KYUNGPOOK Math. J. 51(2011), 241-250
http://dx.doi.org/10.5666/KMJ.2011.51.3.241

Coefficient Inequalities for Certain Subclasses of Analytic Functions Defined by Using a General Derivative Operator

Serap Bulut
Kocaeli University, Civil Aviation College, Arslanbey Campus, 41285, İzmit-Kocaeli, Turkey
e-mail : serap.bulut@kocaeli.edu.tr

Abstract. In this paper, we define new classes of analytic functions using a general derivative operator which is a unification of the Sălăgean derivative operator, the Owa-Srivastava fractional calculus operator and the Al-Oboudi operator, and discuss some coefficient inequalities for functions belong to this classes.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}$.

The following definition of fractional derivative by Owa [8] (also by Srivastava and Owa [18]) will be required in our investigation.

The fractional derivative of order γ is defined, for a function f, by

$$D_z^{\gamma} f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\gamma}} d\xi \quad (0 \leq \gamma < 1), \tag{1.2}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\gamma}$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.

It readily follows from (1.2) that

$$D_z^{\gamma} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} z^{k-\gamma} \quad (0 \leq \gamma < 1, k \in \mathbb{N} = \{1, 2, \ldots\}).$$

Received December 2, 2009; revised November 9, 2010; accepted September 14, 2011.
2010 Mathematics Subject Classification: 30C45, 30C50.
Key words and phrases: Analytic function, Sălăgean derivative operator, Owa-Srivastava fractional calculus operator, Al-Oboudi operator.

Using $D_z^\gamma f$, Owa and Srivastava [10] introduced the operator $\Omega^\gamma : \mathcal{A} \to \mathcal{A}$, which is known as an extension of fractional derivative and fractional integral, as follows:

$$\begin{aligned} \Omega^\gamma f(z) &= \Gamma(2-\gamma) z^\gamma D_z^\gamma f(z), \quad \gamma \neq 2,3,4,\ldots \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} a_k z^k. \end{aligned} \tag{1.3}$$

Note that

$$\Omega^0 f(z) = f(z).$$

In [3], Al-Oboudi and Al-Amoudi defined the linear multiplier fractional differential operator $D_\lambda^{n,\gamma}$ as follows:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D_\lambda^{1,\gamma} f(z) &= (1-\lambda)\,\Omega^\gamma f(z) + \lambda z \left(\Omega^\gamma f(z)\right)' \\ &= D_\lambda^\gamma \left(f(z)\right), \quad \lambda \geq 0,\ 0 \leq \gamma < 1, \\ D_\lambda^{2,\gamma} f(z) &= D_\lambda^\gamma \left(D_\lambda^{1,\gamma} f(z)\right), \end{aligned} \tag{1.4}$$

$$\vdots$$

$$D_\lambda^{n,\gamma} f(z) = D_\lambda^\gamma \left(D_\lambda^{n-1,\gamma} f(z)\right), \quad n \in \mathbb{N}. \tag{1.5}$$

If f is given by (1.1), then by (1.3), (1.4) and (1.5), we see that

$$D_\lambda^{n,\gamma} f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,n}(\gamma,\lambda)\, a_k z^k, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

where

$$\Psi_{k,n}(\gamma,\lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)}\left(1+(k-1)\lambda\right)\right]^n.$$

Remark 1.1. (i) When $\gamma = 0$, we get Al-Oboudi differential operator [2].

(ii) When $\gamma = 0$ and $\lambda = 1$, we get Sălăgean differential operator [14].

(iii) When $n = 1$ and $\lambda = 0$, we get Owa-Srivastava fractional differential operator [10].

Let us define the classes $\mathcal{S}_{\gamma,\lambda}^n(\beta,b)$ and $\mathcal{K}_{\gamma,\lambda}^n(\beta,b)$.

Let $\mathcal{S}_{\gamma,\lambda}^n(\beta,b)$ be the class of functions $f \in \mathcal{A}$ satisfying

$$\Re\left\{1 + \frac{1}{b}\left(\frac{z\left(D_\lambda^{n,\gamma} f(z)\right)'}{D_\lambda^{n,\gamma} f(z)} - 1\right)\right\} > \beta$$

for all $z \in \mathbb{U}$, where $b \in \mathbb{C} - \{0\}$ and $0 \leq \beta < 1$.

Let $\mathcal{K}^n_{\gamma,\lambda}(\beta,b)$ be the class of functions $f\in\mathcal{A}$ satisfying

$$\Re\left\{1+\frac{1}{b}\frac{z\left(D^{n,\gamma}_{\lambda}f(z)\right)''}{\left(D^{n,\gamma}_{\lambda}f(z)\right)'}\right\}>\beta$$

for all $z\in\mathbb{U}$, where $b\in\mathbb{C}-\{0\}$ and $0\leq\beta<1$.

We note that $f\in\mathcal{K}^n_{\gamma,\lambda}(\beta,b)$ if and only if $zf'\in\mathcal{S}^n_{\gamma,\lambda}(\beta,b)$.

Remark 1.2. We have the classes

(i) $\mathcal{S}^0_{\gamma,\lambda}(\beta,b)\equiv\mathcal{S}^1_{0,0}(\beta,b)\equiv\mathcal{S}^*_{\beta}(b)$ and $\mathcal{K}^0_{\gamma,\lambda}(\beta,b)\equiv\mathcal{K}^1_{0,0}(\beta,b)\equiv\mathcal{C}_{\beta}(b)$ defined by Frasin [6].

(ii) $\mathcal{S}^0_{\gamma,\lambda}(\beta,1)\equiv\mathcal{S}^1_{0,0}(\beta,1)\equiv\mathcal{S}^*(\beta)$ and $\mathcal{K}^0_{\gamma,\lambda}(\beta,1)\equiv\mathcal{K}^1_{0,0}(\beta,1)\equiv\mathcal{K}(\beta)$ which are the classes of starlike functions of order β and convex functions of order β in $\mathbb{U}$, respectively.

(iii) $\mathcal{S}^0_{\gamma,\lambda}(0,1)\equiv\mathcal{S}^1_{0,0}(0,1)\equiv\mathcal{S}^*$ and $\mathcal{K}^0_{\gamma,\lambda}(0,1)\equiv\mathcal{K}^1_{0,0}(0,1)\equiv\mathcal{K}$ which are familiar classes of starlike and convex functions in $\mathbb{U}$, respectively.

(iv) $\mathcal{S}^n_{0,1}(\beta,1)\equiv\mathcal{S}_n(\beta)$ which is the class of n-starlike functions of order β defined by Sălăgean [14].

Observe that if $f\in\mathcal{S}^n_{\gamma,\lambda}(\beta,b)$ (or $\mathcal{K}^n_{\gamma,\lambda}(\beta,b)$), then $D^{n,\gamma}_{\lambda}f\in\mathcal{S}^*_{\beta}(b)$ (or $\mathcal{C}_{\beta}(b)$).

Now we define new classes by means of the generalized Al-Oboudi differential operator $D^{n,\gamma}_{\lambda}$ as follows:

A function $f\in\mathcal{A}$ is in the class $\mathcal{SD}^n_{\gamma,\lambda}(\alpha,\beta,b)$ if

$$\Re\left\{1+\frac{1}{b}\left(\frac{z\left(D^{n,\gamma}_{\lambda}f(z)\right)'}{D^{n,\gamma}_{\lambda}f(z)}-1\right)\right\}>\alpha\left|\frac{1}{b}\left(\frac{z\left(D^{n,\gamma}_{\lambda}f(z)\right)'}{D^{n,\gamma}_{\lambda}f(z)}-1\right)\right|+\beta\quad(z\in\mathbb{U})\tag{1.6}$$

where $\alpha\geq0$, $\beta\in[-1,1)$, $\alpha+\beta\geq0$ and $b\in\mathbb{C}-\{0\}$.

A function $f\in\mathcal{A}$ is in the class $\mathcal{KD}^n_{\gamma,\lambda}(\alpha,\beta,b)$ if

$$\Re\left\{1+\frac{1}{b}\frac{z\left(D^{n,\gamma}_{\lambda}f(z)\right)''}{\left(D^{n,\gamma}_{\lambda}f(z)\right)'}\right\}>\alpha\left|\frac{1}{b}\frac{z\left(D^{n,\gamma}_{\lambda}f(z)\right)''}{\left(D^{n,\gamma}_{\lambda}f(z)\right)'}\right|+\beta\qquad(z\in\mathbb{U})\tag{1.7}$$

where $\alpha\geq0$, $\beta\in[-1,1)$, $\alpha+\beta\geq0$ and $b\in\mathbb{C}-\{0\}$.

We note that $f\in\mathcal{KD}^n_{\gamma,\lambda}(\alpha,\beta,b)$ if and only if $zf'\in\mathcal{SD}^n_{\gamma,\lambda}(\alpha,\beta,b)$.

Geometric interpretation. $f\in\mathcal{SD}^n_{\gamma,\lambda}(\alpha,\beta,b)$ and $f\in\mathcal{KD}^n_{\gamma,\lambda}(\alpha,\beta,b)$ if and only if $1+\frac{1}{b}\left(\frac{z\left(D^{n,\gamma}_{\lambda}f(z)\right)'}{D^{n,\gamma}_{\lambda}f(z)}-1\right)$ and $1+\frac{1}{b}\frac{z\left(D^{n,\gamma}_{\lambda}f(z)\right)''}{\left(D^{n,\gamma}_{\lambda}f(z)\right)'}$, respectively, take all values in the conic domain $R_{\alpha,\beta}$ which is included in the right half plane such that

$$R_{\alpha,\beta}=\left\{u+iv:u>\alpha\sqrt{(u-1)^2+v^2}+\beta\right\}.$$

From elementary computations we see that $\partial R_{\alpha,\beta}$,

$$\partial R_{\alpha,\beta} = \left\{ u + iv : u^2 = \left(\alpha \sqrt{(u-1)^2 + v^2} + \beta \right)^2 \right\},$$

represents the conic sections symmetric about the real axis. Thus $R_{\alpha,\beta}$ is an elliptic domain for $\alpha > 1$, a parabolic domain for $\alpha = 1$, a hyperbolic domain for $0 < \alpha < 1$ and a right half plane $u > \beta$ for $\alpha = 0$.

By virtue of (1.6), (1.7) and the properties of the domain $R_{\alpha,\beta}$, we have, respectively

$$\Re \left\{ 1 + \frac{1}{b} \left(\frac{z \left(D_\lambda^{n,\gamma} f(z) \right)'}{D_\lambda^{n,\gamma} f(z)} - 1 \right) \right\} > \frac{\alpha + \beta}{\alpha + 1}$$

and

$$\Re \left\{ 1 + \frac{1}{b} \frac{z \left(D_\lambda^{n,\gamma} f(z) \right)''}{\left(D_\lambda^{n,\gamma} f(z) \right)'} \right\} > \frac{\alpha + \beta}{\alpha + 1},$$

which means that

$$f \in \mathcal{SD}_{\gamma,\lambda}^n(\alpha, \beta, b) \Rightarrow D_\lambda^{n,\gamma} f \in \mathcal{S}_{\gamma,\lambda}^n \left(\frac{\alpha + \beta}{\alpha + 1}, b \right)$$

and

$$f \in \mathcal{KD}_{\gamma,\lambda}^n(\alpha, \beta, b) \Rightarrow D_\lambda^{n,\gamma} f \in \mathcal{K}_{\gamma,\lambda}^n \left(\frac{\alpha + \beta}{\alpha + 1}, b \right).$$

Remark 1.3. We have the classes

(i) $\mathcal{SD}_{\gamma,\lambda}^n(0, \beta, b) \equiv \mathcal{S}_{\gamma,\lambda}^n(\beta, b)$ and $\mathcal{KD}_{\gamma,\lambda}^n(0, \beta, b) \equiv \mathcal{K}_{\gamma,\lambda}^n(\beta, b)$.

(ii) $\mathcal{SD}_{\gamma,\lambda}^n(\alpha, \beta, 1) \equiv \mathcal{SP}_{\gamma,\lambda}^n(\alpha, \beta)$ and $\mathcal{KD}_{\gamma,\lambda}^n(\alpha, \beta, 1) \equiv \mathcal{UCV}_{\gamma,\lambda}^n(\alpha, \beta)$ (Al-Oboudi and Al-Amoudi [3]).

(iii) $\mathcal{SD}_{\gamma,\lambda}^0(\alpha, \beta, 1) \equiv \mathcal{SD}_{0,0}^1(\alpha, \beta, 1) \equiv \mathcal{SD}(\alpha, \beta)$ and $\mathcal{KD}_{\gamma,\lambda}^0(\alpha, \beta, 1) \equiv \mathcal{KD}_{0,0}^1(\alpha, \beta, 1) \equiv \mathcal{KD}(\alpha, \beta)$ (Shams et al. [15]).

(iv) $\mathcal{SD}_{0,1}^n(\alpha, \beta, 1) \equiv \mathcal{US}_n(\alpha, \beta)$ which is the class of n-uniform starlike functions of order β and type α (Acu and Owa [1]).

(v) $\mathcal{SD}_{\gamma,\lambda}^0(0, \beta, b) \equiv \mathcal{SD}_{0,0}^1(0, \beta, b) \equiv \mathcal{S}_\beta^*(b)$ and $\mathcal{KD}_{\gamma,\lambda}^0(0, \beta, b) \equiv \mathcal{KD}_{0,0}^1(0, \beta, b) \equiv \mathcal{SD}_{0,1}^1(0, \beta, b) \equiv \mathcal{C}_\beta(b)$ (Frasin [6]).

(vi) $\mathcal{SD}_{\gamma,\lambda}^0(0, \beta, 1) \equiv \mathcal{SD}_{0,0}^1(0, \beta, 1) \equiv \mathcal{S}^*(\beta)$ and $\mathcal{KD}_{\gamma,\lambda}^0(0, \beta, 1) \equiv \mathcal{KD}_{0,0}^1(0, \beta, 1) \equiv \mathcal{SD}_{0,1}^1(0, \beta, 1) \equiv \mathcal{K}(\beta)$.

(vii) $\mathcal{SD}_{\gamma,\lambda}^0(0, 0, 1) \equiv \mathcal{SD}_{0,0}^1(0, 0, 1) \equiv \mathcal{S}^*$ and $\mathcal{KD}_{\gamma,\lambda}^0(0, 0, 1) \equiv \mathcal{KD}_{0,0}^1(0, 0, 1) \equiv \mathcal{SD}_{0,1}^1(0, 0, 1) \equiv \mathcal{K}$.

(viii) $\mathcal{SD}_{\gamma,\lambda}^0(\alpha, \beta, 1) \equiv \mathcal{SD}_{0,0}^1(\alpha, \beta, 1) \equiv \mathcal{SP}(\alpha, \beta)$ and $\mathcal{KD}_{\gamma,\lambda}^0(\alpha, \beta, 1) \equiv \mathcal{KD}_{0,0}^1(\alpha, \beta, 1) \equiv \mathcal{SD}_{0,1}^1(\alpha, \beta, 1) \equiv \mathcal{UCV}(\alpha, \beta)$ which are uniformly starlike and convex functions, respectively, of order β and type α (Bharati et al. [4]).

(ix) $\mathcal{SD}^0_{\gamma,\lambda}(1,\beta,1) \equiv \mathcal{SD}^1_{0,0}(1,\beta,1) \equiv \mathcal{SP}(\beta)$ and $\mathcal{KD}^0_{\gamma,\lambda}(1,\beta,1) \equiv \mathcal{KD}^1_{0,0}(1,\beta,1) \equiv \mathcal{SD}^1_{0,1}(1,\beta,1) \equiv \mathcal{UCV}(\beta)$ (Rønning [12]).

(x) $\mathcal{SD}^0_{\gamma,\lambda}(1,0,1) \equiv \mathcal{SD}^1_{0,0}(1,0,1) \equiv \mathcal{SP}$ (Rønning [13]) and $\mathcal{KD}^0_{\gamma,\lambda}(1,0,1) \equiv \mathcal{KD}^1_{0,0}(1,0,1) \equiv \mathcal{SD}^1_{0,1}(1,0,1) \equiv \mathcal{UCV}$ which is the class of uniformly convex functions (Goodman [7]).

(xi) $\mathcal{SD}^1_{\gamma,0}(0,\beta,1) \equiv \mathcal{ST}_\gamma(\beta)$ (Srivastava et al. [16]).

(xii) $\mathcal{SD}^1_{\gamma,0}(1,0,1) \equiv \mathcal{SP}_\gamma$ (Srivastava and Mishra [17]).

Observe that if $f \in \mathcal{SD}^n_{\gamma,\lambda}(\alpha,\beta,b)$ (or $\mathcal{KD}^n_{\gamma,\lambda}(\alpha,\beta,b)$), then $D^{n,\gamma}_\lambda f \in \mathcal{SD}(\alpha,\beta,b)$ (or $\mathcal{KD}(\alpha,\beta,b)$).

For the classes $\mathcal{SD}(\alpha,\beta)$ and $\mathcal{KD}(\alpha,\beta)$, Shams et al. [15] have shown some sufficient conditions for f to be in the classes $\mathcal{SD}(\alpha,\beta)$ and $\mathcal{KD}(\alpha,\beta)$.

In [9], Owa et al. have investigated coefficient inequalities for f belonging to the classes $\mathcal{SD}(\alpha,\beta)$ and $\mathcal{KD}(\alpha,\beta)$.

The purpose of this paper is to generalize the results of [9] using generalized Al-Oboudi differential operator.

2. Main results

Theorem 2.1. *If $f \in \mathcal{SD}^n_{\gamma,\lambda}(\alpha,\beta,b)$ with $0 \le \alpha \le \beta$, then $f \in \mathcal{S}^n_{\gamma,\lambda}(\delta,b)$ where $\delta = \frac{\beta-\alpha}{1-\alpha}$.*

Proof. Let $f \in \mathcal{SD}^n_{\gamma,\lambda}(\alpha,\beta,b)$. Then we have

$$\Re\left\{1+\frac{1}{b}\left(\frac{z\left(D^{n,\gamma}_\lambda f(z)\right)'}{D^{n,\gamma}_\lambda f(z)}-1\right)\right\} > \alpha\Re\left\{\frac{1}{b}\left(\frac{z\left(D^{n,\gamma}_\lambda f(z)\right)'}{D^{n,\gamma}_\lambda f(z)}-1\right)\right\}+\beta$$

or equivalently

$$(1-\alpha)\,\Re\left\{1+\frac{1}{b}\left(\frac{z\left(D^{n,\gamma}_\lambda f(z)\right)'}{D^{n,\gamma}_\lambda f(z)}-1\right)\right\} > \beta-\alpha \qquad (z\in\mathbb{U}).$$

If $0 \le \alpha \le \beta$, then we get

$$0 \le \frac{\beta-\alpha}{1-\alpha} < 1.$$

□

Corollary 2.2. *If $f \in \mathcal{KD}^n_{\gamma,\lambda}(\alpha,\beta,b)$ with $0 \le \alpha \le \beta$, then $f \in \mathcal{K}^n_{\gamma,\lambda}(\delta,b)$ where $\delta = \frac{\beta-\alpha}{1-\alpha}$.*

Theorem 2.3. *If $f \in \mathcal{SD}^n_{\gamma,\lambda}(\alpha,\beta,b)$ with $0 \le \alpha \le \beta$, then*

$$|a_2| \le \frac{2\,|b|\,(1-\beta)}{\Psi_{2,n}\,(\gamma,\lambda)\,(1-\alpha)} \tag{2.1}$$

and

$$|a_k| \leq \frac{2|b|(1-\beta)}{\Psi_{k,n}(\gamma,\lambda)(k-1)(1-\alpha)} \prod_{j=1}^{k-2} \left(1 + \frac{2|b|(1-\beta)}{j(1-\alpha)}\right) \qquad (k \geq 3). \tag{2.2}$$

Proof. We note that for $f \in \mathcal{SD}^n_{\gamma,\lambda}(\alpha,\beta,b)$ with $0 \leq \alpha \leq \beta$,

$$\Re\left\{1 + \frac{1}{b}\left(\frac{z\left(D^{n,\gamma}_\lambda f(z)\right)'}{D^{n,\gamma}_\lambda f(z)} - 1\right)\right\} > \frac{\beta-\alpha}{1-\alpha} \qquad (z \in \mathbb{U}).$$

Let us define the function $p(z)$ by

$$p(z) = \frac{(1-\alpha)\left[1 + \frac{1}{b}\left(\frac{z\left(D^{n,\gamma}_\lambda f(z)\right)'}{D^{n,\gamma}_\lambda f(z)} - 1\right)\right] - (\beta-\alpha)}{1-\beta} \qquad (z \in \mathbb{U}).$$

Hence $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$ and $\Re\{p(z)\} > 0$ $(z \in \mathbb{U})$. Let

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots .$$

So we obtain

$$1 + \frac{1}{b}\left(\frac{z\left(D^{n,\gamma}_\lambda f(z)\right)'}{D^{n,\gamma}_\lambda f(z)} - 1\right) = 1 + \frac{1-\beta}{1-\alpha}\left(p_1 z + p_2 z^2 + \cdots\right)$$

or equivalently

$$z\left(D^{n,\gamma}_\lambda f(z)\right)' - D^{n,\gamma}_\lambda f(z) = b\frac{1-\beta}{1-\alpha}\left(D^{n,\gamma}_\lambda f(z)\right)\left(p_1 z + p_2 z^2 + \cdots\right).$$

The last equality implies that

$$\begin{aligned}\Psi_{k,n}(\gamma,\lambda)(k-1)a_k \quad = \quad & \frac{b(1-\beta)}{1-\alpha}\{p_{k-1} + \Psi_{2,n}(\gamma,\lambda)\, a_2 p_{k-2} + \Psi_{3,n}(\gamma,\lambda)\, a_3 p_{k-3} \\ & + \cdots + \Psi_{k-1,n}(\gamma,\lambda)\, a_{k-1} p_1\}.\end{aligned}$$

Applying the coefficient estimates $|p_k| \leq 2$ $(k \geq 1)$ for Carathéodory functions [5], we get

$$\begin{aligned}|a_k| \quad \leq \quad & \frac{2|b|(1-\beta)}{\Psi_{k,n}(\gamma,\lambda)(k-1)(1-\alpha)}\{1 + \Psi_{2,n}(\gamma,\lambda)|a_2| \\ & +\Psi_{3,n}(\gamma,\lambda)|a_3| \\ & + \cdots + \Psi_{k-1,n}(\gamma,\lambda)|a_{n-1}|\}.\end{aligned} \tag{2.3}$$

For $k = 2$,

$$|a_2| \leq \frac{2|b|(1-\beta)}{\Psi_{2,n}(\gamma,\lambda)(1-\alpha)},$$

which proves (2.1).

For $k=3$,

$$|a_3| \leq \frac{2\,|b|\,(1-\beta)}{\Psi_{3,n}\,(\gamma,\lambda)\,2\,(1-\alpha)}\left(1+\frac{2\,|b|\,(1-\beta)}{1-\alpha}\right).$$

Therefore (2.2) holds for $k=3$.

Assume that (2.3) is true for $k=m$. Then we obtain

$$\begin{aligned}|a_{m+1}| &\leq \frac{2\,|b|\,(1-\beta)}{\Psi_{m+1,n}\,(\gamma,\lambda)\,m\,(1-\alpha)}\left\{1+\frac{2\,|b|\,(1-\beta)}{1-\alpha}\right.\\ &\quad+\frac{2\,|b|\,(1-\beta)}{2\,(1-\alpha)}\left(1+\frac{2\,|b|\,(1-\beta)}{1-\alpha}\right)\\ &\quad\left.+\cdots+\frac{2\,|b|\,(1-\beta)}{(m-1)\,(1-\alpha)}\prod_{j=1}^{m-2}\left(1+\frac{2\,|b|\,(1-\beta)}{j\,(1-\alpha)}\right)\right\}\\ &= \frac{2\,|b|\,(1-\beta)}{\Psi_{m+1,n}\,(\gamma,\lambda)\,m\,(1-\alpha)}\prod_{j=1}^{m-1}\left(1+\frac{2\,|b|\,(1-\beta)}{j\,(1-\alpha)}\right).\end{aligned}$$

So (2.2) is true for $k=m+1$.

Consequently, using mathematical induction, we have proved that (2.2) holds true for all $k\geq 3$. □

Corollary 2.4. *Setting $\alpha=0$ in Theorem 2.3, we have*

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}\,(j+2\,|b|\,(1-\beta))}{\Psi_{k,n}\,(\gamma,\lambda)\,(k-1)!} \qquad (k\geq 2).$$

Corollary 2.5. *If we set $n=0$, $|b|=1$ or $n=1$, $\gamma=\lambda=0$, $|b|=1$ in Corollary 2.4, then we have*

$$|a_k| \leq \frac{\prod_{j=2}^{k}\,(j-2\beta)}{(k-1)!} \qquad (k\geq 2)$$

given by Robertson [11].

Theorem 2.6. *If $f\in\mathcal{KD}^{n}_{\gamma,\lambda}(\alpha,\beta,b)$ with $0\leq\alpha\leq\beta$, then*

$$|a_2| \leq \frac{|b|\,(1-\beta)}{\Psi_{2,n}\,(\gamma,\lambda)\,(1-\alpha)}$$

and

$$|a_k| \leq \frac{2\,|b|\,(1-\beta)}{\Psi_{k,n}\,(\gamma,\lambda)\,k\,(k-1)\,(1-\alpha)}\prod_{j=1}^{k-2}\left(1+\frac{2\,|b|\,(1-\beta)}{j\,(1-\alpha)}\right) \qquad (k\geq 3).$$

Corollary 2.7. *Setting* $\alpha = 0$ *in Theorem 2.6, we have*

$$|a_k| \leq \frac{\prod_{j=0}^{k-2} \left(j + 2\left|b\right|\left(1-\beta\right)\right)}{\Psi_{k,n}\left(\gamma,\lambda\right) k!} \qquad (k \geq 2).$$

Corollary 2.8. *If we set* $n = 0$, $|b| = 1$ *or* $n = 1$, $\gamma = \lambda = 0$, $|b| = 1$ *in Corollary 2.7, then we have*

$$|a_k| \leq \frac{\prod_{j=2}^{k} \left(j - 2\beta\right)}{k!} \qquad (k \geq 2)$$

given by Robertson [11].

Theorem 2.9. *If* $f \in \mathcal{SD}^n_{\gamma,\lambda}(\alpha,\beta,b)$ *with* $0 \leq \alpha \leq \beta$, *then*

$$\begin{aligned} &\max\left\{0,\ |z| - \frac{2\left|b\right|\left(1-\beta\right)}{\Psi_{2,n}\left(\gamma,\lambda\right)\left(1-\alpha\right)}\left|z\right|^2 \right.\\ &\left. -\sum_{k=3}^{\infty} \frac{2\left|b\right|\left(1-\beta\right)}{\Psi_{k,n}\left(\gamma,\lambda\right)\left(k-1\right)\left(1-\alpha\right)} \left(\prod_{j=1}^{k-2}\left(1+\frac{2\left|b\right|\left(1-\beta\right)}{j\left(1-\alpha\right)}\right)\right)\left|z\right|^k \right\} \\ \leq\ & |f(z)| \leq |z| + \frac{2\left|b\right|\left(1-\beta\right)}{\Psi_{2,n}\left(\gamma,\lambda\right)\left(1-\alpha\right)}\left|z\right|^2 \\ &+\sum_{k=3}^{\infty} \frac{2\left|b\right|\left(1-\beta\right)}{\Psi_{k,n}\left(\gamma,\lambda\right)\left(k-1\right)\left(1-\alpha\right)} \left(\prod_{j=1}^{k-2}\left(1+\frac{2\left|b\right|\left(1-\beta\right)}{j\left(1-\alpha\right)}\right)\right)\left|z\right|^k \end{aligned}$$

and

$$\begin{aligned} &\max\left\{0,\ 1 - \frac{4\left|b\right|\left(1-\beta\right)}{\Psi_{2,n}\left(\gamma,\lambda\right)\left(1-\alpha\right)}\left|z\right| \right.\\ &\left. -\sum_{k=3}^{\infty} \frac{2k\left|b\right|\left(1-\beta\right)}{\Psi_{k,n}\left(\gamma,\lambda\right)\left(k-1\right)\left(1-\alpha\right)} \left(\prod_{j=1}^{k-2}\left(1+\frac{2\left|b\right|\left(1-\beta\right)}{j\left(1-\alpha\right)}\right)\right)\left|z\right|^{k-1} \right\} \\ \leq\ & |f'(z)| \leq 1 + \frac{4\left|b\right|\left(1-\beta\right)}{\Psi_{2,n}\left(\gamma,\lambda\right)\left(1-\alpha\right)}\left|z\right| \\ &+\sum_{k=3}^{\infty} \frac{2k\left|b\right|\left(1-\beta\right)}{\Psi_{k,n}\left(\gamma,\lambda\right)\left(k-1\right)\left(1-\alpha\right)} \left(\prod_{j=1}^{k-2}\left(1+\frac{2\left|b\right|\left(1-\beta\right)}{j\left(1-\alpha\right)}\right)\right)\left|z\right|^{k-1}. \end{aligned}$$

Theorem 2.10. *If* $f \in \mathcal{KD}^n_{\gamma,\lambda}(\alpha,\beta,b)$ *with* $0 \le \alpha \le \beta$*, then*

$$\begin{aligned}
&\max\Bigg\{0,\ |z| - \frac{|b|\,(1-\beta)}{\Psi_{2,n}(\gamma,\lambda)\,(1-\alpha)}\,|z|^2 \\
&\quad -\sum_{k=3}^{\infty} \frac{2\,|b|\,(1-\beta)}{\Psi_{k,n}(\gamma,\lambda)\,k\,(k-1)\,(1-\alpha)} \left(\prod_{j=1}^{k-2}\left(1+\frac{2\,|b|\,(1-\beta)}{j\,(1-\alpha)}\right)\right)|z|^k\Bigg\} \\
\le\ & |f(z)| \le |z| + \frac{|b|\,(1-\beta)}{\Psi_{2,n}(\gamma,\lambda)\,(1-\alpha)}\,|z|^2 \\
&\quad +\sum_{k=3}^{\infty} \frac{2\,|b|\,(1-\beta)}{\Psi_{k,n}(\gamma,\lambda)\,k\,(k-1)\,(1-\alpha)} \left(\prod_{j=1}^{k-2}\left(1+\frac{2\,|b|\,(1-\beta)}{j\,(1-\alpha)}\right)\right)|z|^k
\end{aligned}$$

and

$$\begin{aligned}
&\max\Bigg\{0,\ 1 - \frac{2\,|b|\,(1-\beta)}{\Psi_{2,n}(\gamma,\lambda)\,(1-\alpha)}\,|z| \\
&\quad -\sum_{k=3}^{\infty} \frac{2\,|b|\,(1-\beta)}{\Psi_{k,n}(\gamma,\lambda)\,(k-1)\,(1-\alpha)} \left(\prod_{j=1}^{k-2}\left(1+\frac{2\,|b|\,(1-\beta)}{j\,(1-\alpha)}\right)\right)|z|^{k-1}\Bigg\} \\
\le\ & |f'(z)| \le 1 + \frac{2\,|b|\,(1-\beta)}{\Psi_{2,n}(\gamma,\lambda)\,(1-\alpha)}\,|z| \\
&\quad +\sum_{k=3}^{\infty} \frac{2\,|b|\,(1-\beta)}{\Psi_{k,n}(\gamma,\lambda)\,(k-1)\,(1-\alpha)} \left(\prod_{j=1}^{k-2}\left(1+\frac{2\,|b|\,(1-\beta)}{j\,(1-\alpha)}\right)\right)|z|^{k-1}.
\end{aligned}$$

References

[1] M. Acu and S. Owa, *Convex functions associated with some hyperbola*, J. Approx. Theory Appl., **1**(2005), 37-40.

[2] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Math. Sci. 2004, no. 25-28, 1429-1436.

[3] F. M. Al-Oboudi and K. A. Al-Amoudi, *On classes of analytic functions related to conic domains*, J. Math. Anal. Appl., **339**(2008), 655-667.

[4] R. Bharati, R. Parvatham and A. Swaminathan, *On subclasses of uniformly convex functions and corresponding class of starlike functions*, Tamkang J. Math., **28**(1997), 17-32.

[5] C. Carathéodory, *Über den variabilitätsbereich der Fourier'schen konstanten von possitiven harmonischen funktionen*, Rend. Circ. Palermo, **32**(1911), 193-217.

[6] B. A. Frasin, *Family of analytic functions of complex order*, Acta Math. Acad. Paedagog. Nyházi. (N.S.), **22**(2006), 179-191.

[7] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math., **56**(1991), 87-92.

[8] S. Owa, *On the distortion theorems. I*, Kyungpook Math. J. **18**(1978), 53-59.

[9] S. Owa, Y. Polatoğlu and E. Yavuz, *Coefficient inequalities for classes of uniformly starlike and convex functions*, J. Inequal. Pure Appl. Math., **7**(2006), Article 160, 5 pp.

[10] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math., **39**(1987), 1057-1077.

[11] M. S. Robertson, *On the theory of univalent functions*, Ann. of Math., **37**(2)(1936), 374-408.

[12] F. Rønning, *On starlike functions associated with parabolic regions*, Ann. Univ. Mariae Curie-Sklodowska Sect. A, **45**(1991), 117-122.

[13] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. ,**118**(1993), 189-196.

[14] G. Ş. Sălăgean, Subclasses of univalent functions, Complex Analysis-Fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin, 1983, pp. 362-372.

[15] S. Shams, S. R. Kulkarni and J. M. Jahangiri, *Classes of uniformly starlike and convex functions*, Int. J. Math. Math. Sci., **55**(2004), 2959-2961.

[16] H. M. Srivastava, A. K. Mishra and M. K. Das, *A nested class of analytic functions defined by fractional calculus*, Commun. Appl. Anal., **2**(1998), 321-332.

[17] H. M. Srivastava and A. K. Mishra, *Applications of fractional calculus to parabolic starlike and uniformly convex functions*, Comput. Math. Appl., **39**(2000), 57-69.

[18] H. M. Srivastava and S. Owa, (Eds.), Univalent Functions, Fractional Calculus, and Their Applications, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, UK; JohnWiley & Sons, New York, NY, USA, 1989.