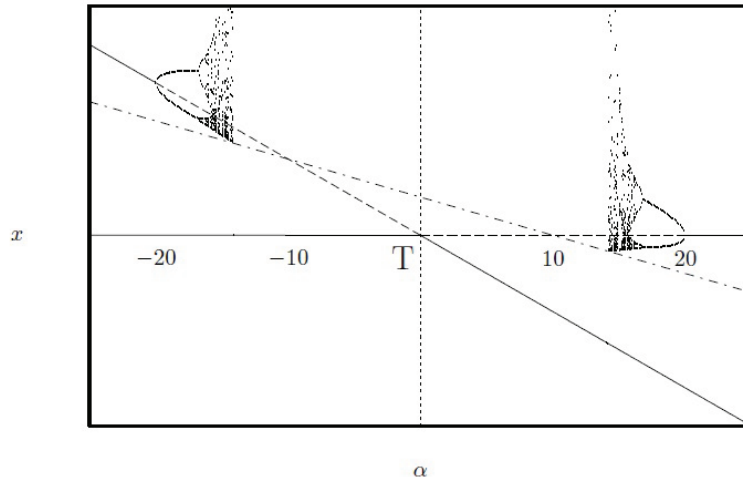


FIGURE 11. Transcritical: $f = \alpha x + x^2, c_1 = 1, h = 0.1$.

Apart from the loss of monotonicity of orbits, resulting in the stable fixed point being converted to a global attractor, the behavior of the map in the neighborhood of the bifurcation is the same as in the corresponding flow. No intermittent behavior exists near the bifurcation point.

3.3. The pitchfork normal form

Discretizing the normal form for pitchfork bifurcation using linearized one-point collocation methods results in a family of maps with singular set $1 - hc_1(\alpha - 3\tilde{s}x^2) = 0$. The nearest approach to the origin in L_2 -norm is $d = \frac{1}{6}\sqrt{\frac{12}{h} - 1}$ at $(x, \alpha; c_1) = (\frac{1}{6}\sqrt{\frac{12}{h} - 2}, \frac{1}{6}; 1)$ for subcritical bifurcation, and $d = 1/h$ at $(x, \alpha; c_1) = (0, 1/h; 1)$ for supercritical bifurcation.

Figure 12 shows the bifurcation diagram ($\tilde{s} = -1, c_1 = 1, h = 0.1$) for the induced map (2.16). In the vicinity of the subcritical pitchfork, there is loss

of monotonicity of orbits and the same type of attractor basin distortion as in the previous cases. Type III intermittency exists for small $\alpha > 0$.

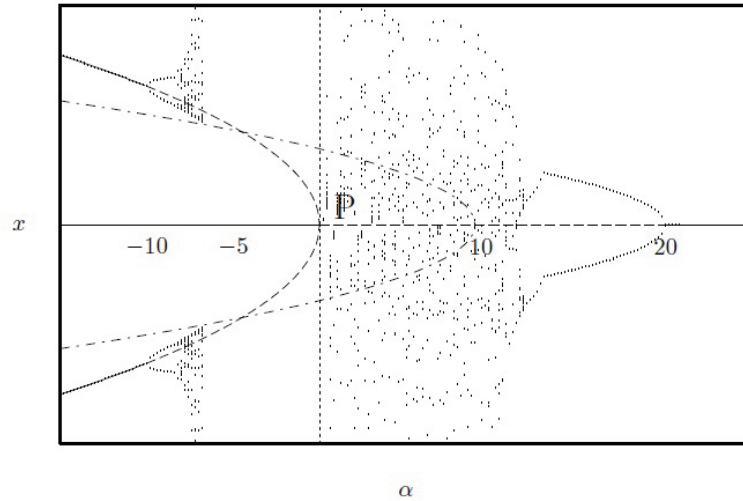


FIGURE 12. Subcritical pitchfork: $f = \alpha x + x^3$, $c_1 = 1$, $h = 0.1$.

3.4. The period doubling normal form

The period doubling bifurcation is, to start with, spurious. The set originating from the singular denominator may modify the approach of orbits to stable attractor basins; however, this effect will almost always be transient because the singular set has measure zero.

With regard to the explicit map (2.20) associated with the normal form of flow causing period doubling, the singular set $1 - hc_1(\alpha_0 - \alpha + 3\tilde{s}x^2) = 0$ intersects the x -axis for $\{c_1 < 1/2, \tilde{s} = 1\}$ and $\{c_1 > 1/2, \tilde{s} = -1\}$. See Figs. 6–9. In this event, the transients and attractor basins associated with solutions in the neighborhood of the period doubling point can be affected. However, the L_2 minimum distance from bifurcation point to singular set is found to be that of the pitchfork bifurcation above, so the singular set cannot disturb the essential period doubling behavior.

4. Discussion

The saddle-node, transcritical and pitchfork bifurcations occur in the family of maps arising from linearized one-point collocation in complete correspondence to their appearance in the originating scalar flow. Spurious period

doubling bifurcation behavior can occur for all $c_1 \neq 1/2$; flows giving rise to spurious period doubling are locally topologically equivalent to

$$\dot{x} = \left(\frac{2}{h(2c_1 - 1)} - \alpha \right) x + \tilde{s}x^3.$$

There is a singular set associated with linearized one-point collocation methods. The singular set has no effect on the dynamics local to the bifurcation points. It does, however, affect the global dynamics by introducing non-monotonic asymptotics including intermittency, spurious period doubling bifurcations and cascades, and distortions in the basins of attraction of locally stable fixed points.

We intend to further study collocation methods from a dynamical systems perspective by examining the common bifurcations in \mathbb{R}^n , $n \geq 2$. We are interested in possible spurious behavior associated with general codimension-1 bifurcation, including the correspondence of Hopf bifurcation in the flow with its induced Neimark-Sacker bifurcation in the map, and the effects of discretization of global bifurcations such as boundary crisis. Also, we are interested in the dynamics of numerics for periodic nonautonomous systems.

Local error control has become routine in practical implementation of algorithms. Higham *et al.* [10] and Aves *et al.* [2] proved results on conditions for eliminating spurious fixed points in Runge-Kutta methods when adaptive time stepping techniques are used. Lamba [17], and Garay and Lee [4], also considered variable stepsize Runge-Kutta methods, and derived upper and lower semicontinuity results for discretized general attractors. We would like to ascertain whether local error control has any dynamical consequences, particularly in eliminating the possibility of spurious period doubling bifurcations, in linearized one-point collocation methods. The bifurcation results found here in §2.4 are h -independent, and this suggests spurious period-doubling behavior may persist for variable stepsize algorithms.

Normal form analysis is worthwhile as a general method for the study of the dynamics of numerics for explicit and implicit numerical methods for ordinary differential equations. Normal forms allow us to consider flows containing state parameters, whereas other methods are restricted to a parametrization with respect to algorithm stepsize. Thus normal forms give us a direct link to the vast theory of *parameter-dependent* dynamical systems. We hope that this type of analysis will be applied to broad classes of ODE solvers, to investigate the stability properties of these methods.

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