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Analyzing Errors Made by Eighth-Grade Students in Solving Geometrical Problems

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In mathematical problem solving, students may make various errors. In order to draw useful lessons from the errors, and then correct them, we surveyed 24 eighth-grade students' performances in geometrical problem solving according to Casey's hierarchy of errors. It was found that:

1. Students' effect can lead to errors at the stage of "comprehension", "strategy selection", and "skills manipulation"; and
2. Students' geometric schemas also influenced their strategy selection".

Keywords: geometrical problem solving, motivation, beliefs, schemas
MESC Classification: D73, G73
MSC2010 Classification: 97D70, 97D20.

1. BACKGROUND

Many researchers have expressed interest in analyzing students' errors from different perspectives (*e. g*., Cox, 1975; De Bock, Van Dooren, Janssens & Verschaffel, 2002; Dai, 1996; Fiori & Zuccheri, 2005; Knifong & Holtan, 1976; Lannin, Barker & Townsend, 2007; Newman, 1977). After literature review, we found many studies have used Newman procedure (Newman, 1977) for analyzing students' errors in problem solving since 1977 (Casey, 1978; Clarkson, 1991; Clements, 1980, 1982; Watson, 1980; Pan & Wu, 2008). It seems that this analysis procedure attracted considerable attention from mathematics education researchers

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Newman (1977) claimed that if a student wished to obtain a correct solution to a one-step word problem, he or she should ultimately proceed according to the following hierarchy:

1. Read the problem (reading),
2. Comprehend what is read (comprehension),
3. Carry out a mental transformation from the words of the question to the selection of an appropriate mathematical strategy (transformation),
4. Apply the process skills demanded by the selected strategy (process skills), and
5. Encode the answer in an acceptable written form (encoding).

Casey (1978), by modifying and extending Newman's hierarchy of errors, produced a more general hierarchy which could be applied to the analysis of errors made on many-step word problems in mathematics:

1. Question form,
2. Question reading,
3. Question comprehension,
4. Strategy selection,
5. Skills selection, and
6. Skills manipulation.

In her hierarchy, "question form" was included as the first stage, and Newman's "transformation" category was redefined in terms of "strategy selection" and "skills selection". Newman (1977) and Casey (1978) claimed that "carelessness" and "motivation" were the two error causes which could lead to various errors at any stage of the problem solving process (Clements, 1980).

In the above-mentioned papers, the analysis technique founded on the Newman procedure was limited to probing students' errors in arithmetical or algebraic word problems. Few studies have been conducted on errors made in solving geometrical problems according to this technique. In this study, we used the analysis technique based on the Casey's hierarchy to explore secondary students' errors in solving many-steps geometrical problems.

2. METHODOLOGY

2.1. Participants

The study was conducted in November 2002, when students finished the midterm in a middle-class suburban junior school (Grades 7 – 9). Twenty-four students from four

Grade 8 classes at this school volunteered to participate in the study. The school located in Nantong Economic and Technological Development Zone near Shanghai, has seven Grade 8 classes. These classes were ranked on the basis of students' final performance (Chinese, mathematics, and English) in the previous year. The twenty-four students were selected from two mid-ranked classes based on their high geometric achievements of the midterm.

2.2. Materials

Mathematics subject at this school included two parts: algebra course and geometry course. Grade 8 students at this school learned algebra course during the previous autumn term (the first term, September 2001 – January 2002). Until the following spring term (the second term, March 2002 – June 2002), their mathematics subject didn't include geometry course. The geometric series textbook used by students was Junior Geometry, which was published by the People's Education Press in 2002. Before the midterm was held in November 2002, Grade 8 students had learned some geometric topics, such as parallel lines, construction with straightedge and compass, congruent triangles, isosceles triangle.

The purpose of the study is to investigate students' errors in solving geometric problems. In order to reduce some influence on students' errors, we would rather choose a geometric topic which hadn't been learned by these students, but also could be understood based on their prior knowledge. Compared with the series textbook Junior Mathematics, published by Shanghai Education Press in 2002, we found that a geometric topic on the theorem "the median to the hypotenuse of a right-angled triangle equals to the half of the hypotenuse" (MHRT theorem) followed close behind "isosceles triangle" in Junior Mathematics. According to Junior Mathematics, the MHRT theorem would be taught in the autumn term of Grade 8 (the first term). However, in Junior Geometry, this theorem would be taught as a corollary of a property of rectangle in the spring term of Grade 8 (the second term). It was assumed that the high-achieving volunteers had necessary knowledge and sufficient competence to understand the MHRT theorem. Hence, it is reasonable that we selected the geometric topic on the MHRT theorem to investigate students' errors made in solving problems on this particular topic.

2.3. Procedure

At the first stage, the purpose of the study is to promote the participants' understanding and application of the MHRT theorem. On the first day, the twenty-four students were interviewed individually during they were asked to read and explain the MHRT theorem, its proof, and an example (see Appendix 1). One researcher (the second author) interviewed with the students individually, the other (the first author) wrote field notes during the interviews. If a student didn't need help, the interviewer asked the student explained what he or she had read. When a student needed help, the interviewer offered help; didn't just give answer—explained how the student could find the answer. If one explanation didn't help, the interviewer should try another, and ask the student to explain his explanation back in order to find out if he or she really understood. Students were required to do exercises individually after they were interviewed (see Appendix 1).

At the second stage, we want to find out error causes of students when they solved geometric problems. The twenty-four students were examined within 45 minutes on the second day. They were required to finish three test questions (see Appendix 2). Each student was interviewed individually based on their performance in the examination (semi-structured interview). All the interviews were audio recorded at the second stage.

3. RESULTS

In this study, students' errors in solving geometrical problems can be classified into four categories.

3.1. Comprehension (includes understanding geometrical figures)

In general, if a student wants to prove a mathematical statement, he has to identify what is the hypothesis, and what is the conclusion in that statement. When to prove a geometrical statement in word sentence, Students are usually asked to draw the corresponding geometrical diagrams, and translate the sentence into the standard format[1] before students prove it.

When Qian confronted TQ 3, he translated the statement into the standard format in which the statement is represented with mathematical symbols: Given that $\angle ACB = 90°$, $\angle A = 30°$, D is the midpoint of AB, prove CD = ½AB.

Episode 1: Interview with Qian

1.	Interviewer:	Can you read the question?
2.	Qian:	Yes. In a right-angled triangle, if an acute angle is thirty degrees, then the leg to the acute angle equals to the half of the hypotenuse.
3.	Interviewer:	Great! What do you want to prove?

[1] In a geometrical statement, the hypothesis is written as "given", and the conclusion is written as "to prove". The mathematical relations (spatial, numerical), and the geometrical elements in the statement are represented by mathematical symbols.

4.	Qian:	To prove… the median to the hypotenuse, yes, equals to the half of the hypotenuse.
5.	Interviewer:	Can you read it again?
6.	Qian:	Oh, wrong! … I think this question is similar as the theorem which I learned yesterday.

Although Qian read the question correctly (Utterance 2), he intended to prove "the median to the hypotenuse equals to the half of the hypotenuse" (Utterance 4). We think that the error was not caused by carelessness, but by his immoderate motivation—Qian considered intensively the theorem which was introduced by researchers on last day (Utterance 6). His intention prevented him from receiving all of the information in the statement.

When Sun confronted TQ 3, the translation errors occurred in her work (see Figure 1). Her translation as following: Given that ∠C = 90°, D is the midpoint of the hypotenuse, an acute angle is 30°, prove the median to the hypotenuse equals to the half of the hypotenuse. Sun replaced "the leg to the acute angle" by "the median to the hypotenuse" in her translation. In the interview with Sun, she explained: "because the sentence is too long, it is difficult to identify the hypothesis… I don't know what to prove." In addition, Sun remained most word representations in her written work, failed to translate them into mathematics symbols. Although, Sun's error seems as Qian's, but the cause is different, Sun misunderstood the statement of TQ 3.

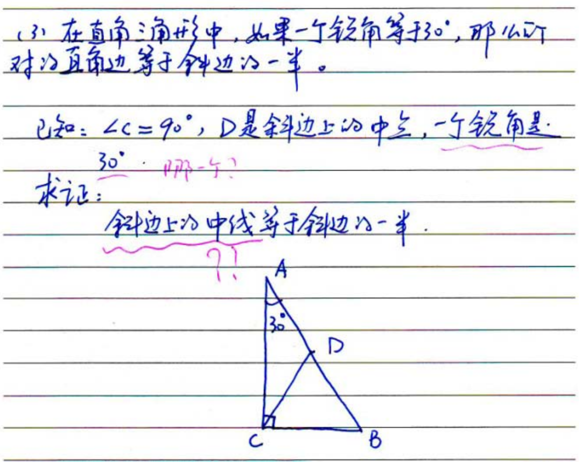

Figure 1. Sun's transformation on TQ 3.

In the process of solving TQ 1 (see Figure 2), Zhao directly employed the hypothesis of "BM = BD". This hypothesis was not provided in the original statement, also hadn't been justified before she used it. In fact, "BM = BD" was unnecessary for solving TQ 1. In our interview with Zhao, She explained: "when I set N as midpoint of side AC, I want to prove two triangles ΔMBD and ΔNCD congruent. At that time, I paid attention to identify the two triangles congruent, and also the two segments BM and BD seemed equivalent, so …" From her explanation, it seems that the visual geometrical figure influenced Zhao's solving process.

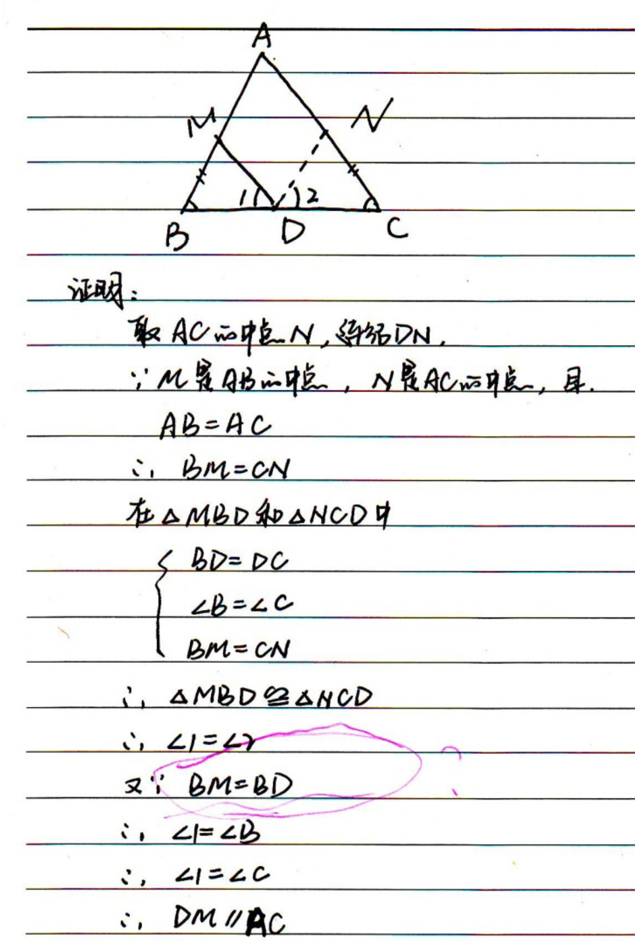

Figure 2. Zhao's solution to TQ 1.

3.2 Strategy selection

When students confronted TQ 2, Three students of them connect point M and N, then try to prove that ΔPNM is an isosceles triangle where ∠NMP =∠MNP. For example, in episode 2, Li failed to prove ∠NMP =∠MNP in ΔPNM. In fact, it is impossible to selecting this strategy to prove MP=NP in TQ 2.

Episode 2: Interview with Li

1.	Interviewer:	Can you explain your solution?
2.	Li:	I connect point M and N... If ∆PNM is an isosceles triangle, then PM=PN. So I want to prove ∠NMP =∠MNP… I just know ∠BAC =120°, ∠BAD =∠CEA = 60° (see Figure 3). How to prove ∠NMP =∠MNP... I don't know…

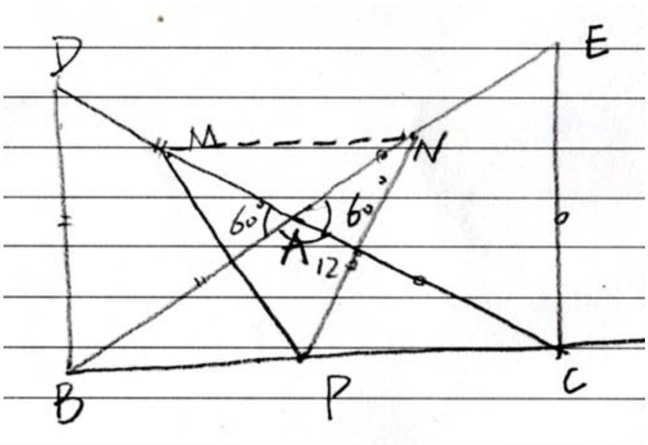

Figure 3. The drawing in Li's solution to TQ 2.

Four students employed another impossible strategy; they want to prove ∆MPB and ∆NPC congruent. Zhou constructed segment BM and NC, and had confidence to prove the two triangles congruent (Utterance 1, Utterance 4). It was unfortunate that ∆MPB and ∆NPC are incongruent when side AB is unequal to side AC in ∆ABC.

Episode 3: Interview with Zhou

1.	Interviewer:	Can you tell me what are you thinking?
2.	Zhou:	Ok. From my drawing (see Figure 4), you see… ∆BMP and ∆NCP seem congruent…
3.	Interviewer:	Can you prove your argument?
4.	Zhou:	You know, I tried again and again, but I think so… I know BP=CP, BM is perpendicular to AD, and CN is also perpendicular to AE (see Figure 4)… Let me try again!

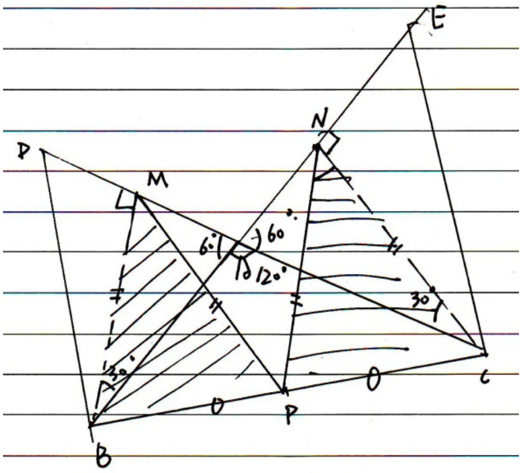

Figure 4. The drawing in Zhou's solution to TQ 2.

Wu constructed a segment between B and M firstly, then tried to prove ΔBMC is a right-angled triangle where ∠BMC=90° (see Figure 5), but he gave up this strategy at last. Wu told us: "if I can justify ∠BMC=90°, then MP equals to the half of the BC. But how to prove ∠BMC=90°, I have not any good idea. … Now, I try to prove that ΔMNP is an isosceles triangle." In fact, Wu turned to another strategy because he failed to connect priority knowledge to prove that BM is perpendicular to side AD in equilateral ΔABD.

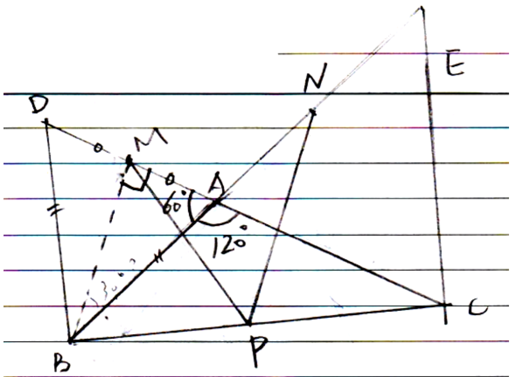

Figure 5. The drawing in Wang's solution to TQ 2.

3.3 Skills selection

In the secondary level, the mathematics syllabus required that students should master the skills of five basic constructions (Ministry of Education, 2000). In the textbook *Junior*

Geometry1, the construction “to construct a segment equal to a given segment” was introduced firstly. The unit of “construction with compass and straightedge” in textbook *Junior Geometry2*, introduced the other four basic constructions:

1. To construct an angle equal to a given angle;
2. To construct a bisector of a given angle;
3. From a given point, to construct a perpendicular to a given line; and
4. To construct a perpendicular bisector of a given segment.

Students can apply the five basic constructions to construct complex figures.

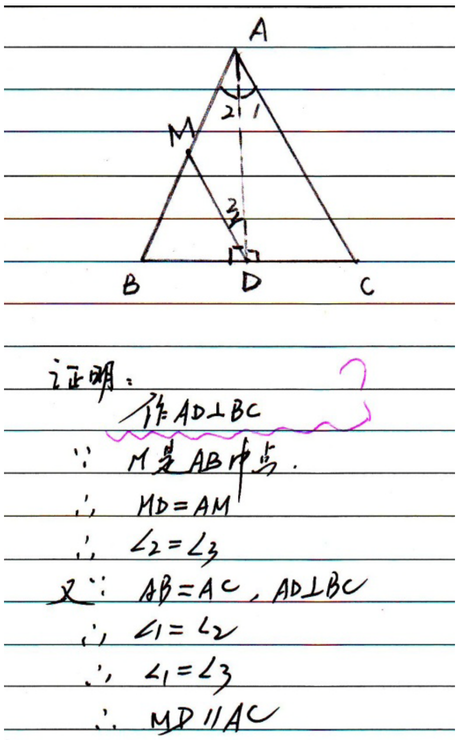

Figure 6. Zhen’s solution to TQ 1.

However, sometimes students will confuse proofs with constructions. They usually employ improper constructions in place of the necessary reasoning. For instance, Zhen wrote “construct $AD \perp BC$” (see Figure 6), then he completed his proof based on the perpendicular relation of AD and BC. But he failed to distinguish connecting a segment between two given points from constructing a perpendicular from a given point. In general, it is impossible to construct a perpendicular to a given line from two different given points except that some particular conditions are confined. Wang’s solution to TQ 3 is an extremely example of the confusion, where he wrote down "construct AD = BC = BD" (D is the midpoint of the hypotenuse BC in right-angled ΔABC). In the interview, Zhen explained: "If you connect point A and D, you see…AD is perpendicular to BC. Yes, it's correct! " It seems that visual feature of the figure conduct the negative effect on his geometrical reasoning.

3.4. Skills manipulation

When Feng solved TQ 1, she justified ΔBMD and ΔCND congruent, then produced an outcome of "∠1=∠C" (see Figure 7**).** In episode 4, we find that Feng confused the corresponding angles in the two congruent triangles (utterance 2). According to her solution, ∠1 in ΔBMD is corresponding to ∠NDC in ΔCND. In fact, "∠1 =∠C" is correct, but can't be deduced directly from ΔBMD and ΔCND congruent. Eight students had the errors similar as Feng.

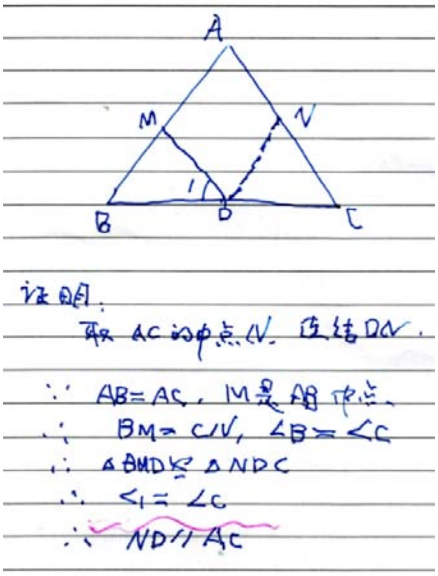

Figure 7. Feng's solution to TQ 1.

Episode 4: Interview with Feng

1.	Interviewer:	Can you tell me why ∠1 =∠C ?
2.	Feng:	Ok. I have proved ΔBMD and ΔCND congruent (see Figure 7). Because the two triangles are congruent, each pair corresponding angles are equal. So, ∠1 =∠C.
3.	Interviewer:	Do you think ∠1 is corresponding to ∠C?

4. Feng: Yes. You see…they look like.

In Chen's solution to TQ 1 (see figure 8), he forgot to connect point D and A, and directly applied the MHRT theorem in the question situation. In the interview, Chen explained: "I tried again and again. I had a good idea when just three minutes left. I worried no time, so I wrote down the solution too quickly…" Chen seems too tense when he wrote down the solution to TQ 1.

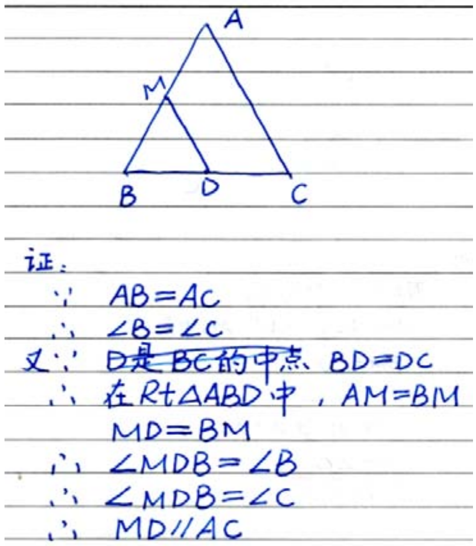

Figure 8. Chen's solution to TQ 1.

4. CONCLUSIONS AND DISCUSSION

This study supported the argument that students' motivation and beliefs influence on their mathematical problem solving (Cobb, 1985; Buchanan, 1987). Motivation was believed to maintain students' information process from their environment in terms of salient goals or values (Ames & Ames, 1984). According to the Yerkes-Dodson law (Yerkes & Dodson, 1908), the relationship between motivational level and behavioral efficiency is an inverted U function (Bregman & Mcallister, 1982). It means that the top performance is achieved at some intermediate level of motivation. Berlyne (1966) also claimed that there is a tendency to maintain an optimal level of motivation to receive external information. The level of motivation below this optimal level attempts to increasing receive information. The motivation above this optimal level creates a need to reduce receiving information. In this study when Qian intensively attempted to apply the theorem (the median to the hypotenuse of a right-angled triangle equals to the half of the hypotenuse) to solve TQ3, his strong intention replaced the external information in comprehension stage. Zhao "paid attention to identify the two triangles congruent" in TQ 1, so that she used visual evidence to prove it. Zhen's intention for success led to his tension, so that he forgot to connect point A and D at the stage of "skills manipulation". These pieces of evidence are provided to confirm that students' motivation above the optimal level in solving mathematics problems can negatively affect their cognitive behaviors.

Schoenfeld (1985) claimed that students' naive empiricism can affect their behaviors in mathematical situations. It described an understanding of separation proofs and constructions when students solved a construction problem using straightedge and compass. They accepted or rejected a potential solution to the problem just according to the accuracy of the construction. As a result of the fact that constructions were graded by how good they looked, proofs were seen as the formal confirmation of results that are already known. In this study, at the stage of "skills selection", students used constructions in place of proofs. Another example is Feng's solution to TQ 1. Although he had proved two triangles congruent, Feng identified two corresponding angles in the congruent triangles according to the construction (looks like). These pieces of evidence supported the claim that students' naive empiricism can affect their mathematical behaviors. Schoenfeld (1988; 1989) found that this belief can be as a direct consequence of their instruction, such as emphasizing repetitive practice, focusing on the mastery of mechanical procedures as isolated skills. From the above analysis, we found that students' effect can lead to errors at the stage of "comprehension", "strategy selection", and "skills manipulation".

One way to organize existing knowledge is through relatively stable, internal networks called schemas. Schemas are provided a framework for interpreting students' difficulties in problem solving (Chinnappan, 1998; Nesher & Hershkovitz, 1994; Sweller, 1989). Within the field of Euclidean geometry, diagrams play a central role. Therefore, Chinnappan (1998) invented the term "geometrical schema", which evolves around a particular shape (*e.g.* a right-angled triangle) connecting other concepts and knowledge about how and when use these concepts. It was claimed that the inside organization of a geometric schema, and an extent of those connections between geometric schemas are important to solve problems (Chinnappan, 1998). The evidence from this study confirmed Chinnappan's claim. When Wu confronted TQ 3, he turned to another strategy as he failed to activate the connections (among "the median to a side", "a perpendicular to the side", and "the bisector of the angle to the side") inside of the equilateral triangle schema. When Li and Zhou solved TQ 3, they activated the isosceles triangle schema and the congruent triangles schema respectively. However, they failed to connect their activated schemas with the right-angled triangles ∆BMC and ∆BNC (see Figures 4 and 5). It seems that the students' "strategy selection" was influenced by the quality of schemas (cf. Huang & Cheng, 2003).

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APPENDIX 1. EXERCISES

Theorem. The median to the hypotenuse of a right-angled triangle equals to the half of the hypotenuse.

Given that in ΔABC, $\angle ACB = 90°$, CD is the median of AB, prove

$$CD = \frac{1}{2} AB.$$

Proof. (Here, the proof is omitted).

Example: Given that in ΔABC, $\angle B = \angle C$, AD is the bisector of $\angle BAC$, E, F are the midpoints of side AB and AC respectively, prove $DE = DF$.

Proof. (Here, the proof is omitted).

Exercises.

1. Given that in ΔABC, $AD \perp BC$, M, N are the midpoints of side AB and AC respectively, $AM = DN$, prove $AB = AC$.
2. Given that in ΔABC, BD is a perpendicular to side AC, D on the side AC, CE is a perpendicular to side AB, E on the side AB, M is the midpoint of BC, prove $MD = ME$.
3. Given that $\angle ABC = \angle ADC = 90°$, E is the midpoint of AC, prove $\angle EBD = \angle EDB$.

APPENDIX 2. TEST QUESTIONS

Test Questions.

1. Given that in ΔABC, $AB = AC$, M is the midpoint of side AB, D is the midpoint of side BC, prove $MD // AC$.
2. Given that in ΔABC, $\angle BAC = 120°$, construct equilateral ΔABD and ΔACE adjunct to and outside the given ΔABC, M is the midpoint of side AD, N is the midpoint of side AE, P is the midpoint of side BC, prove $MP = NP$.
3. In a right-angled triangle, if an acute angle is 30°, then the leg to the acute angle equals to the half of the hypotenuse.