International Journal of Fuzzy Logic and Intelligent Systems, vol. 11, no. 3, September 2011, pp. 143−148
http://dx.doi.org/10.5391/IJFIS.2011.11.3.143

Backstepping Control and Synchronization for 4-D Lorenz-Stenflo Chaotic System with Single Input

Sung-Hun Yu*, Chang-Ho Hyun+ and Mignon Park*+**

*** School of Electrical and Electronic Engineering, Yonsei University**
**** Division of Electrical Electronic and Control Engineering, Kongju National University**

Abstract

In this paper, a backstepping design is proposed to achieve stabilization and synchronization for the Lorenz-Stenflo (LS) chaotic system. The proposed method is a recursive Lyapunov-based scheme and provides a systematic procedure to design stabilizing controllers. The proposed controller enables stabilization of the chaotic motion and synchronization of two identical LS chaotic systems using only a single control input. Numerical simulations are presented to validate the proposed method.

Keywords: Chaos Control, Chaos synchronization, Backstepping design, Lorenz-Stenflo chaotic system

1. Introduction

Since the pioneering work by Ott et al. [1] and Pecora and Carroll [2], chaos control and synchronization have been investigated extensively. Because of the theoretical and practical applications of chaos in mathematics, physics, secure communications, life sciences, heart beat regulations, human brain dynamics and biomedical communities, etc., how to achieve chaos control and synchronization of complex chaotic systems is an interesting and challenging issue.

Until now, various methods [3–13] have been proposed to achieve control and synchronization of chaotic systems. Among the aforementioned control methods, the backstepping design technique [12,13] has been widely recognized as the powerful method for chaos control. The backstepping method is a recursive Lyapunov-based scheme. The technique guarantees global stabilities and tracking properties for the class of strict feedback systems [14]. This approach provides a systematic process to derive a control input, following step by step algorithms.

The backstepping design for chaos control can be used for only a few chaotic systems which have state equation as strict feedback form. Also, the backstepping design for chaos synchronization can be used for only a few chaotic systems which have synchronization error dynamics as the strict feedback form. Since the state equation and synchronization error dynamics of most 4-D chaotic systems can't be derived as the strict feedback form, it is hard to apply the ordinary backstepping design for 4-D chaos control and synchronization. However, in case of a 4-D Lorenz–Stenflo (LS) chaotic system [15], the problem is solved by rearranging the original state equation.

The LS chaotic system is one of the 4-D chaotic systems. The original state equation of the LS chaotic system isn't strict feedback form. Obviously, the backstepping design can't be used directly to control and synchronize the LS chaotic system. However, the original state equation the LS chaotic system can be rearranged as strict feedback form. Therefore, the backstepping design can be applied to control and synchronize the LS chaotic systems by using the rearranged state equation.

In this paper, the backstepping control and synchronization method are proposed to stabilize the chaotic motion to the origin and synchronize the two identical LS systems. The problem of chaos control and synchronization is solved by applying only a single control input.

The rest of the paper is organized as follow: Section 2 introduces a brief description of the LS chaotic system, Section 3 presents chaos control scheme for the LS chaotic system using the backstepping method, Section 4 presents chaos synchronization scheme between two identical LS chaotic systems using the backstepping method and Section 5 provides conclusions.

2. System Description

The LS system was formulated by Stenflo from a low-frequency short-wavelength gravity wave equation [15]. It comprises the following differential equation system.

$$\begin{aligned} \dot{x} &= ay - ax + dw \\ \dot{y} &= cx - xz - y \\ \dot{z} &= xy - bz \\ \dot{w} &= -x - aw \end{aligned} \tag{1}$$

where $a(>0)$, $c(>0)$, $d(>0)$ and $b(>0)$ are respectively the Rayleigh number, Prandtl number, rotation

Manuscript received Jul. 20, 2011; revised Aug. 23, 2011; accepted Sep. 8, 2011.
+Corresponding author
This research project was supported by the Sports Promotion Fund of Seoul Olympic Sports Promotion Foundation from Ministry of Culture, Sports and Tourism.

number and geometric parameter. With $a = 1.0$, $b = 0.7$, $c = 26.0$ and $d = 1.5$, the LS system exhibits the chaotic attractor shown in Fig. 1.

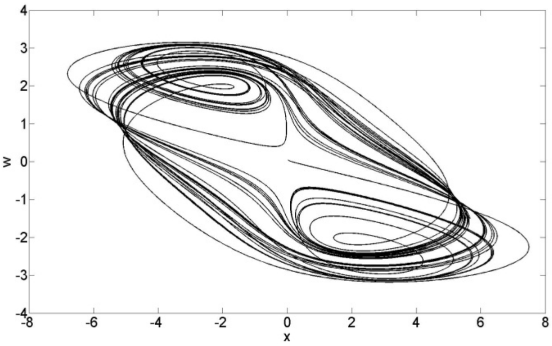


Fig. 1. $x{-}w$ phase portrait of the LS chaotic system

Obviously, the original LS chaotic system is not a strict feedback form and the ordinary backstepping technique can't be used for chaos control and synchronization. In order to apply the backstepping method, we rearrange the original LS chaotic system (1).

Let $x_1 = w$, $x_2 = x$, $x_3 = y$ and $x_4 = z$, then the system (1) is transformed into the following form

$$\begin{aligned} \dot{x}_1 &= -x_2 - ax_1 \\ \dot{x}_2 &= ax_3 - ax_2 + dx_1 \\ \dot{x}_3 &= cx_2 - x_2x_4 - x_3 \\ \dot{x}_4 &= x_2x_3 - bx_4. \end{aligned} \tag{2}$$

We notice that the rearranged LS system (2) is strict feedback form. Since the system (2) is strict feedback form, the backstepping method can be applied. The following chaos control and synchronization design procedure is derived by using the system (2).

3. Control of the LS chaotic System

3.1. Backstepping Control

In this section, we design the controller based on the backstepping method to stabilize the LS chaotic system. The problem of control is addressed by applying only a single control input.

In order to stabilize the LS chaotic system to the origin, we add a control input u_1 to the third equation of the system (2). Then the controlled LS chaotic system become as follows

$$\begin{aligned} \dot{x}_1 &= -x_2 - ax_1 \\ \dot{x}_2 &= ax_3 - ax_2 + dx_1 \\ \dot{x}_3 &= cx_2 - x_2x_4 - x_3 + u_1 \\ \dot{x}_4 &= x_2x_3 - bx_4. \end{aligned} \tag{3}$$

Our object is to find u_1 that makes the states x_1, x_2, x_3 and x_4 stable at the origin as time t goes to infinity.

The backstepping design procedure is recursive. At the i-th step, the i-th order subsystem is stabilized with respect to a Lyapunov function V_i by the design of a virtual input α_i. At the final n-th step, the entire system is stabilized with respect to a Lyapunov function V_n by an actual control input u.

Now we begin to design the control input u_1 based on the backstepping method.

Step 1. Let $z_1 = x_1$, then we can obtain its derivative as follows

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 \\ &= -x_2 - az_1, \end{aligned} \tag{4}$$

where $x_2 = \alpha_1(z_1)$ is regarded as the virtual control input.

For the design of α_1 to stabilize the z_1-subsystem (4), we choose the Lyapunov function V_1 as

$$V_1 = \frac{z_1^2}{2}. \tag{5}$$

Then, the derivative of V_1 is as following

$$\begin{aligned} \dot{V}_1 &= z_1\dot{z}_1 \\ &= -az_1^2 - z_1\alpha_1. \end{aligned} \tag{6}$$

If we choose $\alpha_1 = 0$, then $\dot{V}_1 = -az_1^2 < 0$ makes the z_1-subsystem (4) asymptotically stable.

Since the virtual controller α_1 is estimative, define the error between e_2 and α_1 as

$$z_2 = e_2 - \alpha_1 \tag{7}$$

Then, we can obtain the following (z_1, z_2)-subsystem

$$\begin{aligned} \dot{z}_1 &= -az_1 - z_2 \\ \dot{z}_2 &= dz_1 - az_2 + ax_3, \end{aligned} \tag{8}$$

where $e_3 = \alpha_2(z_1, z_2)$ is regarded as the virtual input.

Step 2. In this step, we will stabilize the (z_1, z_2)-subsystem (8).

We can choose the Lyapunov function V_2 as follows

$$V_2 = V_1 + \frac{z_2^2}{2}. \tag{9}$$

Its derivative is given by

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 \\ &= -az_1^2 - z_1z_2 + z_2(dz_1 - az_2 + a\alpha_2) \\ &= -az_1^2 - az_2^2 + z_2\{(d-1)z_1 + a\alpha_2\}. \end{aligned} \tag{10}$$

If we choose $\alpha_2 = \frac{1-d}{a}z_1$, then $\dot{V}_2 = -az_1^2 - az_2^2 < 0$ makes the (z_1, z_2)-subsystem (8) asymptotically stable.

Similarly, define the error variable between e_3 and α_2

$$z_3 = e_3 - \alpha_2. \tag{11}$$

Then, we can derive the following (z_1, z_2, z_3) subsystem

$$\begin{aligned}\dot{z}_1 &= -az_1 - z_2 \\ \dot{z}_2 &= z_1 - az_2 + az_3 \\ \dot{z}_3 &= \frac{(a-1)(1-d)}{a} z_1 + (c - \frac{d-1}{a} - x_4) z_2 - z_3 + u_1.\end{aligned} \tag{12}$$

Step 3. In order to stabilize the (z_1, z_2, z_3) -subsystem (12), we can choose the Lyapunov function V_3 as follows

$$V_3 = V_2 + \frac{z_3^2}{2}. \tag{13}$$

Its derivative is given by

$$\begin{aligned}\dot{V}_3 &= \dot{V}_2 + z_3 \dot{z}_3 \\ &= -az_1^2 - az_2^2 + az_2 z_3 + z_3 \{\frac{(a-1)(1-d)}{a} z_1 \\ &\quad + (c - \frac{d-1}{a} - x_4) z_2 - z_3 + u_1\} \\ &= -az_1^2 - az_2^2 - z_3^2 + z_3 \{\frac{(a-1)(1-d)}{a} z_1 \\ &\quad + (a + c - \frac{d-1}{a} - x_4) z_2 + u_1\}.\end{aligned} \tag{14}$$

It is clear that $\dot{V}_3 = -az_1^2 - az_2^2 - z_3^2 < 0$ and the (z_1, z_2, z_3) -subsystem (12) is asymptotically stable by choosing the control input as follows

$$u_1 = \frac{(a-1)(d-1)}{a} z_1 - (a + c - \frac{d-1}{a} - x_4) z_2. \tag{15}$$

Since $\dot{V}_3$ is negative definite, it follows that the equilibrium (0, 0, 0) of the subsystem (12) is globally asymptotically stable. In view of $z_1 = x_1$, $z_2 = x_2$ and $z_3 = x_3 - \frac{1-d}{a} z_1$, this implies that x_1, x_2 and x_3 go to zero asymptotically.

According to $x_1 \to 0$, $x_2 \to 0$, $x_3 \to 0$ and the fourth equation in the rearranged LS chaotic system (2), we get that (x_1, x_2, x_3, x_4) in the system (2) tends to (0, 0, 0, 0) as $t \to \infty$. In other words, the controlled system (2) is asymptotically stable at the origin with the proposed control input (15).

3.2. Numerical Results

In this subsection, numerical simulation results are given to verify the effectiveness of the proposed control method.

We set $a = 1.0$, $b = 0.7$, $c = 26.0$ and $d = 1.5$, as in Fig. 1, to ensure chaotic behavior. The initial conditions for the system (2) are $x_1(0) = 1, x_2(0) = 2, x_3(0) = 3$ and $x_4(0) = 4$.

Fig. 2 shows the time response of states for the rearranged LS chaotic system (2) with the proposed control input. The control input is added to the system at $t = 20$. As expected, it shows that the controlled LS chaotic system is stabilized to the origin in 10 seconds.

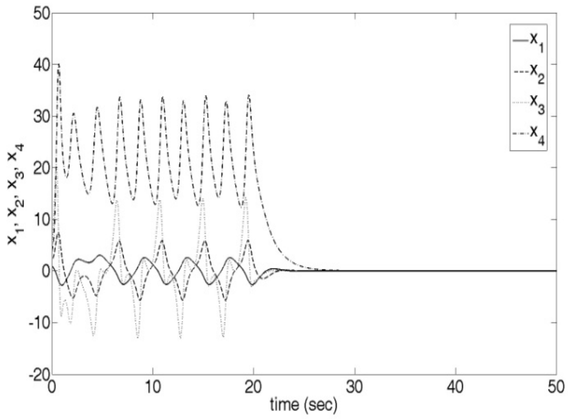


Fig.2. Time response of states for the LS chaotic system

4. Synchronization of the LS chaotic System

4.1. Backstepping Synchronization

In this section, we design the controller based on the backstepping method to synchronize the two identical LS chaotic systems.

In order to observe the synchronization behavior in the LS chaotic system, we assume that the system (2) is the drive system. Then, the response system is

$$\begin{aligned}\dot{y}_1 &= -y_2 - ay_1 \\ \dot{y}_2 &= ay_3 - ay_2 + dy_1 \\ \dot{y}_3 &= cy_2 - y_2 y_4 - y_3 + u_2 \\ \dot{y}_4 &= y_2 y_3 - by_4,\end{aligned} \tag{16}$$

where u_2 is the control input.

Here, we aim at determining the control input u_2 which is required for the response system (16) to synchronize with the drive system (2). For this purpose, let the error states between the state variables of the response system (16) and the drive system (2) be

$$e_1 = y_1 - x_1, e_2 = y_2 - x_2, e_3 = y_3 - x_3, e_4 = y_4 - x_4. \tag{17}$$

Subtracting (16) from (3), we obtain the error dynamics

$$\begin{aligned}\dot{e}_1 &= -e_2 - ae_1 \\ \dot{e}_2 &= ae_3 - ae_2 + de_1 \\ \dot{e}_3 &= ce_2 - e_2 e_4 - e_3 - x_4 e_2 - x_2 e_4 + u_2 \\ \dot{e}_4 &= e_2 e_3 - be_4 + x_3 e_2 + x_2 e_3.\end{aligned} \tag{18}$$

Now our object becomes to find u_2 that makes the error vector $e = [e_1, e_2, e_3, e_4]^T$ converge to zero as time t goes to infinity. This implies that the trajectory of the response system (3) asymptotically approaches the drive system (2).

Again, we begin to design the control input u_2 based on the backstepping method outlined in subsection 3.1.

Step 1. Let $z_1 = e_1$, then we can obtain its derivative

$$\begin{aligned} \dot{z}_1 &= \dot{e}_1 \\ &= -e_2 - az_1 \end{aligned} \tag{19}$$

where $e_2 = \alpha_1(z_1)$ is regarded as the virtual control input.

For the design of α_1 to stabilize the z_1-subsystem (19), we choose the Lyapunov function V_1 as

$$V_1 = \frac{z_1^2}{2}. \tag{20}$$

The derivative of V_1 is as following

$$\begin{aligned} \dot{V}_1 &= z_1 \dot{z}_1 \\ &= -az_1^2 - z_1\alpha_1. \end{aligned} \tag{21}$$

If we choose $\alpha_1 = 0$, then $\dot{V}_1 = -az_1^2 < 0$ makes the z_1-subsystem (19) asymptotically stable.

Since the virtual controller α_1 is estimative, define the error between e_2 and α_1 as

$$z_2 = e_2 - \alpha_1. \tag{22}$$

Then, we can obtain the following (z_1, z_2)-subsystem

$$\begin{aligned} \dot{z}_1 &= -az_1 - z_2 \\ \dot{z}_2 &= dz_1 - az_2 + ae_3, \end{aligned} \tag{23}$$

where $e_3 = \alpha_2(z_1, z_2)$ is regarded as the virtual input.

Step 2. In this step, we will stabilize the (z_1, z_2)-subsystem (23).

We can choose the Lyapunov function V_2 as follows

$$V_2 = V_1 + \frac{z_2^2}{2}. \tag{24}$$

Its derivative is given by

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 \\ &= -az_1^2 - z_1z_2 + z_2(dz_1 - az_2 + a\alpha_2) \\ &= -az_1^2 - az_2^2 + z_2\{(d-1)z_1 + a\alpha_2\}. \end{aligned} \tag{25}$$

If we choose $\alpha_2 = \frac{1-d}{a} z_1$, then $\dot{V}_2 = -az_1^2 - az_2^2 < 0$ makes the (z_1, z_2)-subsystem (23) asymptotically stable.

Similarly, define the error variable between e_3 and α_2

$$z_3 = e_3 - \alpha_2. \tag{26}$$

Then, we can derive the following (z_1, z_2, z_3)-subsystem

$$\begin{aligned} \dot{z}_1 &= -az_1 - z_2 \\ \dot{z}_2 &= z_1 - az_2 + az_3 \\ \dot{z}_3 &= \frac{(a-1)(1-d)}{a} z_1 + (c + \frac{1-d}{a} - x_4 - e_4) z_2 \\ &\quad - z_3 - x_2e_4 + u_2. \end{aligned} \tag{27}$$

Step 3. In order to stabilize the (z_1, z_2, z_3)-subsystem (27), we can choose the Lyapunov function V_3 as follows

$$V_3 = V_2 + \frac{z_3^2}{2}. \tag{28}$$

Its derivative is given by

$$\begin{aligned} \dot{V}_3 &= \dot{V}_2 + z_3\dot{z}_3 \\ &= -az_1^2 - az_2^2 + az_2z_3 + z_3\{\frac{(a-1)(1-d)}{a} z_1 \\ &\quad + (c + \frac{1-d}{a} - x_4 - e_4)z_2 - z_3 - x_2e_4 + u_2\} \\ &= -az_1^2 - az_2^2 - z_3^2 + z_3\{\frac{(a-1)(1-d)}{a} z_1 \\ &\quad + (a + c + \frac{1-d}{a} - x_4 - e_4)z_2 - x_2e_4 + u_2\}. \end{aligned} \tag{29}$$

It is clear that $\dot{V}_3 = -az_1^2 - az_2^2 - z_3^2 < 0$ and the (z_1, z_2, z_3)-subsystem (29) is asymptotically stable by choosing the control input as follows

$$\begin{aligned} u_2 &= \frac{(a-1)(d-1)}{a} z_1 \\ &\quad - (a + c - \frac{d-1}{a} - x_4 - e_4)z_2 + x_2e_4. \end{aligned} \tag{30}$$

Since $\dot{V}_3$ is negative definite, it follows that the equilibrium (0, 0, 0) of the subsystem (27) is globally asymptotically stable. In view of $z_1 = e_1$, $z_2 = e_2$ and $z_3 = e_3 - \frac{1-d}{a} z_1$, this implies that e_1, e_2 and e_3 go to zero asymptotically.

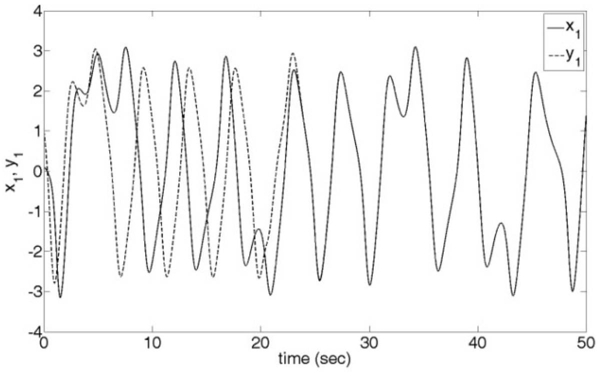
According to $e_1 \to 0$, $e_2 \to 0$, $e_3 \to 0$ and the fourth equation in the synchronization error dynamics (18), we get we get that (e_1, e_2, e_3, e_4) of the controlled system (18) go to (0, 0, 0, 0) as $t \to \infty$. In other words, the trajectory of the controlled response system (16) asymptotically approaches the trajectory of the drive system (2) with the proposed control input (30).

4.2. Numerical Results

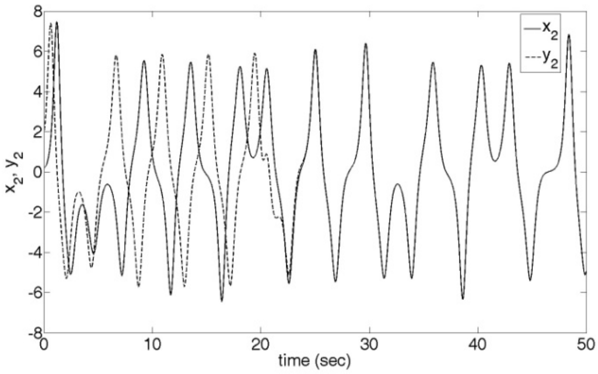
Similarly, we set $a = 1.0$, $b = 0.7$, $c = 26.0$ and $d = 1.5$, as in Fig. 1, to ensure chaotic behavior. The initial conditions for the drive system (2) are $x_1(0) = 0.1$, $x_2(0) = 0.2$, $x_3(0) = 0.3$, $x_4(0) = 0.4$ and the response system (16) are $y_1(0) = 1$, $y_2(0) = 2$, $y_3(0) = 3$, $y_4(0) = 4$. The control input is added to the response system at $t = 20$.

Fig. 3 (a)-(d) show the trajectories of states for the drive system and the response system with the proposed control input. The trajectories of synchronization errors for the drive system and the response system are shown in Fig. 4.

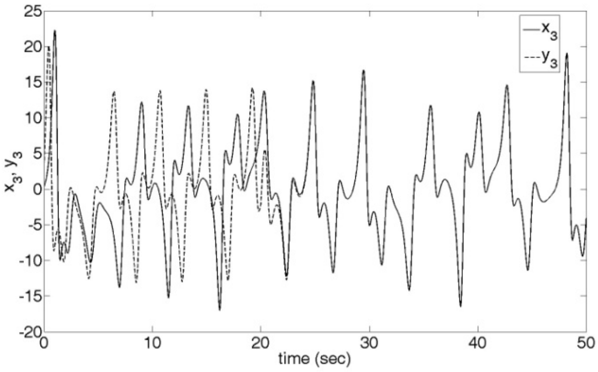
As expected, it shows that all the state variables are synchronized and the synchronization errors converge to zero in 10 seconds with the proposed control input (30).



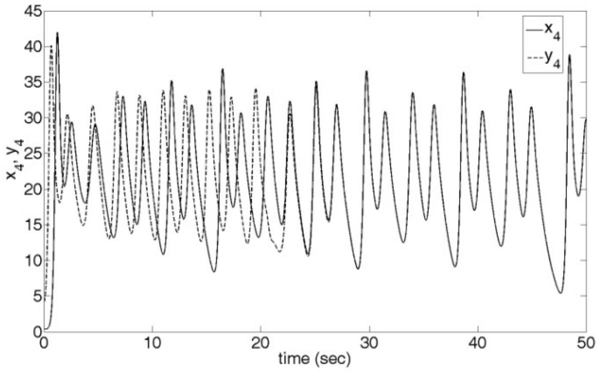
(a) Time response of x_1 and y_1



(b) Time response of x_2 and y_2



(c) Time response of x_3 and y_3



(d) Time response of x_4 and y_4

Fig. 3. Time response of states for the drive system (2) and the response system (3)

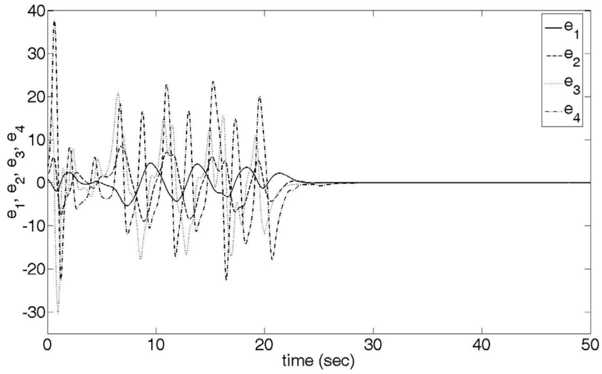


Fig. 3. Time response of synchronization error states

5. Conclusions

This paper has examined the synchronization of the LS chaotic system using the backstepping method. The proposed scheme is a recursive Lyapunov-based scheme and provides a systematic procedure to design controller. The proposed approach enables stabilization of chaotic motion and synchronization of the two identical LS chaotic systems using only a single control input. Numerical simulations were also carried out to illustrate the effectiveness of the proposed approach

References

[1] E. Ott, C. Grebogi and J.A. Yorke, "Controlling chaos," *Phys. Rev. Lett.*, vol.64, pp.1196–1199, 1990.

[2] L.M. Pecora and T.L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.*, vol.64, pp.821-824, 1990.

[3] T.L. Liao and S.H. Tsai, "Adaptive synchronization of chaotic systems and its application to secure communications," *Chaos, Solitons & Fractals*, vol.11, pp.1387-1396, 2000.

[4] H.N. Agiza and M.T. Yassen, "Synchronization of Rossler and Chen dynamical systems using active control," *Phys. Lett. A*, vol.278, pp.191-197, 2001.

[5] M. Jang, C. Chen and C. Chen, "Sliding mode control of hyperchaos in Rossler systems," *Chaos Solitons & Fractals*, vol.14, pp.1465–1476, 2002.

[6] S. Sarasola, F.J. Torredea and A. D'anjou, "Feed-back synchronization of chaotic systems," *Int. J. Bifurc. Chaos*, vol.13, pp.177-191, 2003.

[7] X. Liao, "Chaos synchronization of general Lur'e systems via time-delay feed-back control," *Int. J. Bifurc. Chaos*, vol.13, pp.207-213, 2003.

[8] M.T. Yassen, "Chaos Synchronization between two different chaotic system using active control," Chaos, Solitons & Fractals, vol.23, pp.131-140, 2005.

[9] Y. Lei, W. Xu and W. Xie, "Synchronization of Two Chaotic Four-dimensional Systems using Active Control"

Chaos, Solitons & Fractals, vol.32, no.5, pp.1823-1829, 2007.
[10] A. N. Njah and O. D. Sunday, "Synchronization of Identical and Non-identical 4-D Chaotic Systems via Lyapunov Direct Method," *Chaos, International Journal of Nonlinear Science*, vol.8, pp.3-10, 2009.
[11] R.A. Tang, Y.L. Liu and J.K. Xue, "An extended active control for chaos synchronization," *Phys. Lett. A*, vol.373, pp.1449-1454, 2009.
[12] M.T. Yassen, "Chaos Control of Chaotic Dynamical Systems using Backstepping Design", Chaos, Solitons & Fractals vol.27, no.2, pp.537-548, 2006.
[13] B. Wang and G.J. Wen, "On the synchronization of a class of chaotic systems based on backstepping method," *Phys. Lett. A*, vol.370, pp.35-39, 2007..
[14] M. Krstic, I. Kanellakopoulus, P. Kokotovic, *Nonlinear and Adaptive Control Design*, John Wiley & sons, Inc., 1995.
[15] L. Stenflo "Generalized Lorenz Equations for Acoustic-Gravity Waves in the Atmosphere," *Physica Scr*, vol.53, pp.83-84, 1996.

Sung-Hoon Yu received the B.S. and the M.S degrees in electronic engineering from Yonsei University, Seoul, Korea, in 2005 and 2007. He is currently a Ph. D. candidate of Dept. of electrical and electronic engineering in Yonsei University. His current research interests include intelligent control, robust control, adaptive control, chaos system and robot manipulate.

E-mail : bboong7991@yonsei.ac.kr

Chang-Ho Hyun received the B.S. degrees in control and instrumentation engineering from Kwangwoon University, Seoul, Korea, , and the M.S. and Ph.D. degrees in electrical and electronic engineering from Yonsei University, Seoul, Korea, in 1999. 2002, 2008. From 2008 to 2009, he was a senior engineering in Samsung Electronics. Since 2009, he has joined the faculties of the School of Electrical, Electronic and Control Engineering at Kongju National University, where he is currently a full-time lecturer. His current research interests include intelligent control and application, nonlinear control, robotics, mobile robots.

E-mail : hyunch@kongju.ac.kr

Mignon Park received a B.S. and an M.S. in electronics from Yonsei University, Seoul, Korea, in 1973 and 1977, and a Ph.D. from the University of Tokyo, Tokyo, Japan, in 1982. He was a researcher in the Institute of Biomedical Engineering, University of Tokyo, from 1972 to 1982, as well as at the Massachusetts Institute of Technology, Cambridge, and the University of California Berkeley, in 1982. He was a visiting researcher in the Robotics Division, Mechanical Engineering Laboratory Ministry of International Trade and Industry, Tsukuba, Japan, from 1986 to 1987. He has been a Professor in the Department of Electrical and Electronic Engineering, Yonsei University, since 1982. His research interests include fuzzy control and applications, robotics, and fuzzy biomedical systems.

E-mail : mignpark@yonsei.ac.kr