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Congruent Triangles Sufficient and Insufficient Conditions Suggested Milestones for Inquiry and Discussion

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In this paper we propose an inquiry task on the subject of congruent triangles. The task deals with conditions that are sufficient for congruency, and conditions that are insufficient. The aim of the task is to find the minimal number of identical components in two triangles that is sufficient to ensure congruency.

Keywords: sufficient condition, necessary condition, congruent triangles, inquiry task, respectively identical components
MESC Classification: B53, B54, B59, D59, E49, G13, G19
MSC2000 Classification: 93B27, 97B50

INTRODUCTION

One of the aims of teaching Euclidean geometry in secondary schools is the development of logical thinking and the fostering of a culture of formulating conjectures and explaining them. In *Principles and standards for school mathematics* (National Council of Teachers of Mathematics, 2000, p. 310) it is noted:

> Students should enter high school understanding the properties of, and relationship among, basic geometric objects. This knowledge can be extended and applied in various ways. Students should become increasingly able to use deductive reasoning to establish or refute conjectures and should be able to use established knowledge to deduce information about other situations.

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This aim can be achieved at different levels. Usually in class teaching we are satisfied with a level that includes acquaintance with a limited collection of conjectures, and development of ability to solve two types of problems: those that involve calculation and those that require proof based on the collection of conjectures. In these cases problems are worded to direct the students' thinking towards those specific conjectures. Proofs by negation, existence theorems, construction problems, and different interpretations of learned theorems are usually not dealt with in the framework of regular studies, and in many cases they are avoided and considered to be "outside" the syllabus

The learning of Euclidean geometry could be compared to a tour of a site rich with interesting objects and artifacts, where visitors are not allowed to see all the treasures. Rather, they are lead to believe that they can see everything by sticking to the marked path through the site, and that anything outside that path is a "minefield." In other words, students develop misconceptions about the site. According to Patkin (1994, 1996) it is necessary to mould students' thinking by exposure to different types of proofs presented in different forms, and by the use of mistakes and misconceptions.

The acquisition of the understanding of a concept is a gradual process (Van Hiele, 1986; 1999). Learners should be helped to get to a stage where they can differentiate between concept image and concept definition (Vinner, 1991). Care should be taken that the collection of examples accumulated during their studies will include more than prototype examples (Hershkovitz, 1998). Correct use of examples is likely to influence and improve learners' mathematical knowledge.

In our opinion it is important to vary methods of teaching geometry, and to integrate with these methods examples and questions that might be considered non-conventional, where the learner is encouraged to ask questions and to investigate them. The process of raising questions and discussing them is important from a didactical point of view and serves as a means of encouraging learners to arrive independently at definitions of concepts and wording of conjectures. In this case, the teacher acts as discussion guide or investigation leader along relevant paths that include evaluating decisions and conclusions. It is important to note that the tools that are available to the learners do not always enable them to find an answer to every question. Sometimes, for the same reason, teachers find it difficult to explain answers to students. However, the advantages of this type of instruction outweigh the disadvantages. The acquirement of understanding is managed in a gradual way, and a process is developed that corresponds with the constructivist approach, according to which the learner builds knowledge for himself. Thus it is possible to reduce difficulties that cause the development of mistakes and the growth of misconceptions (Hershkovitz, 1987).

SUGGESTED MILESTONES FOR INQUIRY AND DISCUSSION

In this paper we propose milestones for inquiry and discussion in the instruction of the subject congruent triangles. It is possible to use these milestones at the initial stages of learning the concept “congruent triangles”, and also later, in learning other branches of mathematics.

The goal of the inquiry task proposed below is to achieve four aims:

a. Understanding the concepts “necessary condition” and “sufficient condition” and the difference between them.
b. Deducing that three is the minimum number of identical components in two triangles that is sufficient for congruency, but that not every combination of three identical components in two triangles is a sufficient condition for congruency.
c. Acquiring experience in construction problems.
d. Intuitive understanding of equivalent conditions, production of equivalent conditions through familiar connections between the different components of a triangle, and the opposite – deducing the existence of connections between the components based on the equivalence of the conditions.

The concept “congruent triangles” is taught in secondary school. In their learning of the concept “congruency” students learn that congruent triangles are triangles that can be placed one on top of the other such that they match at every point. If two triangles are congruent to each other, then there are six equalities: the three sides are respectively equal and the three angles are respectively equal. These six conditions are necessary for the two triangles to be congruent. The sufficient conditions, learned in secondary school and called the congruency theorems, are different combinations of three conditions from the six necessary conditions.

Now follow four conjectures about congruency theorems. The first conjecture is: if two sides of one triangle are equal respectively to two sides of a second triangle, and the angle contained by these sides in one triangle is equal to the angle contained by these sides in the second triangle, then the two triangles are congruent. This conjecture is known as the first congruency theorem, and in short is called “side, angle, side” (SAS). The second conjecture is: if one side of a triangle is equal to a side of another triangle, and the two angles adjacent to this side in the first triangle are equal respectively to the two angles adjacent to the side in the second triangle, then the two triangles are congruent. This conjecture is known as the second congruency theorem, in short “angle, side, angle” (ASA). The third conjecture deals with respective equality between three sides in one triangle and those in another triangle, and is known as the third congruency theorem, “side,

side, side" (SSS). The fourth and last congruency theorem is called "side, side, and the angle opposite the longer side" (SSA). The conjecture is: two triangles, in which two sides and the angle opposite the longer of the two sides are equal respectively, are congruent. The emphasis on the particular angle derives from the fact that two triangles, in which two sides and the angle opposite the shorter of the two sides are equal respectively, are not necessarily congruent.

In this paper we suggest a way in which, after the definition of the concept "congruent triangles" and the discovery of the six equalities between the respective sides and the respective angles deriving from the congruency, it is possible to reach the classic congruency theorems by means of an inquiry task. The aim of the inquiry task is to find the minimum number of identical components in the two triangles that is sufficient for congruency. The inquiry task can be divided into three stages:

Stage A: Can one identical component in two triangles be sufficient for congruency?

Stage B: Can two components respectively equal in two triangles be a sufficient condition for congruency?

Stage C: Can three components respectively equal in two triangles be a sufficient condition for congruency?

Students can arrive at negative answers in stages A and B by constructing counter-examples. In stage B they have to deal with all the possible combinations of two identical components in the triangles, which are:

B1—side, side;
B2—angle, angle;
B3—side, angle adjacent to the side; and
B4—side, angle opposite the side.

In stage C students first have to define with all the possible combinations of three components, which are:

C1—three angles;
C2—three sides;
C3—two sides and the angle contained by the sides;
C4—two sides and the angle opposite one of the sides;
C5—one side and its two adjacent angles;
C6—one side and two angles, one angle adjacent to the side and one opposite it.

In this stage students will identify the four congruency theorems SAS, SSS, ASA, and SSA, and will also conclude that not every combination of three respectively equal components constitutes a sufficient condition for congruent triangles. After the discovery of the theorem SSA opposite the longer of the two sides, stage C of the inquiry task is completed.

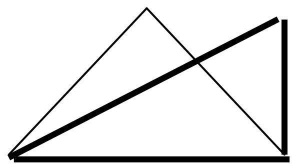
In subsequent geometry classes it is possible to expand the list of necessary conditions for congruent triangles. For example, in congruent triangles all the following components are also respectively equal: medians, altitudes, angle bisectors, radii of inscribed and circumscribed circles, area, and perimeter. This fact raises the questions:

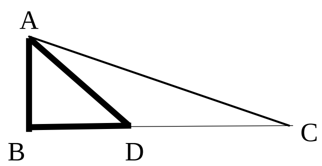
1. Is it possible to formulate additional congruency theorems where the sufficient conditions for congruency will include also equality between some of these other components?
2. Do negative answers to the questions in stages A and B of the inquiry task remain valid when relating to identical components from among these additional components?

We will now present several examples that can be used in stages A–C of the inquiry task in relation to the subject of sufficient and insufficient conditions for congruency.

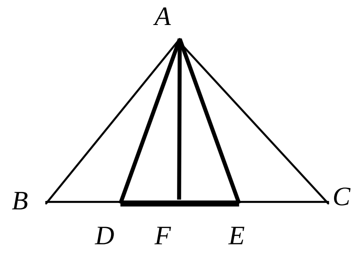
Stage A: Existence of one equal component in two triangles is insufficient for congruency

Examples:

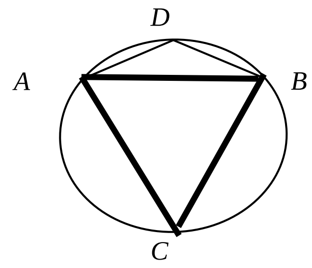
A1. Two triangles that have one common side but are not congruent. One of the triangles is acute-angled and isosceles, and the other is right-angled such that one of the perpendicular sides coincides with the base of the isosceles triangle.	

A2. Two triangles that have one common angle but are not congruent. ΔABC and ΔABD are two right-angled triangles that are not congruent. (The same angle is in each triangle.)	

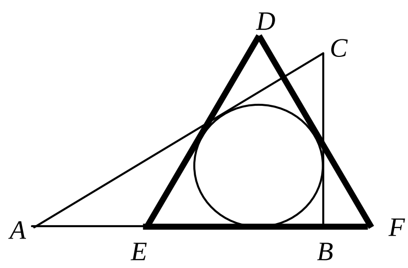
A3. Two triangles that have a common altitude (median or angle bisector) but are not congruent. The point F is the midpoint of segments *DE* and *BC* (points *B*, *D*, *E*, and *C* are on the same straight line). Segment AF is perpendicular to *BC*, so AF is an altitude (median and angle bisector) in each of the isosceles triangles *ABC* and *ADE*, which are not congruent.



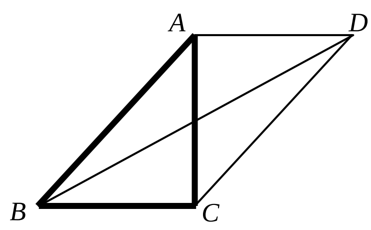
A4. Two triangles that are inscribed in the same circle—with the same radius—but are not congruent.
AB is a chord of the circle but not a diameter. Points C and D are on the circle at opposite sides of chord AB. Triangles ABC and ABD are inscribed in the same circle but are not congruent.

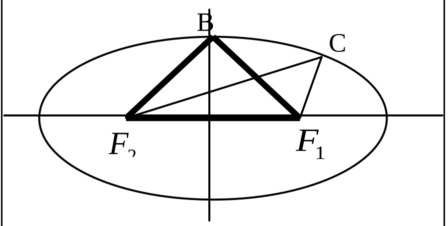


A5. Two triangles that circumscribe the same circle—with the same radius—but are not congruent. *ABC* is a right-angled triangle and *EDF* is an equilateral triangle. The two triangles circumscribe the same circle but are not congruent.



A6. Two triangles that have equal area but are not congruent.
ABCD is a parallelogram and ΔABC and ΔBDC are triangles of equal area but are not congruent (one of them is acute-angled and the other is obtuse-angled).



A7. Two triangles that have equal perimeter but are not congruent. F_1 and F_2 are foci of an ellipse, B is one of the vertices of the ellipse (the endpoint of the short axis), and C is a point on the ellipse. Triangles F_1F_2C and F_1F_2B have equal perimeter but are not congruent (one of the triangles is isosceles and the other is not).	

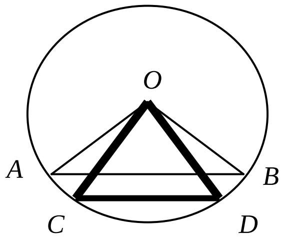
Stage B: Existence of two respectively equal components in two triangles is insufficient for congruency

Examples:

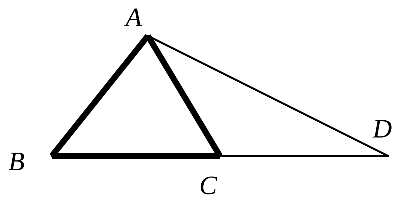
B1. Example A6 is also an example for two triangles with common side and equal area that are not congruent.

B2. Example A7 is also an example for two triangles with common side and equal perimeter that are not congruent.

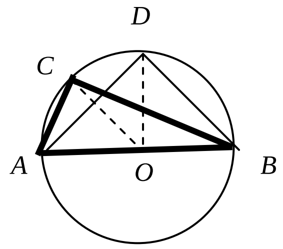
B3. Example A4 is also an example for two triangles with common side and inscribed in the same circle that are not congruent.

B4. Two triangles that are equal respectively in two sides but are not congruent. Points *A*, *B*, *C*, and *D* are on circle *O* such that points *C* and *D* are on arc *AB* whose size is less than 180^o. *OD* = *OB* = *CD* = *AO*, all radii. Thus, two sides of triangle AOB are equal respectively to two sides of triangle *COD* but the triangles are not congruent.	

B5. Two triangles that are equal respectively in one side and an adjacent angle but are not congruent.
ΔABC and ΔABD have a common side AB and a common angle adjacent to the side ($\angle ABD = \angle ABC$) but the triangles are not congruent.



B6. Two triangles that are equal respectively in one side and the opposite angle, or a side and the radius of the circumscribing circle, or an angle and the radius of the circumscribing circle, or a side and the median to that side but are not congruent.

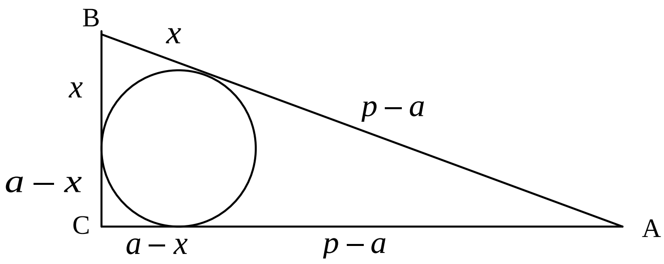


Points *A*, *B*, *C*, and *D* are on circle *O*, *AB* is a diameter of the circle and *DO* is perpendicular to AB.

In the framework of this paper, due to the mass of examples, we will not relate to all the combinations of two respectively equal components in two non-congruent triangles. We leave it to the reader to try to construct suitable examples. It is important to note that, in constructing examples, it is necessary to be meticulous about the process so that there can be no doubt about the existence of the constructed shapes. Of course it is also important to ensure that the triangles whose components are respectively equal are in fact not congruent.

We would like to present just one more example, belonging to stage *B*, that in our opinion is less common or less trivial than the other examples:

B7. Two triangles that have equal perimeter and equal area but are not congruent.
Let us say that we are given a triangle with area S and perimeter $2p$. If r is the radius of the circle inscribed in the triangle, then from the formula $S = pr$ it follows that r, the radius of the circle inscribed in the triangle, is defined unambiguously.
Therefore finding two triangles of equal perimeter and area that are not congruent is equivalent to constructing two non-congruent triangles of equal area that circumscribe the same circle.
Let us say that triangle *ABC* has area S and circumscribes a circle of radius *r*, where the sides of the triangle are $AB = C$, $BC = a$, and $AC = b$.



$$a+b+c=\frac{2S}{r}=2p$$

From this it follows that side *BC* is divided into two segments of which one is x and the other is $a-x$, side *AC* is divided into segments $a-x$ and $b-a+x$ that equals $p-a$, and side *AB* is divided into two segments x and $p-a$. We can also use Heron's formula:

$$S=\sqrt{p(p-a)(p-b)(p-c)}$$

$$\Updownarrow$$

$$p^2\cdot r^2=p(p-a)(p-b)(p-c)$$

$$\Updownarrow$$

$$Sr=(p-a)(p-b)(p-c)$$

$$\Updownarrow$$

$$Sr=(p-a)\cdot x\cdot(a-x)$$

$$\Updownarrow$$

$$x^2-ax+\frac{Sr}{p-a}=0$$

If there are two solutions to this equation, then they must be positive since $a>0$ and also

$$\frac{Sr}{p-a}>0\,.$$

Thus, to ensure the existence of a triangle according to S, r and p, a sufficient condition would be that the parameter a satisfies

$$a^2-\frac{4Sr}{p-a}>0\,.$$

For example, let us say that $S = 6$ (units of area), $p = 6$ (units of length), and $r = 1$ (unit of length). We will find two triangles that satisfy these conditions (equal areas and equal perimeters) but that are not congruent.

1. A right-angled triangle with sides $a_1 = 3,\ b_1 = 4,\ c_1 = 5$ (units of length) satisfies the conditions.
2. Let us say that in a triangle that satisfies the condition "one of the sides is $a_2 = 3.5$ (units of length)". Inserting the values into the quadratic equation results in the equation: $x^2 - 3.5x + 2.4 = 0$ whose solutions are

$$x_{1,2} = \frac{3.5 \pm \sqrt{2.65}}{2}$$

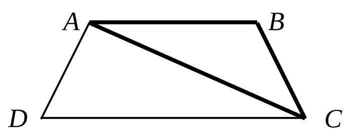
$$\Downarrow$$

$$b_2 = p - x_1 = 6 - \frac{3.5 + \sqrt{2.65}}{2} \approx 3.44$$
$$c_2 = p - a + x_1 = 2.5 + \frac{3.5 + \sqrt{2.65}}{2} \approx 5.064$$

Stage C: Existence of three respectively equal components in two triangles is not always sufficient for congruency

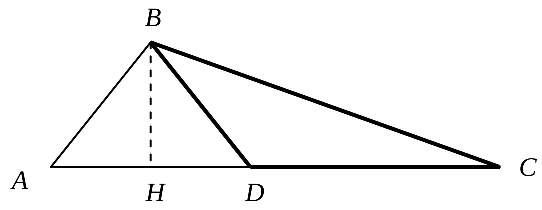
Examples:

C1. If three angles in one triangle are equal respectively to three angles in another triangle, the triangles are not necessarily congruent.

C2. Two sides and the angle opposite the shorter side. *ABCD* is an isosceles trapezoid (*AB* is parallel to *CD* and *BC* = *AD*). In ΔABC and ΔADC two sides are equal respectively (*AC* = *AC* and *BC* = *AD*) and the angles opposite one of these sides are also equal ($\angle BAC = \angle ACD$), but the triangles are not congruent.	

C3. If two sides in one triangle are equal respectively to two sides in another triangle, and the altitudes to the third side are also equal in the two triangles, the triangles are not necessarily congruent.

ΔBDC BDC is obtuse-angled ($\angle BDC > 90°$), BH is the altitude to side DC, $HD = AH$ (point A is on the continuation of *CH*) and *BD* = *AB*. That is to say, triangles *ABC* and *BDC* (that are not congruent) have two respectively equal sides (*BD* = *AB*) and *BC* = *BC*), and the altitude to the third side BH is common to both the triangles.



In the framework of expanding the task in stage C it is possible to find combinations of three respectively equal components in two triangles that constitute sufficient conditions for congruency. For example, it is possible to prove additional congruency theorems like side, median, side and median, median, median.

Theorem (side, median, side): If two sides of one triangle are respectively equal to two sides of a second triangle, and the medians to the third side are also equal in both triangles, then the two triangles are congruent.

Given: ΔABC and ΔKLM ; *BD* is median to side *AC* in ΔABC and *LN* is median to side *KM* in ΔKLM ; *LM* = *AB*; *KL* = *BC*; *LN* = *BD*.

We need to prove: $\Delta ABC \cong \Delta MLK$

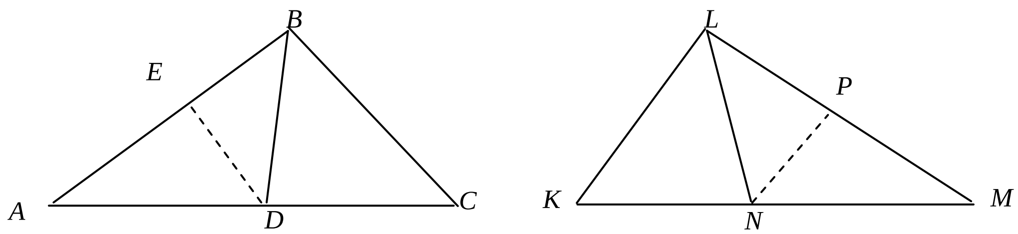


Figure 1. $\Delta ABC \cong \Delta MLK$

Proof. Constructions:

1. In triangle *ABC* we draw through point *D* a straight line parallel to side BC. *E* is the

intersection point of that line with side *AB*.

2. In triangle *KLM* we draw through point *N* a straight line parallel to side *KL*. *P* is the intersection point of that line with side *LM*.

According to the theorem—if a segment bisects one side of a triangle and is parallel to another side then it bisects the third side—we reach the conclusion that *ED* joins the midpoints of two sides in triangle *ABC* and *NP* joins the midpoints of two sides in triangle *KLM*. From here, according to the theorem—if a segment joins the midpoints of two sides in a triangle then it is half the length of the third side—it follows that

$$DE = \frac{1}{2}BC, NP = \frac{1}{2}KL\ .$$

It was given that $KL = BC$, and so $NP = DE$. From here, it follows that $\Delta DBE \cong \Delta NLP$ according to the theorem SSS. From this congruency it follows that $\angle EBD = \angle PLN$ (corresponding angles in congruent triangles). Thus $\Delta ABD \cong \Delta MLN$ according to the theorem SAS, and from here it follows that $NM = AD$ (corresponding sides in congruent triangles). But

$$AD = \frac{1}{2}AC, NM = \frac{1}{2}KM\ ,$$

and so $KM = AC$.

Thus triangles ABC and MLK are congruent according to the theorem SSS (QED).

(In proving the theorem side, median, side we relied on theorems about a segment joining the midpoints of two sides in a triangle, and on the standard congruency theorems.)

Using this theorem and the theorem about the intersection point of medians in a triangle, it is possible to prove another congruency theorem (median, median, median): If three medians in one triangle are respectively equal to three medians in a second triangle then the triangles are congruent.

Just before we finish …

After confirming the fact that three respectively equal components in two triangles do not always constitute a sufficient condition for congruency, it is possible to discuss instances where there exist in two non-congruent triangles more than three respectively equal components.

Instance 1: Existence of four respectively equal components in two triangles does not always constitute a sufficient condition for congruency. In example *C*3, in triangles *ABC* and *BDC*, the respectively equal components are two sides $(BC = AB,\ BC = BC)$, the altitude to the third side (BH), and the angle opposite the smaller of the two sides ($\angle BCD = \angle BCA$).

Instance 2: In conclusion we will give an example of two non-congruent triangles with five equal components (although there is not complete correspondence between all the equal components). Triangle *ABC* is isosceles (*AC* = *AB*).

$\angle BAC = 36°$, $\angle ABC = \angle ACB = 72°$, *BD* bisects angle $\angle ABC$, and *BH* is the altitude to side AC. Therefore in ΔBDC $\angle DBC = 36°$, $\angle BDC = \angle BCD = 72°$. The two triangles have a common side $(BC = BC)$, three angles equal ($\angle BAC = \angle DBC$, $\angle ACB = \angle BCD$, $\angle ABC = \angle BDC$), and the altitude to one of the sides $(BH = BH)$. Further examples of "5-con" triangles appear in Pawley (1967) and Burke (1990).

CONCLUSION

In this paper we presented an example of an inquiry task on the subject of congruent triangles. This activity facilitates understanding of the subject itself and also helps to accomplish "meta" goals, such as the development of mathematical thinking through the use of a prolonged inquiry that integrates different mathematical contents. The examples constitute an additional layer that clarifies for learners the distinction between their concept image and the actual concept definition of congruency, and contribute to a distinction between sufficient and necessary conditions. Experience with this kind of inquiry task encourages the development of intuitive understanding of equivalent and non-equivalent conditions, with one all-inclusive aim—to deepen the learner's understanding and knowledge.

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