International Journal of Fuzzy Logic and Intelligent Systems, vol. 11, no. 3, September 2011, pp. 149−152
http://dx.doi.org/10.5391/IJFIS.2011.11.3.149

Fixed Point Theorem for Common Property(E.A.) and Weak Compatible Functions in Intuitionistic Fuzzy Metric Space

Jong Seo Park*

***Department of Mathematic Education, Chinju National University of Education, Jinju 660-756, South Korea**

Abstract

In this paper, we obtain common fixed point theorem for common property(E.A.) and weakly compatible functions in intuitionistic fuzzy metric space.

Key words : Implicit function, Weakly compatible function, Property(E.A.).

1. Introduction

In 1997, George and Veeramani[2] modified the concept of fuzzy metric space introduced by Kramosil and Michalek[6] with a view to obtain a Hausdorff topology on fuzzy metric spaces, and this has recently found very fruitful applications in quantum particle physics. In recent years, many authors have proved fixed point theorems in fuzzy metric spaces, and we observed some common fixed point theorems in fuzzy metric space which improved many known results([4], [12] etc). Sessa[13] introduced the notion of weakly commuting mappings which was further enlarged by Jungck[5] by defining compatible mapping in fuzzy metric space. Park[8] studied some properties for compatible map in intuitionistic fuzzy metric space. Also, Park[9] defined the intuitionistic fuzzy contraction, and obtain some fixed point theorem using common property(E.A.) and weakly compatibility in intuitionistic fuzzy metric space.

In this paper, we obtain common fixed point theorem for common property(E.A.) and weakly compatible functions in intuitionistic fuzzy metric space.

2. Preliminaries

In this part, we recall some definitions, properties and known results in the intuitionistic fuzzy metric space as following :

Let us recall(see [11]) that a continuous $t-$norm is a operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ which satisfies the following conditions: (a)$*$ is commutative and associative, (b)$*$ is continuous, (c)$a * 1 = a$ for all $a \in [0,1]$, (d)$a * b \le c * d$ whenever $a \le c$ and $b \le d$ $(a, b, c, d \in [0,1])$. Also, a continuous $t-$conorm is a operation $\diamond : [0,1] \times [0,1] \rightarrow [0,1]$ which satisfies the following conditions: (a)$\diamond$ is commutative and associative, (b)$\diamond$ is continuous, (c)$a \diamond 0 = a$ for all $a \in [0,1]$, (d)$a \diamond b \ge c \diamond d$ whenever $a \le c$ and $b \le d$ $(a, b, c, d \in [0,1])$.

Definition 2.1. ([7])The $5-$tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t-$norm, $\diamond$ is a continuous $t-$conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions; for all $x, y, z \in X$, such that

(a)$M(x, y, t) > 0$,
(b)$M(x, y, t) = 1$ if and only if $x = y$,
(c)$M(x, y, t) = M(y, x, t)$,
(d)$M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
(e)$M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous,
(f)$N(x, y, t) > 0$,
(g)$N(x, y, t) = 0$ if and only if $x = y$,
(h)$N(x, y, t) = N(y, x, t)$,
(i)$N(x, y, t) \diamond N(y, z, s) \ge N(x, z, t + s)$,
(j)$N(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous.

Note that (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Definition 2.2. ([10]) Let X be an intuitionistic fuzzy metric space.

Manuscript received Jun. 27, 2011; revised Sep. 6, 2011; accepted Sep. 6, 2011.
*This paper is supported by the Chinju National University of Education Research Fund in 2011.
2000 *Mathematics Subject Classification.* 46S40, 47H10, 54H25

(a) $\{x_n\}$ is said to be convergent to a point $x \in X$ by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$, $\lim_{n\to\infty} N(x_n, x, t) = 0$ for all $t > 0$.

(b) $\{x_n\}$ is called a Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1, \quad \lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$$

for all $t > 0$ and $p > 0$.

(c) X is complete if every Cauchy sequence converges in X.

Definition 2.3. ([8]) A pair of self mappings (f, g) defined on an intuitionistic fuzzy metric space X is said to be compatible(or asymptotically commuting) if for all $t > 0$,

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1, \quad \lim_{n\to\infty} N(fgx_n, gfx_n, t) = 0$$

where $\{x_n\} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$. Also, the pair (f, g) is called non-compatible, if there exists a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, but either $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$, $\lim_{n\to\infty} N(fgx_n, gfx_n, t) \neq 0$ or the limit does not exists.

Definition 2.4. [9] A pair of self mappings (f, g) defined on an intuitionistic fuzzy metric space X is said to satisfy the property(E.A.) if there exists a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Definition 2.5. [9] Two pairs of self mappings (A, S) and (B, T) defined on an intuitionistic fuzzy metric space X are said to share common property(E.A.) if there exist sequences $\{x_n\}, \{y_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ for some $z \in X$.

Definition 2.6. [8] Two self mappings f and g on an intuitionistic fuzzy metric space X are called weakly compatible if they commute at their point of coincidence. That is, $fx = gx$ implies $fgx = gfx$.

Implicit relations on fuzzy metric spaces have been used in many articles([1], [3] etc). Let $\Psi = \{\phi, \psi\}$ be implicit functions set, $I = [0, 1]$, $\phi, \psi : I^4 \to R$ be continuous functions following conditions :

(I)ϕ is decreasing and ψ is increasing in four variables.

(II)If $\phi(u, v, u, v) \geq 0$ or $\phi(u, v, v, u) \geq 0$ for $u > 0$ and $v \geq 0$, then $u > v$,

If $\psi(u, v, u, v) \leq 1$ or $\psi(u, v, v, u) \leq 1$ for $u > 0$ and $v \geq 0$, then $u < v$.

3. Main Result

Lemma 3.1. Let A, B, S and T be self mappings of an intuitionistic fuzzy metric space X. Assume that, for all distinct $x, y \in X$ and $t > 0$, there exists $\phi, \psi \in \Psi$ with

$$\begin{aligned} &\phi(M(Ax, By, t), M(Sx, Ty, t), \\ &\qquad M(Sx, Ax, t), M(By, Ty, t)) \geq 0, \quad (1) \\ &\psi(N(Ax, By, t), N(Sx, Ty, t), \\ &\qquad N(Sx, Ax, t), N(By, Ty, t)) \leq 1. \end{aligned}$$

Also, suppose that the pairs (A, S) and(or) (B, T) satisfies the property(E.A.), and $A(X) \subset T(X)$ and(or) $B(X) \subset S(X)$. Then the pair (A, S) and (B, T) share the common property(E.A.).

Proof. Since (A, S) satisfies the property(E.A.), there exists a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$. Also, since $A(X) \subset T(X)$, there exists $y_n \in X$ such that $Ax_n = Ty_n$ for each x_n. Hence $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ty_n = z$. Therefore, we have $\lim_{n\to\infty} Ax_n = z$, $\lim_{n\to\infty} Sx_n = z$ and $\lim_{n\to\infty} Ty_n = z$. Now, let $\lim_{n\to\infty} By_n = z$. We note that $\lim_{n\to\infty} By_n = z$ if and only if $\lim_{n\to\infty} M(Ax_n, By_n, t) = 1$.

Assume that $\lim_{n\to\infty} By_n \neq z$, then there exists a subsequence $\{By_{n_k}\} \subset \{By_n\}$ such that $\lim_{n\to\infty} M(Ax_{n_k}, By_{n_k}, t) = u < 1$. By (1), we have

$$\begin{aligned} &\phi(M(Ax_{n_k}, By_{n_k}, t), M(Sx_{n_k}, Ty_{n_k}, t), \\ &\qquad M(Sx_{n_k}, Ax_{n_k}, t), M(By_{n_k}, Ty_{n_k}, t)) \geq 0, \\ &\psi(N(Ax_{n_k}, By_{n_k}, t), N(Sx_{n_k}, Ty_{n_k}, t), \\ &\qquad N(Sx_{n_k}, Ax_{n_k}, t), N(By_{n_k}, Ty_{n_k}, t)) \leq 1. \end{aligned}$$

Since as $n \to \infty$, $\phi(u, 1, 1, u) \geq 0$, $\psi(u, 0, 0, u) \leq 1$. Hence $u > 1$ and $u < 0$ from implicit function. This is a contradiction. Therefore $\lim_{n\to\infty} By_n = z$ which shows that (A, S) and (B, T) share the common property(E.A.). □

Theorem 3.2. Let A, B, S and T be self mappings of an intuitionistic fuzzy metric space X. Assume that, for all distinct $x, y \in X$ and $t > 0$, there exists $\phi, \psi \in \Psi$ with

$$\begin{aligned} &\phi(M(Ax, By, t), M(Sx, Ty, t), \\ &\qquad M(Sx, Ax, t), M(By, Ty, t)) \geq 0, \quad (2) \\ &\psi(N(Ax, By, t), N(Sx, Ty, t), \\ &\qquad N(Sx, Ax, t), N(By, Ty, t)) \leq 1. \end{aligned}$$

Also, suppose that (A, S) and (B, T) share the common property(E.A.) and $S(X), T(X)$ are closed subsets of X. Then the pair (A, S) as well as (B, T) have a point of coincidence each. Furthermore, if the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. Since (A, S) and (B, T) share the common property(E.A.), then there exist sequences $\{x_n\}, \{y_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ for some $z \in X$. Also, since $S(X)$ is a closed subset of X, $\lim_{n\to\infty} Sx_n = z \in S(X)$. Hence, there exist a point $u \in X$ such that $Su = z$. Suppose that $Au \neq Su$, then by (2), we have

$$\begin{aligned}&\phi(M(Au, By_n, t), M(Su, Ty_n, t),\\ &\qquad M(Su, Au, t), M(By_n, Ty_n, t)) \geq 0,\\ &\psi(N(Au, By_n, t), N(Su, Ty_n, t),\\ &\qquad N(Su, Au, t), N(By_n, Ty_n, t)) \leq 1.\end{aligned}$$

As $n \to \infty$, we have

$$\begin{aligned}&\phi(M(Au, Su, t), 1, M(Au, Su, t), 1) \geq 0,\\ &\psi(N(Au, Su, t), 0, N(Au, Su, t), 0) \leq 1.\end{aligned}$$

Hence $M(Au, Su, t) > 1$ and $N(Au, Su, t) < 0$. This is a contradiction, and hence $Au = Su$. Therefore u is a coincidence point of the pair (A, S).

Since $T(X)$ is a closed subset of X, $\lim_{n\to\infty} Ty_n = z \in T(X)$. Therefore, there exists a point $v \in X$ such that $Tv = z$. Now, we prove $Bv = Tv$. If $Bv \neq Tv$, then by (2), we have

$$\begin{aligned}&\phi(M(Ax_n, Bv, t), M(Sx_n, Tv, t),\\ &\qquad M(Sx_n, Ax_n, t), M(Bv, Tv, t)) \geq 0,\\ &\psi(N(Ax_n, Bv, t), N(Sx_n, Tv, t),\\ &\qquad N(Sx_n, Ax_n, t), N(Bv, Tv, t)) \leq 1.\end{aligned}$$

As $n \to \infty$, we have

$$\begin{aligned}&\phi(M(Tv, Bv, t), 1, 1, M(Bv, Tv, t)) \geq 0,\\ &\psi(N(Tv, Bv, t), 0, 0, N(Bv, Tv, t)) \leq 1.\end{aligned}$$

Hence $M(Tv, Bv, t) > 1$ and $N(Tv, Bv, t) < 0$. This is a contradiction, and hence $Tv = Bv$. Therefore v is a coincidence point of the pair (B, T). Since (A, S) is weakly compatible and $Au = Su$, we have $Az = ASu = SAu = Sz$.

Now, we prove that z is a common fixed point of (A, S). Suppose that $Az \neq z$, then by (2), we have

$$\begin{aligned}&\phi(M(Az, Bv, t), M(Sz, Tv, t),\\ &\qquad M(Sz, Az, t), M(Bv, Tv, t)) \geq 0,\\ &\psi(N(Az, Bv, t), N(Sz, Tv, t),\\ &\qquad N(Sz, Az, t), N(Bv, Tv, t)) \leq 1.\end{aligned}$$

That is,

$$\begin{aligned}&\phi(M(Az, z, t), M(Az, z, t), 1, 1) \geq 0,\\ &\psi(N(Az, z, t), N(Az, z, t), 0, 0) \leq 1.\end{aligned}$$

Hence $M(Az, z, t) > 1$ and $N(Az, z, t) < 0$. This is a contradiction, and hence $Az = z$. By weakly compatibility of (B, T) and (2), we have $Bz = z = Tz$. Hence z is a common fixed point of (A, S) and (B, T). Uniqueness of fixed point is an easy consequence of (2). This completes the proof. □

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Jong Seo Park
Professor of Chinju National University of Education
Research Area: Fuzzy mathematics, Fuzzy fixed point theory, Fuzzy differential equation
E-mail : parkjs@cue.ac.kr